① Quantum Computation & Quantum Information (QIC) 2016.02.23 教3-304 3学分

focus simple / difficult

Lecture I Quantum Mechanics (QM)

I.1. About the Course

1a. Class meeting 周二下午 教3-304

1b. Instructor Yong ZHANG

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Teaching assistant

To be Affirmaed

1c. lec. notes ≠ class notes → ur notes, for examination

teacher ↓

Yong Zhang 'Introduction to QIC. pdf' (version 5) → to be updated

↓ 老师自己的讲义 [illegible] V4

1d. Reference

Main reference: is for majority of you

Michael Nielsen & Issac Chuang, 'Quantum Computation and Quantum Information'

PDF in homepage of instructor

Hard Copy RMB 80

↓ 与 lecture notes 不同, [illegible] focus

Teaching plan is by Main reference

Recommended reference for excellent (difficult!)

John Preskill (Caltech), 'Online lec. notes on QIC'

PDF → Instructor's homepage
→ preskill's homepage

1e. Homeworks

NC's problem sets (mostly) OR Preskill's problem sets (few cases)

1f. Research

Ask the Instructor for it (If you have very×3 (strong) interest in QIC)

Do something in theoretical physics

1g. Evaluation (ur grading)

Do Not have middle exam

homeworks (50分) + Final exam (50分)

is an open test,
open to ur → homeworks
→ class notes

I.2. Quantum Mechanics is Weird (不可思议) (most think they understand, but don't understand...)

I.2.a. Classical world (our daily life)

The particle picture: Newtonian dynamics
Thermodynamics
Statistical Mechanics

The wave picture: the wave theory

Sound ↙ ↓ gravity ↘ electrodynamics

I.2.b. Crisis in Classical Laws

eg. The atom is NOT stable, because the moving electron will spiral into neucleus (有$\vec{a}$)

(particle picture)

eg. 康普顿散射, 黑体辐射
(wave picture)

I.2.c. Quantum world (Every where)

- The atom & tiny objects obey QM

The particle picture → NOT correct
The wave picture →

- The probability wave amplitude is Right. The particle-like object (有粒子性, [illegible]) has Momentum and Energy, $\vec{p} = \hbar\vec{k}$, $E = \hbar\omega$ associated with wavenumber $\vec{k}$ and frequency ω

([illegible])

The particle like picture is Right (不是 wave-like)

- The probability wave amplitude $\psi(\vec{x}) = \langle \vec{x} | \psi \rangle$ (bra | ket), $|\psi(\vec{x})|^2$: 表示 the probability of the Object (event) at space $\vec{x}$

eg. plane wave: $\psi(x) \propto e^{i\vec{k}\cdot\vec{x}} \Rightarrow$ free particle with definite momentum $\vec{p} = \hbar\vec{k}$

(why strange) $|\psi(x)|^2 = 1 \Rightarrow$ this particle (and object) can show up everywhere (with equal p)

确定的动量 → 完全不确定的位置

- Heisenberg's uncertainty relation

$\Delta p = 0 \quad \Delta x = +\infty$

$\Delta x \Delta p \geq \frac{\hbar}{2}$ (与经典世界完全不同)

I.2.d. Conventional QM.

partial differential Eq.

Apply QM to understand natural Object in nature. Do we understand natural object in nature?

- Condensed Matter Physics (Super Conductor ...)
- High Energy Physics, particle physics (Standard Model)
  Quantum Field Theory (next semester)
- Quantum Gravity
- ...

[illegible]

I.2.e. Modern quantum mechanics

Question: Do we understand QM itself?

Richard P. Feynman → 'I think I can safely say that Nobody understands QM'

Answer 1: QIC represents further modern development of QM, as a new type of Advanced QM between QM and Quantum Field theory.

(QIC是在理解QM究竟是什么?)

Answer 2: QIC focuses on the understandings and applications of fundamental principles of QM, instead of always solving partial differential eqs.

(理解QM的方法, 不再cheating 方法)

Answer 3: QIC aims at designing quantum systems to perform a task, instead of understanding natural objects in nature.

(design human beings? 先去理解QM. 再design)

I.3 History of QIC

QM

Computer Science

Information Science

Cryptography (Security)

Electronic Engineering

understand → QIC ← help

I.3.a QM (1970s–1980s) (–present)

Schrödinger's QM, Heisenberg's QM: 1920s

Bohr's QM (1910s)

Planck's QM (1900s)

1970s–1980s: No-cloning theorem (1980s) → No copy machine in quantum world

No-copying (Copy machine in Daily life)

Thm. An unknown quantum state cannot be perfectly copied on a quantum copy machine. (quantum copy is forbidden)

Remark: If thm wrong, what happen?

→ Uncertainty relation wrong! (一旦可copy, 可copy 很多, so many copies, so make precise measurement)

→ Special relativity wrong! (A signal cannot be transmitted faster than c.

→ 若可 copy:

$v > c \Rightarrow$ Space-like region, go to the past

$\Rightarrow$ we know the quantum state $\Rightarrow$ we can copy

Quantum Information is essentially different from classical Information

1990s – now: In experiments Complete control of single quantum Systems

(Control an atom, an electron 巨大的技术进步)

Question: How to design and control Composite quantum Systems to perform a task.

I.3.b Computer Science

Computer (Daily life)

- Turing machine (1930s) 图灵机
- Von Neumann machine (1940s)
- Electronic devices (such as transistors)

Question: when Electronic devices become smaller & smaller, quantum effects must be considered. (导体, 电路)

Answer: QIC!

lecture II Historical development of Quantum Computation & Quantum Information

II.1. philosophy for doing homeworks (personality) (Rules)

Doing Homeworks = Doing Research

perspective: Focus on a small problem, in two weeks

Eg. 解决一个problem

persistence: Solve ur problem completely

patience: write down ur solution step by step

power: Quote ur reference carefully

II.2 History of QC & QI

QM

Computation

Information

QC & QI

Cryptography

II.2.a Quantum Mechanics

AIM: We want to understand QM itself

Approach: performing information processing & Computational tasks in Quantum mechanical Systems.

Result:

Intuition about QM

(讲述对量子力学有直观的认识)

↓

predictive power of human minds (without 计算, 推导)

Ex1. Quantum No-cloning Thm

Remark: No Quantum copy machine is compatible with quantum world ... classical world copy machine

① linear superposition principle

② The Heisenberg uncertainty relation

③ The causality principle in Special Relativity. (USC)

Ex2. We are able to Control a single quantum system (An atom, An Electron)

In experiment, we can build up small-scale quantum computer.

Remark: QC & QI is Experiment!

④

II.2.b Computer Science [illegible]
(Electronic computer)

II2b-1 : Fate (Destiny) of electronic computer

Def. Turing machine (1938, Turing, Computer [illegible]) ······ [illegible] Turing machine

Theoretical model of classical deterministic computation

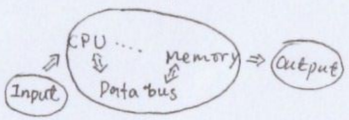
Note: Digital circuit model $\stackrel{\text{ }}{=}$ Turing machine

Def. Universal Turing machine

which can simulate other Turing machines

The Von Neumann machine (World War II)

A realistic model of the Turing machine using electronic devices



Albert Einstein → special relativity
one stone → general relativity (Classical physics) – Achievement positive

Failure, Negative (Are we understand ?)
QI & QC

John Von Neumann

achievement
1. Father of Modern Computer
2. pioneer of quantum mechanics ······ Failure
(Density matrix
Quantum measurement theorem
The Von Neumann entropy)
3. He has nothing with QC & QI

Von Neumann
left hand: Computer
right hand: QM
QC & QI

Einstein
left hand: SR
Right hand: QM
Quantum Field Theory

lesson: Do you really understand ur talents, and do you really want to use them?

No, because Von Neumann didn't.

Remark: When electronic components (such as transistor) are becoming smaller and smaller, the function of this components must be changed by quantum effects. The fate of Electronic Computer is QC.

II2b-2. The Universe is a quantum computer or NOT?
theoretical physics

△ The Church – Turing thesis (viewpoints)

Any Algorithmic process can be simulated by the Turing machine.

Algorithmic process which can be performed on a physical device.
physical concept

Turing machine : mathematical notion.

Remark: Doing research on Turing Machine. (This is Computer Science)

The Church-Turing thesis is foundation of modern computer science.

physics = mathematics

△ The Strong Church – Turing Thesis

Any algorithmic process (physics) can be Efficiently simulated by Turing machine.

Def. Efficient Algorithm runs in space and time polynomial in size of the problem.

Inefficient Algorithm runs in space and time super polynomial (Typically exponential) in the size of the problem solved.

Ex. The problem size $N=100$

Time (Efficient Algorithm) $= N^2 = 100^2 = 10000$

Time (Inefficient ...) $= 2^N = 2^{100}$

△ The modefined Strong Church – Turing thesis

Randomized (probabilistic) algorithm (ex. Monte Carlo Simulation, Random number)

Any algorithmic process can be simulated Efficiently by probabilistic Turing machine.

△ The modefined modefied ··· (1985) (David Deutsch's proposal) conjecture

Any physical process can be efficiently simulated by quantum computer.

Remark: If ture, the universe is quantum computer!

([illegible])

II2b-3. Quantum Algorithms Exceeding classical Algorithms. (much better)

	Quantum Computer	Classical computer
physics ([illegible])	Quantum mechanics	classical physics
mathematics ([illegible])	probability amplitude	Deterministic probabilistic

There indeed exist some algorithmic process which can not be efficiently simulated on classical computers but can be efficiently simulated on quantum computer.

eg1. Deutsch's algorithm (1988)

eg2. Shor's algorithm for Factoring (1993) > [illegible] ≒ classical computation [illegible]

eg3. The Grover's algorithm for search. (1994)

quantum simulation (Feymann, 1982)
Some quantum systems

Conjecture.

– Quantum systems cannot be efficiently simulated on a classical computer, but can be efficient simulated on

II 2b-4 Fault-tolerant quantum computation

QC - Environment
disturbance
noise

计算机每秒产生错误.
但可以给出正确的结果 (Fault-tolerant)

errors
faults ...... Quantum Error-correction Codes .

II.2.c. Information

Communication ... Classical

Cryptography

:

quantum

II.2.d Quantum Entanglement

1. first due to Einstein
Schrödinger Cat

2. Distinguish quantum mechanics
from classical physics

3. widely used in QC & QI

lec III will about Qubits

⑤ 2016.03.08

Lecture Ⅲ Qubits

Ⅲ.1. Definitions

什么是信息(经典,量子) 什么是量子信息
计算(经典,量子)

Def Information is physical, (Rolf Landauer 1961) 信息的物理含义：物理即信息是观念上的改变

and is something encoded in physical systems.

"quantization"? Classical Inf → encoded in Classical system ↓ Quantization
Quantum Inf → encoded in Quantum system

Def Computation is a physical process (David Deutsch 1985)

and can be performed on an actual physical system.

David Deutsch, 1985

what computers can or cannot compute is determined by the physics alone, and not by mathematics.

computer	←一一对应→	physical system
computation		motion
Input		Initial state
output		Final state
program (Rules)		Laws of motion

双态系统 物理 ⟺ 计算机

Classical computation ←processed— classical device
"quantization"? ↓ quantization
quant. comput. ←processed— quant. system

Def QIC is the study of using the fundamental principles of QM to perform Information processing and computational tasks. (theoretical, 严格的, exact right)

QIC is the study of perform Information processing & computational tasks in quantum mechanics systems. (experimental)

如何控制? 量子计算机 (classical) (quant.)

Def. (Appreciated in this course)

QIC is the study of combining classical system and quantum system to perform Information processing & computational task.

计算→如何计算的变化? 计算物理

Ⅲ.2. Research Topics in QIC 量子信息的研究内容

Δ Think about computation / Inf physically.

Δ Think about physics computationally.

Δ Design quant. systems (such as quantum computer) between the single & simple quant. systems (in particle physics) and complex quant. systems (in chemistry)

open question 1: QM + SR = QFT (Quant. Field Theory) next semester — 这些物理的结合 < 量子, SR
↓ 量子场论

QIC + SR = ?

QIC + GR = ?

QM + GR = Quant. Gravity

open question 2:

classical systems can not in general simulate quantum systems efficiently

(如何证明??)

open question 3: Did universe is quant. computer OR NOT?

Ⅲ.3. Single qubit

Single Qubit: 定义, $\mathcal{H}_2$, computational basis, 态矢.
测量 < outcome, prob., post-m state
实现
几何表示: Bloch sphere $(\theta,\phi) \Leftrightarrow |\psi_+(\theta,\phi)\rangle$ 是 σ_n 的本征态 (+1)
$|\psi_-(\theta,\phi)\rangle$ (−1)

Ⅲ.3a. BIT (bit):

Full name: binary digit

Def: Smallest unit of classical Information & computation.

Range set: $\{0,1\}$

State: either 0 or 1.

measurement (output): 0 or 1

Physical Implementation - - - - → Electric voltage: Higher is "1", Lower is "0"

Qubit (Quantum bit):

Def ⟺ Indivisible unit of quant. Inf. & computation.

↓ Two-dimensional Hilbert space $\mathcal{H}_2$

$\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$ ($\mathcal{H}_2$ (Hilbert space) { 线性空间 2-dim, 复的, 内积的 })

(另外的基矢, $|0\rangle, |1\rangle$ 是 basis)

$|0\rangle, |1\rangle$ Computational basis

↳ Orthonormal basis

$\langle 0|0\rangle = 1, \ \langle 1|1\rangle = 1, \ \langle 0|1\rangle = 0$

$|0\rangle\langle 0| + |1\rangle\langle 1| = I_2$ (completness relation)

State: a complex vector $|\psi\rangle \in \mathcal{H}_2$
$= \alpha|0\rangle + \beta|1\rangle, \ \alpha, \beta \in \mathbb{C}$

(Linear superposition, principle)

归一化条件 $\langle\psi|\psi\rangle = 1 \Leftrightarrow |\alpha|^2 + |\beta|^2 = 1$, α, β: probability amplitude.

$\alpha = \langle 0|\psi\rangle$
$\beta = \langle 1|\psi\rangle$

measurement: collapse of wave function — 没有人说得清 measurement! 现在没有人说得清是为什么.

state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Born's rules → collapse, → probability

measurement output (out come)	0 or 1	
post measurement state	$\|0\rangle$	$\|1\rangle$
probability	$\|\alpha\|^2$	$\|\beta\|^2$

测量和别的系统是相互 可以改变的事

physical Implementation : 1° the spin of a single electron

2° Two independent polarization of a photon

3° nucleu spin in magnetic field

4° atom : ground state $|0\rangle$ ⋯→ 在目前来看，实现qubit的途径很多
excited state $|1\rangle$

实验：如 electron, photon, nucleau 等形式

因此也是大的 computer.

Remark1. Bit Qubit

physical object

Abstract mathematical object 大部分时候，仅谈理论 ⋯→ Turing machine

↖ 抽象的数学概念

Remark2. Bit : 0 or 1

Qubit : $\alpha|0\rangle+\beta|1\rangle \Leftrightarrow (\alpha \text{ or } \beta)$ Continuous complex number

Information in a bit is 2

Information in a qubit is ∞ ⋯→ 一个量子比特，可以隐藏无限的信息.

(Hidden, potential)

Remarks3. observed information in a bit : 2 但一但测量，wave collapse.

A single measurement on a qubit : 2 就只有2个结果

↖ beautiful idea

but reality is ugly

$\infty \to 2$ 为什么会坍缩?!!

的坍缩问题 量子力学尚无法解释

我们只能接受这一现象

QM is Counter Intuitive

$\infty \xrightarrow{\text{bits}} 1$ bit

A jump between unobserved states and observed states

→ 与宏观、引力、联有关?

对量子力学的一种挑战

Ⅲ.3b. The Bloch Sphere representation of a single qubit

$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$, $\alpha,\beta\in\mathbb{C}$ → 4 real number　　qubit的记法

$\langle\psi|\psi\rangle=1$ → $|\alpha|^2+|\beta|^2=1$ → 3 real number γ, θ, ϕ

The global phase $e^{i\gamma}$ $(\gamma\in\mathbb{R})$ has no physical meaning ← ⋯ probability $\overset{\shortparallel}{|\;|^2}$ 不受到影响 $|e^{i\gamma}|^2$

$|\psi\rangle = e^{i\gamma}|\psi_+\rangle = e^{i\gamma}|\psi_+(\theta,\phi)\rangle$

$|\psi_+(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$0\le\theta\le\pi$, $0\le\varphi<2\pi$

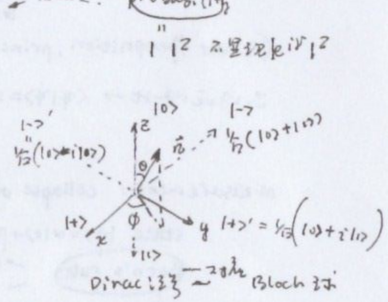
$(\theta,\phi) \Leftrightarrow$ The Bloch sphere

$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{n}=\hat{e}_z=(0,0,1)|_{\theta=0}$, $|\psi_+(\theta,\phi)\rangle=|0\rangle$

$\vec{n}=-\hat{e}_z=(0,0,-1)|_{\theta=\pi}$, $|\psi_+(\theta,\phi)\rangle=e^{i\phi}|1\rangle$

$\vec{n}=\hat{e}_x=(1,0,0)|_{\theta=\frac{\pi}{2},\phi=0}$. $|\psi_+(\theta,\phi)\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) := |+\rangle$



Ⅲ.3c. The stabilizer formalism of a qubit

(operator formulation)

$\{|\psi_+(\theta,\phi)\rangle\} = \{|\psi\rangle \mid \sigma_n|\psi\rangle = |\psi\rangle\}$

用算符的本征值(特征)的语言来表示

$|\psi_+(\theta,\phi)\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$

用 Bloch sphere 表示态没有这么直观

Pauli matrix

$\sigma_1=\sigma_x=X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\sigma_3=\sigma_z=Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\sigma_2=\sigma_y=iXZ=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$\sigma_n=\vec{\sigma}\cdot\vec{n}=\sigma_x n_x+\sigma_y n_y+\sigma_z n_z=\begin{pmatrix}\cos\theta_2 & \sin\theta_2 e^{i\phi}\\ e^{i\phi}\sin\theta_2 & -\cos\theta_2\end{pmatrix}$

Check: $\vec{n}=e_x$, $\sigma_n=\sigma_x$, $|\psi(\vec{e}_x)\rangle=|+\rangle$　　$\vec{n}=\vec{e}_y$, $|\psi(\hat{e}_y)\rangle=|+\rangle'$

$\sigma_x|+\rangle=|+\rangle$　　$\sigma_y|+\rangle'=|+\rangle'$

$\vec{n}=-\vec{e}_x$, $\sigma_n=-\sigma_x$, $|\psi(-\vec{e}_x)\rangle=|-\rangle$　　$\vec{n}=\vec{e}_z$　$\sigma_z|0\rangle=|0\rangle$

$-\sigma_x|-\rangle=|-\rangle$

用 + 记号也是表明其是 $+\sigma_x$ 的本征矢.

Note: $|\psi_-(\theta,\phi)\rangle = -e^{-i\phi}\sin\frac{\theta}{2}|0\rangle+\cos\frac{\theta}{2}|1\rangle$

$\sigma_n|\psi_-(\theta,\phi)\rangle=-|\psi_-(\theta,\phi)\rangle$

记 $|\psi_+\rangle$ 是 σ_n 的本征态

用 $|\psi_+\rangle$ 表示态，可以更好地知道它是哪个 σ_n 的 +1 本征态；用 $|\psi_-\rangle$ 表示态，⋯ 它是哪个 σ_n 的 −1 本征态

Check online lec.notes. (详讨 $|\psi_+\rangle, |\psi_-\rangle$)

IV. Two-qubit-system

Two-Qubit { $\mathcal{H}_2\otimes\mathcal{H}_2$, 基, 态矢(1对一)
测量 { 结果, 概率, post-m.state

Two-bit : x_1x_2 = 00, 01, 10, 11

Two-qubit : $|x_1x_2\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle$ (orthonormal basis $\mathcal{H}_4=\mathcal{H}_2\otimes\mathcal{H}_2$)

张量积

A general state vector of $\mathcal{H}_2\otimes\mathcal{H}_2$:

$$|\psi\rangle = \sum_{x_1,x_2=0}^{1} \alpha_{x_1x_2}|x_1x_2\rangle \longrightarrow \text{probability amplitude}.$$

$$= \alpha_{00}|00\rangle+\alpha_{01}|01\rangle+\alpha_{10}|10\rangle+\alpha_{11}|11\rangle$$

$$\langle\psi|\psi\rangle=1 \Leftrightarrow \sum_{x_1,x_2=0}^{1}|\alpha_{x_1x_2}|^2=1$$

Measurement of the first qubit (partial measurement)

$|0\rangle\otimes I_2$ → Indentity

the first qubit 0

post-measurement state : $\dfrac{\alpha_{00}|00\rangle+\alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2+|\alpha_{01}|^2}}$

probability amplitude : $\langle 0|\otimes I_2|\psi\rangle$

probability $|(\langle 0|\otimes I_2)|\psi\rangle|^2 = |\alpha_{00}|^2+|\alpha_{01}|^2$

$|1\rangle\otimes I_2$ → the first qubit : 1

$\dfrac{\alpha_{10}|10\rangle+\alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2+|\alpha_{11}|^2}}$

$|\langle 1|\otimes I_2|\psi\rangle|^2 = |\alpha_{10}|^2+|\alpha_{11}|^2$

Hidden information $\infty\times\infty=\infty^2$

observed information 00 01 10 11 — two bit

⑦

2016. 3.15

Lecture IV Multi-qubits

Homework 1st & 2nd Due March 29 (two weeks)

Home work: A single qubit $|\psi_+(\theta,\varphi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

The Bloch vector $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_n \equiv \vec{\sigma}\cdot\vec{n}$

problem 1. prove $\sigma_n|\psi_+(\theta,\varphi)\rangle = |\psi_+(\theta,\varphi)\rangle$

problem 2. prove $tr(\rho(\vec{\sigma}\cdot\vec{m})) = \vec{m}\cdot\vec{n}$

ρ called the density matrix, $\rho = |\psi_+(\theta,\varphi)\rangle\langle\psi_+(\theta,\varphi)|$

problem 3. The rotation around the $\hat{n}$-axis about θ is given by $U_{1/2}(\vec{n},\theta) = \exp(-i\frac{\theta}{2}\sigma_n)$

$|\psi_+(\theta,\varphi)\rangle = U_{1/2}(\vec{e}_z \to \vec{n})|0\rangle$, drive the explicit form of $U_{\frac{1}{2}}(\vec{e}_z \to \vec{n})$

and interpret its geometric meaning.

problem 4. $|\psi_-(\theta,\varphi)\rangle = U_{\frac{1}{2}}(\vec{e}_z \to \vec{n})|1\rangle$

derive the form of $|\psi_-(\theta,\varphi)\rangle$ and prove $\sigma_n|\psi_-(\theta,\varphi)\rangle = -|\psi_-(\theta,\varphi)\rangle$

Research Project: Try to solve the following problem with several methods, eg, 3 methods.

"Quantum correlations in a mixed state" (不是纯的状态)

consider a density matrix (一般的,可以是混合态) : $\rho = \frac{1}{8}I_4 + \frac{1}{2}|\psi^-\rangle\langle\psi^-|$, where I_4: 4x4

identity matrix, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Suppose we measure the first spin along

the $\hat{n}$-axis, and the second spin along the $\hat{m}$-axis, where $\hat{n}\cdot\hat{m} = \cos\theta$, what

is the probability that both spin are spin-up along their respective axes?

Hint: $prob(\hat{n},\hat{m}) = tr\left[\rho(\sigma_n \otimes \sigma_m)\right]$ (at least & methods)

IV.1 Single-qubit

Projective measurement of Single Qubit

意义: 本征态 $\{|a_i\rangle\}$, projectors P_{a_i}, density matrix ρ

测量结果: $prob(a_i) = tr(\rho P_{a_i})$, post state: $\frac{P_{a_i}|\psi\rangle}{[Prob(a_i)]^{1/2}}$

$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, α_0, α_1: probability amplitude

$\alpha_0, \alpha_1 \in \mathbb{C}$

$|\alpha_0|^2 + |\alpha_1|^2 = 1$.

$|\psi\rangle \in \mathcal{H}_2$

对未知(物理量)的测量

已转换得到对 basis 上的测量

(投影测量)

projective measurement on the computational basis.

$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, 基于 computational basis $\{|0\rangle, |1\rangle\}$

$|0\rangle$: $|\alpha_0|^2$; $|1\rangle$: $|\alpha_1|^2$

1° orthonormal

2° $|0\rangle\langle 0| + |1\rangle\langle 1| = I_2$

投影测量的测量理论 ↑

projective measurement on $\mathcal{H}_2 = Span\{|+\rangle, |-\rangle\}$, 其中 $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

(change measurement basis)

① orthonormal $\langle +|-\rangle = 0$, $\langle +|+\rangle = \langle -|-\rangle = 1$

$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \frac{\alpha_0+\alpha_1}{\sqrt{2}}|+\rangle + \frac{\alpha_0-\alpha_1}{\sqrt{2}}|-\rangle$

② $|+\rangle\langle +| + |-\rangle\langle -| = I_2$

$|+\rangle$: $\frac{|\alpha_0+\alpha_1|^2}{2}$; $|-\rangle$: $|\alpha_0-\alpha_1|^2/2$

project measure $\mathcal{H}_2 = Span\{|a\rangle, |b\rangle\}$

① orthonormal $\langle a|a\rangle = \langle b|b\rangle = 1$, $\langle a|b\rangle = 0$

② $|a\rangle\langle a| + |b\rangle\langle b| = I_2$

Born's measurement.

这么测量(measurement)是什么?

① out come

② probability.

③ post-measurement state

$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

out come	a	b
probability	$\lvert\langle a\vert\psi\rangle\rvert^2$	$\lvert\langle b\vert\psi\rangle\rvert^2$
post-measurement state	$\vert a\rangle$	$\vert b\rangle$

$|\psi\rangle$ collapse / jump

or $\frac{|b\rangle\langle b|\psi\rangle}{|\langle b|\psi\rangle|}$; $\frac{|a\rangle\langle a|\psi\rangle}{|\langle a|\psi\rangle|}$

projective measurement:

projectors: $P_a = |a\rangle\langle a|$, $|b\rangle\langle b| = P_b$, satisfying $P_a^2 = P_a$, $P_b^2 = P_b$, $P_aP_b = 0$ $\quad (P_a + P_b = I_2)$

density matrix: $\rho = |\psi\rangle\langle\psi|$

$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

$P_a = |a\rangle\langle a|$; $P_b = |b\rangle\langle b|$

outcome: a ; b

probability: $prob(a) = \langle\psi|P_a|\psi\rangle = tr(\rho P_a) = |\langle a|\psi\rangle|^2$; $prob(b) = tr(\rho P_b)$

投影算子的平方, 即 $\langle\psi|P_aP_a|\psi\rangle$

(mathematical)

数学上的美

P_a, ρ

tr → beautiful thing.

post-measurement state: $\frac{P_a|\psi\rangle}{Prob(a)}$; $\frac{P_b|\psi\rangle}{Prob(b)}$

Note: The observable theory is NOT explicitely involved!

problem 5: $prob(a,b) = tr\left(\rho(\sigma_n \otimes \sigma_m)\right) \times$ (答案3)

$= tr\left(\rho(E_n \otimes E_m)\right)$

E_n, E_m are projectors. $E_n = |\psi(\vec{n},\cancel{\vec{m}})\rangle\langle\psi(\vec{n},\cancel{\vec{m}})| = \frac{1}{2}(I_2 + \vec{n}\cdot\vec{\sigma})$

$E_m = |\psi(\vec{m})\rangle\langle\psi(\vec{m})| = \frac{1}{2}(I_2 + \vec{m}\cdot\vec{\sigma})$

IV.2. Two bits and Two qubits

推广到一般的测量理论

而不是用 computational basis

表示.

$|\psi\rangle = \sum_{x_1,x_2=0}^{1} \alpha_{x_1x_2}|x_1x_2\rangle$

Projectors: $P_{00} = |00\rangle\langle 00|$, $P_{01} = |01\rangle\langle 01|$, $P_{10} = |10\rangle\langle 10|$, $P_{11} = |11\rangle\langle 11|$ $\Rightarrow P_{x_1x_2} = |x_1x_2\rangle\langle x_1x_2|$

The axioms defining projectors: $P_{x_1x_2}P_{x_1'x_2'} = \delta_{x_1x_1'}\delta_{x_2x_2'}P_{x_1x_2}$

density matrix: $\rho = |\psi\rangle\langle\psi|$ $\quad \sum_{x_1x_2\in\{0,1\}} P_{x_1x_2} = I_2 \otimes I_2 = I_4$

out come: (x_1x_2)

probability: $Prob(x_1,x_2) = tr(\rho P_{x_1x_2})$

post-measurement state: $\frac{P_{x_1x_2}|\psi\rangle}{Prob(x_1,x_2)}$

⑧ **IV.3 The Bell State & the Bell Basis**

Bell态和Bell基　1. 量子关联的Bell态　2. Bell basis (EPR pairs) 定义 $|\beta_{xy}\rangle$

特点 { ① 是正交的一组，可以用来测量　② 有通式，均可由 $|\beta_{00}\rangle$ 生成　③ x, y 的意义: x … $X\otimes X$ … phase; y … $Z\otimes Z$ … parity

The Bell state (The EPR state, or The EPR Pairs)
Einstein

(3a). The state vector formulation: $|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (最典型的 (0,0) Bell态)

(3b). The stabilizer formalism

$\{|\beta_{00}\rangle\} = \{|\psi\rangle \mid \overset{\sigma_x}{X} \otimes X|\psi\rangle = |\psi\rangle,\ Z\otimes Z|\psi\rangle = |\psi\rangle\}$ (是 $X\otimes X$, and, $Z\otimes Z$ 的共同本征态).

• 注: 由 stabilizer 可以确定到 $|\psi\rangle$ 仅相差一个相位, 即 $|\psi\rangle = \frac{1}{\sqrt{2}}e^{i\gamma}(|00\rangle+|11\rangle)$

(State formulation & stabilizer (operator) formulation 是等价的)

notation:

$X\otimes X_1 = \sigma_x \otimes \sigma_x = \sigma_1 \otimes \sigma_1$ — phase-bit operator

$Z\otimes Z = \sigma_z \otimes \sigma_z = \sigma_3 \otimes \sigma_3$ — parity-bit operator

(3c) The Bell theorem

(这里讨论的是, 不讲[illegible])

Alice (Wuhan) ← - - - $|\beta_{00}\rangle_{AB}$ - - - → Bob (Beijing)

Step I. A & B prepare $|\beta_{00}\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$ in Wuhan.

Step II. Alice stays in Wuhan with her qubit.

Step III. Bob goes to PKU with his qubit.

Step IV. Alice measure her qubit and Bob has done Nothing

Out come :	0	1
prob. :	$\frac{1}{2}$	$\frac{1}{2}$
post-measurement state :	$\|00\rangle_{AB}$	$\|11\rangle_{AB}$

Step V. Bob performs his measurement, Alice done Nothing

out come :	0	1
prob. :	$\frac{1}{2}$	$\frac{1}{2}$
State:	$\|00\rangle_{AB}$	$\|11\rangle_{AB}$

observation: Alice's measurement result is the same as Bob's (相关的)

① Why? 不 (超光)

② How? What?

The measurement results are correlated called quantum correlation.

The Bell theorem:

The quantum measurement correlations are much stronger than could ever exist in classical (correlations) system.

量子纠缠: 本质的是 quantum correlations.

Remark: Such quantum Correlations (in the Bell State) can be exploited (应用) in QC & QI. Which is the first intimation that QC & QI allows informational processing beyond classical Computation & classical information.

(3d) The Bell basis

$\mathcal{H}_4 = \mathcal{H}_2 \otimes \mathcal{H}_2 = \text{span}\{|x_1x_2\rangle,\ x_1x_2 = 0,1\}$

$= \text{span}\{|\beta_{x_1x_2}\rangle,\ x_1x_2 = 0,1\}$ ← Bell basis

$|\beta_{00}\rangle = (|00\rangle+|11\rangle)/\sqrt{2}$, $|\beta_{01}\rangle = (|01\rangle+|10\rangle)/\sqrt{2}$, $|\beta_{10}\rangle = (|01\rangle-|10\rangle)/\sqrt{2}$, $|\beta_{11}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

• 可以验证 $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ 确实是线性无关的.

Note 1. Bell States (QC & QI); EPR Pairs (QM) OR EPR States
↑
theorem　贝尔理论 (EPRB states)

Note 2: $|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\left(|0,y\rangle + (-1)^x|1,\bar{y}\rangle\right)$　$\bar{y} = 1\oplus y$ (binary addition) (modulo 2)

Note 3: $|\beta_{xy}\rangle = (I_2 \otimes X^y Z^x)|\beta_{00}\rangle$　(X: σ_x, Z: σ_z)　一旦知道 $|\beta_{00}\rangle$ 可得到其他的态

Note 4: "x", "y" 的意义是什么? Observables?

$X\otimes X|\beta_{xy}\rangle = (-1)^x|\beta_{xy}\rangle$ … $(-1)^x$ 是 $X\otimes X$ 的本征值

$Z\otimes Z|\beta_{xy}\rangle = (-1)^y|\beta_{xy}\rangle$ … $(-1)^y$ 是 $Z\otimes Z$ 的本征值.

prove: $X|y\rangle = |\bar{y}\rangle$, $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$

$Z|y\rangle = (-1)^y|y\rangle$.

• 注: $X|x\rangle = |\bar{x}\rangle$, $Y|x\rangle = (-1)^x i|\bar{x}\rangle$, $Z|x\rangle = (-1)^x|x\rangle$

$(X\otimes X)\frac{1}{\sqrt{2}}\left(|0,y\rangle + (-1)^x|1,\bar{y}\rangle\right) = \frac{1}{\sqrt{2}}\left(|1,\bar{y}\rangle + (-1)^x|0,y\rangle\right) = (-1)^x|\beta_{xy}\rangle$

x. phase bit: $x=0 \Rightarrow \frac{1}{\sqrt{2}}\left(|0,y\rangle + |1,\bar{y}\rangle\right)$, superposition "+" relative phase "1"

($X\otimes X$, phase-bit operator) $x=0+1 \Rightarrow \frac{1}{\sqrt{2}}\left(|0,y\rangle - |1,\bar{y}\rangle\right)$, superposition "−" relative phase "−1"

y. parity-bit 奇偶: $y=0 \Rightarrow \frac{1}{\sqrt{2}}\left(|0,0\rangle + (-1)^x|1,1\rangle\right)$　two-qubit must be aligned.

$y=1 \Rightarrow \frac{1}{\sqrt{2}}\left(|0,1\rangle + (-1)^x|1,0\rangle\right)$　每个qubit 怎一致是一样的. two-qubit are anti-aligned.

Note 5: $\{|\beta_{xy}\rangle\}$ orthonormal basis (called "Bell basis")

prove: $\langle\beta_{xy}|\beta_{x'y'}\rangle = \delta_{xx'}\delta_{yy'}$

$\sum_{x,y\in\{0,1\}} |\beta_{xy}\rangle\langle\beta_{xy}| = I_2 \otimes I_2$

Note 5. Projective measure on Bell basis.

$P_{xy} = |\beta_{xy}\rangle\langle\beta_{xy}|$

Out comes: (x,y)

Probability: $tr(\rho P_{xy})$

State (post): $\frac{P_{xy}|\psi\rangle}{prob(x,y)}$

④ IV.3 Three bits & Three qubits

$x_1x_2x_3 = 000, 001, \cdots$

$\mathcal{H}_8 = \mathcal{H}_2^{\otimes 3} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 = \mathrm{Span}\{ |x_1x_2x_3\rangle, x_1x_2x_3 = 0,1 \}$

$|\psi\rangle = \sum_{x_1x_2x_3 \in \{0,1\}} \alpha_{x_1x_2x_3} |x_1x_2x_3\rangle$

eg. The GHZ State:

$|GHZ\rangle_3 = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

The GHZ theorem;

IV.4. n bits & n qubits

$x_1x_2\cdots x_n = \underbrace{00\cdots0}_{n}, \cdots, \underbrace{11\cdots1}_{n}$, $\#^{\text{number}}\{x_1, \cdots x_n\} = 2^n$

$\mathcal{H}_{2^n} = \mathcal{H}_2^{\otimes n} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2 = \mathrm{Span}\{ |x_1x_2, \cdots x_n\rangle \}$

$|\psi\rangle = \sum_{x_1 \cdots x_n \in \{0,1\}} \alpha_{x_1\cdots x_n} |x_1 \cdots x_n\rangle$

n-qubit GHZ state

$|GHZ\rangle_n = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\right)$

The GHZ Thm, The GHZ basis.

Remark: when $n=500$, $\dim(\mathcal{H}_{2^n}) = 2^{500}$

$\# \alpha_{x_1 \cdots x_n} = 2^{500}$

500 atoms. ok!

$2^{500} \ggg$ Number of the atoms in the visible universe.

Classical System can not in general simulate quantum systems efficiently.

Lecture V. Quantum Circuit model (I)

import
fundamental subject

Homework 1st & 2nd Due next Tuesday, 29/3

Focus on small problems in 2 weeks.

Solve in problems completely. Answer on lecture note.

Write down your solution step by step.

Quote your reference carefully.

VI.

经典电路模型
量子电路模型 > 类比 (wires, gates)

quant. gate : unitary transformation U

Def. Classical circuit model consists of wires and logical gates.

Def. Quant. Circuit model consists of wires and quantum gates.

	Wires	Gates
classical circuit model	bits	logical gates
Quant. circuit model	Qubits	quant. gates

Remark: Turing machine → theory

Circuit model (practice) > · Different computational models · Equivalent

Def. A quantum gate(s) is a quantum transformation describing a change of a quantum state vector. Satisfying

当我们谈及 state 时,往往指 density matrix

1° linearity. $\forall |\phi\rangle, |\psi\rangle \in \mathcal{H}$

$U(\alpha|\phi\rangle + \beta|\psi\rangle) = \alpha U(|\phi\rangle) + \beta U(|\psi\rangle)$

Remark: why? without linearity, violates special relativity → Time travel, → faster than light communication

thermodynamics, second law.

从其他角度回答U的线性性的原因, U的线性与相对论等的关系 上的问题.

2° probability conservation (total prob. = 1)

$\langle\psi|\psi\rangle = 1$

$|\psi'\rangle = U|\psi\rangle$

$\langle\psi'|\psi'\rangle = 1 \Rightarrow U^\dagger U = UU^\dagger = \mathbb{I}$.

U: unitary matrix.

Def. A quantum gate is a unitary matrix.

1-bit gate → id, NOT
1-qubit gate > 类比
↳ X, Z, $H = H^\dagger = H^{-1} = \frac{1}{\sqrt{2}}(X+Z)$, $R_x(\theta)\, R_Y(\gamma)\, R_z(\delta)$

$\begin{cases} H|0\rangle = |+\rangle \\ H|1\rangle = |-\rangle \\ H|i\rangle = \frac{1}{\sqrt{2}}\left((-1)^i|i\rangle + |\bar{i}\rangle\right) \end{cases}$

V.2 One-bit gate and single qubit gate

分解定理: 任何的U可分解为R_x, R_Y, R_z的乘积. (根据Bloch球旋转理解)

V.2.a one-bit gate

Identity gate: $Id: x \mapsto x$, $x = 0,1$ (trivial but important in GT)

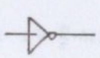
NOT gate: $NOT: x \mapsto \bar{x} = 1 \oplus x$ (Binary addition)

Truth table

In	out
1	0
0	1

NOT gate is a one-by-one mapping, so reversible gate.

NOT gate:



V.2.b quantum NOT gate

NOT gate: $NOT|0\rangle = |1\rangle$, $NOT|1\rangle = |0\rangle$ & $NOT(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$

quantum NOT gate $= \sigma_x = X$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\sigma_x = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$

NOT gate : —[X]— or —⊕—

NOT : $|i\rangle \mapsto |\bar{i}\rangle = |1\oplus i\rangle$, $i=0,1$

经典的门：只有2个 I, NOT

V.2.C There are infinitely many single-qubit gates.

Def. A single qubit gate is a 2×2 unitary matrix, and vice versa. (所有矢量都有无穷多个)

↓ terrible!

eg2.

The Pauli-z gate $Z=\sigma_z$, $Z|i\rangle = (-1)^i|i\rangle$, $i=0,1$

—[Z]—

eg3.

The Hadamard gate (between x & z) : $H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$

$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$

Hadamard gate : Unitary basis transformation from $\{|0\rangle, |1\rangle\}$ to $\{|+\rangle, |-\rangle\}$,

$\{|0\rangle, |1\rangle\}$: eigenstates of $\sigma_z = Z$

$\{|+\rangle, |-\rangle\}$: eigenstates of $\sigma_x = X$

$\underset{X}{\{|+\rangle, |-\rangle\}} \xleftarrow{U} \underset{Z}{\{|0\rangle, |1\rangle\}}$

X 的表示

$U Z U^\dagger = X$

X 的表示

H : similarity transf. $HXH = Z$

$H^2 = I$, $H^\dagger = H^{-1} = H$

—[H]—

$H|i\rangle = \frac{1}{\sqrt{2}}\left((-1)^i|i\rangle + |\bar{i}\rangle\right)$

V.2.d. Decomposition of a single-qubit gate U

Rotational gate around the z-axis :

$R_z(\delta) = e^{-i\frac{\delta}{2}\sigma_z} = \begin{pmatrix} e^{-i\frac{\delta}{2}} & \\ & e^{i\frac{\delta}{2}} \end{pmatrix}$

(初的名字)

—[$R_z(\delta)$]—

Rotational gate around the y-axis

$R_y(\gamma) = e^{-i\gamma\frac{\sigma_y}{2}} = \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2}\\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2}\end{pmatrix}$

Rotational gate around the x-axis

$R_x(\theta) = e^{-i\theta\frac{\sigma_x}{2}} = \cos\frac{\theta}{2} I_2 - i\sin\frac{\theta}{2}\sigma_x = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$

- 市学的将U分成为多个 fundamental gate.

  里是系的.

- single-qubit → bloch球上的一点

  2×2 U → 一个球上的旋转 on bloch sphere

  2×2 U → decomposition with $R_z(\delta)$, $R_y(\gamma)$, $R_x(\theta)$

Thm. (Decomposition of a single-qubit gate)

Any singlequbit gate U can be decomposed as

$U = ph(\alpha) R_z(\beta) R_y(\gamma) R_z(\delta)$

phase $ph(\alpha) = e^{i\alpha}$

$\alpha, \beta, \gamma, \delta$ are real valued.

- R_x, R_y, R_z 只有两个独立

证明 important $\begin{cases} SU_2 \text{ 群论} \\ \text{online lecture note} \\ \text{最好看下的证明} \end{cases}$

proof : online lec. notes.

Example : Decomposition of Hadamard gate ?

$H = ph(\frac{\pi}{2}) R_x(\pi) R_y(\frac{\pi}{2})$

$= ph(\frac{\pi}{2}) R_y(\frac{\pi}{2}) R_z(\pi)$

$= \cdots$ online lec. notes

- Bloch球的旋转 不能旋转 single-qubit 的 gate

check: $R_x(\pi) = -i\sigma_x$

$R_y(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

$ph(\frac{\pi}{2}) = i$

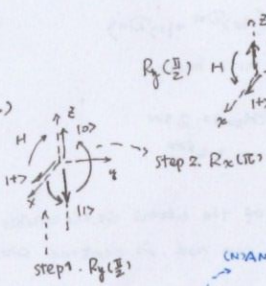
$ph(\frac{\pi}{2}) R_x(\pi) R_y(\frac{\pi}{2}) = -i \cdot i\sigma_x \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

interpret: $H|0\rangle = |+\rangle$

$H|+\rangle = |0\rangle$

(因 $|0\rangle = H^+|+\rangle$)



$ph(\frac{\pi}{2})$与U(1)无意义

V.Ⅲ. Two-bit gate and Two-qubit gate

two-bit gate
two-qubit gate > 关系

→ (N)AND. (N)OR. XOR.

定义: 4×4 U

→ $CNOT_{12}|ij\rangle = |i\rangle \otimes |i \oplus j\rangle$

图示, 矢量符

V.3.a Two-bit gate

AND : $(x,y) \mapsto x\wedge y = x\cdot y$ (Binary multiplication)

truth table	
00	0
01	0
10	0
11	1

not 1-by-1 mapping , IRReversible gate 2bit → 1bit.

(信息有损失)

OR : $(x,y) \mapsto x\vee y = x+y-x\cdot y$ (=1最少一个, 不可积之)

XOR: $(x,y) \mapsto x \oplus y$ (Exclusive gate)

NAND: $(x,y) \mapsto \overline{x\wedge y} = \overline{x\cdot y} = 1+xy$

NOR: $(x,y) \mapsto \overline{x\vee y} = \underbrace{x+y-x\cdot y}_{OR} + \underset{N}{1}$

(11)

V.3.b Two-qubit gate

Def. a 2-qubit gate is a $2^2 \times 2^2 = 4\times4$ unitary matrix, and vice versa. (also infinity)

eg. The CNOT gate (controlled-NOT gate)

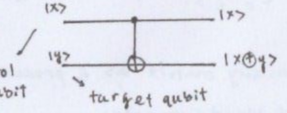
quantum generalization of XOR gate (useful) : $CNOT\,|x\rangle|y\rangle = \underset{\text{control}}{|x\rangle}\,\underset{\text{exclusive gate}}{|x\oplus y\rangle}$

$x=0$, $CNOT\,|0\rangle|y\rangle = |0\rangle|y\rangle$

$x=1$, $CNOT\,|1\rangle|y\rangle = |1\rangle|1\oplus y\rangle$ (flip)

(对y的control)

CNOT =



· 当 $x=0$，y 不变
当 $x=1$，y flip (发生翻转)

$CNOT = \underset{\text{zero}}{|0\rangle\langle 0|} \otimes \underset{\text{不变}}{I_2} + |1\rangle\langle 1| \otimes \underset{\text{bit flip}}{X}$ (du 相关 表示)

$= \begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$= \begin{pmatrix}1&0&&\\0&1&&\\&&0&1\\&&1&0\end{pmatrix}$

V.4 Quantum measurement

量子测量的 circuit 好的表示 "黑匣子"

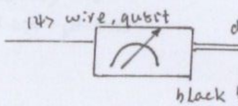
a single-qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

collapse ↓ measurement, jump ← 信息角度理解测量

a classical probabilistic bit "M"

$M = \begin{cases} 0, & prob(0) = |\alpha|^2 \\ 1, & prob(1) = |\beta|^2 \end{cases}$

· 量子的测量问题 如何在电路中表现 (这也是比测量问题)



double line, M (classical bit)

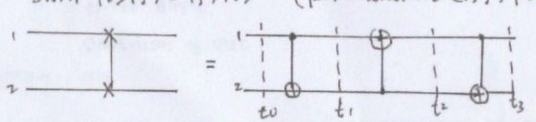
black box measurement (black-hole)

V.5 Example for simplest quant. circuit gate.

典型的 circuit gate $\begin{cases} SWAP_{12} \text{ (重要)} \\ B \text{ (制造 } \beta_{00}\ \beta_{01}\ \beta_{10}\ \beta_{11}) \end{cases}$

eg1. $SWAP_{12} = CNOT_{12}\, CNOT_{21}\, CNOT_{12}$

$SWAP\,|i\rangle|j\rangle = |j\rangle|i\rangle$ (permutation $(i,j)\mapsto(j,i)$)



· CNOT gate 的下标，以区别哪个是控制 gate.

$CNOT_{ij}$: i → control ith qubit; j → target qubit jth qubit

证明：一般直接把电路写成证明
读图: From left to right
读算符: From right to left

proof: from left to right

$|\psi(t_0)\rangle = |x\rangle \otimes |y\rangle$

$|\psi(t_1)\rangle = CNOT_{12}\,|\psi(t_0)\rangle = CNOT_{12}\,|x\rangle|y\rangle = |x\rangle\,|x\oplus y\rangle$

$|\psi(t_2)\rangle = CNOT_{21}\,|\psi(t_1)\rangle = |\underbrace{x\oplus y\oplus x}_{0}\rangle\,|x\oplus y\rangle = |y\rangle\,|x\oplus y\rangle$

$|\psi(t_3)\rangle = CNOT_{12}\,|\psi(t_2)\rangle = |y\rangle\,|x\oplus y\oplus y\rangle = |y\rangle|x\rangle = SWAP\,|x\rangle|y\rangle$

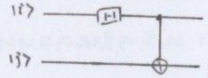
对每个门，都要通过写步骤：写 algebra → 画 circuit → 左→右证明

↑ Quantum program

eg2. The quantum circuit model of the Bell basis

Bell state $|\beta_{ij}\rangle = \frac{1}{\sqrt2}\left(|0j\rangle + (-1)^i|1\bar{j}\rangle\right)$ (产生的方法是?)
i: phase bit, j: parity bit

$|\beta_{ij}\rangle = CNOT_{12} \overset{x}{\otimes} (H\otimes I_2)\,|ij\rangle$ Right to left



left to right

proof: $|\psi(t_0)\rangle = |i\rangle\otimes|j\rangle = |ij\rangle$

$|\psi(t_1)\rangle = (H\otimes I_2)\,|ij\rangle = \frac{1}{\sqrt2}\left((-1)^i|i\rangle + |\bar{i}\rangle\right)\otimes|j\rangle$

$= \frac{1}{\sqrt2}\left((-1)^i|ij\rangle + |\bar{i}j\rangle\right)$

$|\psi(t_2)\rangle = CNOT_{12}\,|\psi(t_1)\rangle = \frac{1}{\sqrt2}\left((-1)^i|i\rangle|i\oplus j\rangle + |\bar{i}\rangle|\bar{i}\oplus j\rangle\right)$

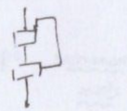
$= \begin{cases} i=0 & \frac{1}{\sqrt2}\left(|0\rangle|j\rangle + |1\rangle|\bar{j}\rangle\right) = \beta_{0j} \\ i=1 & \frac{1}{\sqrt2}\left(-|1\rangle|j\rangle + |0\rangle|\bar{j}\rangle\right) = \beta_{1j} \end{cases}$

$= \beta_{ij}$

V.6 comparison between classical model and quant. circuit model

1. No loops (Feed back) in quant. circuit model

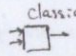
因为量子门是 unitary 的，无法表示



2. Wires cannot come together

classical

quant. gate → unitary



3. No copy gate

copy $x \to (x,x)$

(no cloning theorem → no cloning machine, so no copy gate)

Lec. VI Quantum Circuit Model

(讲授第10、复习第10)

VI.1 Universal Quantum Computation

数字电路 < 经典电路 / 量子电路

classical / quantum universal gate set { 确定 ①②③ / 无法 ① · Computation: 就是如何 运算

approximately universal gate set ①②

Def. Classical circuit model : It's a model of classical Computation, and it is finite sequence (序列) of logical gates acting on a finite sequence of bits.

Def. A Universal gate set is a finite set of elementary logical (computer small) gates which suffic(i)e(nt) to perform any function of a finite bits.

· (有限 gates 作用在有限 bits 上)

Theorem, eg.1. {AND, NOT}. {OR, NOT} is universal gate set.

Theorem, eg.2. {NAND, COPY} is a universal gate set.

· NAND = NOT∘AND

COPY(x) = (x, x)

↳ 以前可以靠实现门, 现在可以实现任何功能 · 用门电路可以做任何事情.

Proof of Thm 1. Lecture Notes (Online)

Proof of Thm 2. $NOT(x) = 1+x = 1+x^2 = 1+AND(x,x) = NAND(x,x) = NAND \circ COPY(x)$

$AND(x,y) = NOT \circ NAND(x,y) = NAND \circ COPY \circ NAND(x,y)$ □

Def. Quantum circuit model : is a model of quant. computation and it is a finite sequence of elementary (universal set) quantum gates acting on a finite number of qubits.

Def. Universal gate set in quant. Comp. is a set of elementary gates if any unitary matrix (operation) can be composed as a product of elementary gates of this set.

modify 修改 | numerical 近似

eg. {CNOT, the set of all single qubit gates}

= {CNOT, all 2×2 unitary matrices} (4×4)

· #{ } = ∞ (order) worse!

Def. Approximately universal gate set : it is a finite set of elementary quantum gates if (A discret universal gate set) (finite) any unitary matrix can be approximately composed as a product of elementary gates from this set to arbitrary precision (accuracy).

eg 1. { CNOT, Hardmard, $\frac{\pi}{8}$ gate }

$$\frac{\pi}{8}\text{ gate} = T\text{ gate} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{8}}\begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix}$$

proof: N & C book.

Corollary: All single-qubit gates (2×2 Unitary matrix) = (近似) { Hardmard gate, $\frac{\pi}{8}$ gate }

· 上面是 computation 的存在性. 后面将讨论如何实现它

eg 2. { Toffoli, Hardmard, phase }

$$\text{Toffoli } |abc\rangle = |a\rangle|b\rangle|c\oplus ab\rangle$$

8×8 unitary matrix

$$\text{phase} = S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = T^2$$

existence; 已解决

construction: · 如何 decomposed 成 universal gate set.

Note. How to decompose a unitary matrix as a product of logical gates in a universal gate set? NP-Hard (NOT EASY).

VI.2

Reversible computation : (Thermodynamics)

可逆计算 {
可逆门 和 不可逆门 { 量子门 => 可逆 / 经典可逆门 1 Id. NOT, 2 & XOR, 3 Tofoli, Fredkin
量子计算 <=> 量子可逆计算
经典可逆计算 { Thm 1 可由 1-bit, 2bit gate 实现 / Thm 2 可由 Tofoli / Thm 3 可由 Fredkin → one by one

Def. Reversible gates : are reversible (have (its) inverse)

Irreversible gates: are irreversible (have no inverse)

e.g. for reversible gates: quantum gates

e.g. for reversible gates in classical computation: one-bit gate (Id, NOT)

two-bit gate (generalized XOR)

generalized XOR $(x, y) \mapsto (x, x\oplus y)$ (one-by-one)

0 0	0 0
0 1 →	0 1
1 0	1 1
1 1	1 0

Lecture VI: Quantum Circuit Model

1. Homework 1st & 2nd Due today, are collected class by class.

2. Review: QM

Overview of QIC

Single qubit

Multiple qubit

quantum gates

(13)

three-bit gate (Toffoli gate: $(x,y,z) \mapsto (x, y, z \oplus xy)$

Fredkin gate: $(x,y,z) \mapsto (x, xz + \bar{x}y, xy + \bar{x}z)$

Landauer's principle: Irreversible logic gates (不可逆的) erages information with irreducible expenditure of power, whereas reversible gates has no expenditure of power in principle.

• (回答: 什么是 reversible 的 classical computation) 根其深刻的认识

proof: Thermodynamics

irreversible process : entropy increase ($\Delta S \neq 0$)

↑

Irreversible gates

$\Delta S \neq 0 \Rightarrow \Delta E \leq 0$

Entire proof on wiki or google

Thm1. The one-bit reversible gates and two-bit reversible gates connot perform classical universal reversible comp.

proof. Lec. notes (online)

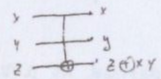
Thm2. The 3-bit Toffoli gate can perform classical universal reversible comp.

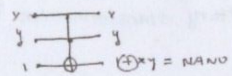
proof: Toffoli gate is reversible (Truth Table)

• 要可逆计算 而由 1,2-bit gate 实现经典的 … 3-bit gate

Toffoli gate 可实现所有经典计算. 它并又可逆, 故可实现可逆经典计算

perform NAND with Toffoli:

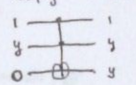




$1 \oplus xy =$ NAND

Toffoli: $(x, y, 1) \mapsto (x, y, \mathrm{NAND}(x,y))$

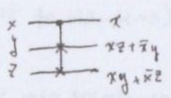
perform copy with Toffoli:



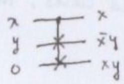
Toffoli: $(1, y, 0) \mapsto (1, \mathrm{copy}(y))$

Thm 3. The Fredkin gate can perform classical universal reversible Comp.

proof: Fredkin gate:

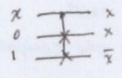


perform AND with Fredkin:



$xy = \mathrm{AND}(x,y)$

perform NOT with Fredkin:



$\} \mathrm{copy}(x)$, $\bar{x}$ NOT(x)

VI.3 classical computation on quantum Computer

control gate ⟨control operation⟩ 量子计算: CNOT, Toffoli, Fredkin

$CU|c\rangle|t\rangle = |c\rangle U^c|t\rangle$

VI.3a Controlled operation

If A true, then do B

If A false, then do C

• controlled by human being

用量子计算机实现经典计算 { 经典的可逆计算 ← 量子 Toffoli / Fredkin; 经典的随机计算 ← 量子 H + 测量 }

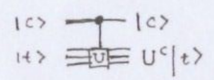
经典 copy ← 量子 CNOT (〈量子不可克隆定理: 量子 COPY〉)

Def. Controlled-Unitary gate (CU)

$CU : |c\rangle|t\rangle \mapsto |c\rangle U^c|t\rangle$

c: control qubit

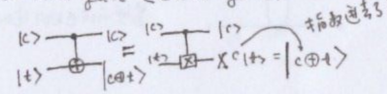
t: target qubit (一个或多个)



$U^0 = \mathrm{Id}$

$$\begin{cases} c=0 & |0\rangle\otimes|t\rangle \mapsto |0\rangle\otimes|t\rangle \\ c=1 & |1\rangle\otimes|t\rangle \mapsto |1\rangle\otimes U|t\rangle \end{cases}$$

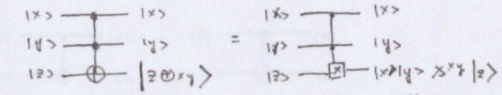
e.g. Controlled-NOT gate (CNOT gate)



$X^c|t\rangle = |c \oplus t\rangle$ 标准的表示!

e.g. Quantum Toffoli gate = Controlled-controlled-NOT gate

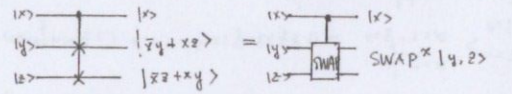
Quantum Toffoli =



$|x\rangle|y\rangle X^{xy}|z\rangle = |xy \oplus z\rangle = |z \oplus xy\rangle$

(beautiful. X 的 power 巧妙 (xy 为 0 → 1))

e.g. Quantum Fredkin gate = Controlled-SWAP gate



$|x\rangle$, $|\bar{x}y + xz\rangle$, $|xz + xy\rangle$ = $|x\rangle$, $\mathrm{SWAP}^x|y,z\rangle$

• controlled operation 很易理解 Toffoli / Fredkin gate.

check: $x=0$: LHS $|0\rangle|y\rangle|z\rangle \mapsto$ RHS $|0\rangle|y\rangle|z\rangle$; $x=1$: LHS $|1\rangle|y\rangle|z\rangle \longmapsto$ RHS $|1\rangle|z\rangle|y\rangle$

VI.3b Classical deterministic computation on quantum Computer.

• 经典计算 ⟨ 随机的, 决定的 ⟩

Thm. Quantum Toffoli gate / Fredkin (可进行 deterministic)

• Qm Toffoli ↓ Classical Toffoli ↓ 所有的经典计算

proof: Quantum Toffoli gate can simulate classical Toffoli:

$|000\rangle \leftrightarrow (000)$

$|001\rangle \leftrightarrow (001)$

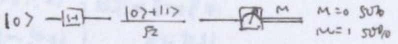
Big set ⋮ small set

$|111\rangle \leftrightarrow (111)$

(14)

VI.3C Classical Random Computation on quant. computer.

Generator of random number ($\leftrightarrow$ classical Random computation)



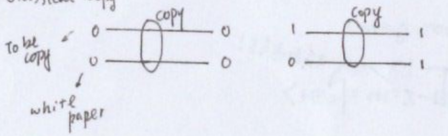
对此态测量,就能产生随机数.

若不测量1回,则还可以产生 $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

VI.3d Classical copy on quantum computer

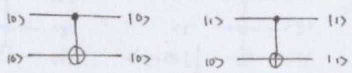
QM computer	Classical computer
↓	↓
No QM copy	Classical Copy

classical copy



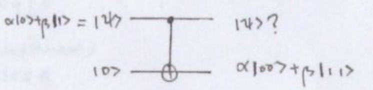
显然可用CNOT完成

quantum copy. No-cloning thm, No perfect copy of unknowing quantum state.



· 经典计算和量子计算的分界情况

Classical copy is OK.



Ideal quantum copy: $|\psi\rangle\otimes|\psi\rangle = \alpha^2|00\rangle+\alpha\beta|01\rangle+\alpha\beta|10\rangle+\beta^2|11\rangle$

realistic quantum copy: $\alpha|00\rangle+\beta|11\rangle$

$\left.\begin{array}{l}\alpha^2=\alpha,\ \alpha\beta=0\\ \beta^2=\beta,\ \alpha\beta=0\end{array}\right\rangle$ $\begin{array}{l}\alpha=1,\ \beta=0\\ \alpha=0,\ \beta=1\end{array}$ 只有当特殊情况时可以copy. (Z方向的本征态)

2016.04.03

LecVII No-cloning thm

LecVIII Quantum Teleportation

Homework 3rd, 4th Due April 21, 2016

Go to the Instructor's English homepage, Download 2-01-homework_Teleportation (paper)

problem 1. Derive Eq.(6) and Eq.(9) (simple question)

problem 2. Derive Eq.(20) and Eq.(23) (look complicated)

problem 3. Derive Eq.(97) and Eq.(99)

problem 4. Derive Eq.(100) and Eq.(102)

No-Cloning Thm $\begin{cases}\text{定理一:详述; 与经典copy的关系}\\ \text{定义: copy machine}\\ \text{Thm1: 不可cloning}\quad\text{Thm2: 真copy} \leftrightarrow \text{正交态}\quad\text{Thm3: 不disturbing, 要区分} \leftrightarrow \text{正交}\end{cases}$

VII.1 No-cloning thm

(经典计算中最基本的东西之一) 经典力学→量子力学的一个极端情况 (special limit)

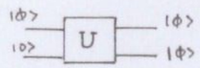
The quantum copy machine able to make a perfect copy of an unknown quantum state. does not exist in principle.

The classical copy machine really exist, because the quantum copy machine capable copying two orthogonal states does exist in principle.

Def. Quantum copy machine is defined as a unitary transformation U satisfying

$$U\left(|\phi\rangle\otimes|0\rangle\right) = |\phi\rangle\otimes|\phi\rangle$$

copied object/state — copying; blank state/document; copied



每个Remark都是good research project

Remark: Non-Unitary transf (Copy machine can be defined as non-unitary) ✓

Remark: Copy of density matrix (ensemble of state vectors) ✓

Remark: Non-perfect copy ✓

Remark: In all mentioned cases, no-cloning thm survives! (等讲带到)

Thm1. The Cloning machine U does not exist. 不误观察知道

proof: suppose U exist, then for the computational basis states $|0\rangle$ and $|1\rangle$

$$U\left[|0\rangle\otimes|0\rangle\right] = |0\rangle\otimes|0\rangle = |00\rangle$$

$$U\left[|1\rangle\otimes|0\rangle\right] = |11\rangle$$

For example, $CNOT|c\rangle|t\rangle = |c\rangle|c\oplus t\rangle$

$CNOT(|0\rangle\otimes|0\rangle) = |00\rangle$, $CNOT(|1\rangle\otimes|0\rangle) = |11\rangle$

注意，$|0\rangle\ |1\rangle$ 是 orthogonal 的

Now for $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$. $\alpha, \beta \in \mathbb{C}$

因此 这种 copy 是 perfect.

$U(|\phi\rangle\otimes|0\rangle) \overset{def}{=} |\phi\rangle\otimes|\phi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$

$U(|\phi\rangle\otimes|0\rangle) = U[(\alpha|0\rangle + \beta|1\rangle)\otimes|0\rangle] = U(\alpha|00\rangle + \beta|10\rangle) = \alpha|00\rangle + \beta|11\rangle$

linear superposition principle

Remark: QM obeys linear superposition principle, quantum copy is non-linear operation.

存在这里，所有 copy machine 都 (不可能的)

Remark, suppose $|\phi\rangle \xrightarrow{U\ OK} |\phi\rangle\otimes 2 \to \cdots\ |\phi\rangle\otimes|\phi\rangle\otimes|\phi\rangle\otimes\cdots\otimes|\phi\rangle$

∞ number of measurement of $|\phi\rangle$ (知道所有信息)

X Uncertainty relation

Thm2. No perfect copy of two non-orthogonal distinct states. (不能 copy 不同的 两个态)

Proof: Suppose $|\phi\rangle, |\psi\rangle$, non-orthogonal $\langle\phi|\psi\rangle \neq 0$.

distinctness $\langle\phi|\psi\rangle \neq 1$

• 注意：单位矢量 $\langle\psi|\phi\rangle = 1 \Leftrightarrow |\psi\rangle = |\phi\rangle$

suppose U exist, $U(|\phi\rangle\otimes|0\rangle) = |\phi\rangle\otimes|\phi\rangle$ (L)

$U(|\psi\rangle\otimes|0\rangle) = |\psi\rangle\otimes|\psi\rangle$ (R)

證本: $|\psi\rangle = \alpha|\phi\rangle + \alpha^{\perp}|\phi^{\perp}\rangle$

$\langle L|R\rangle = \langle 0|\otimes\langle\phi|U^{\dagger}U|\psi\rangle\otimes|\psi\rangle = \langle\phi|\otimes\langle\phi|\psi\rangle\otimes|\psi\rangle$

$\Rightarrow \langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2 \Rightarrow \langle\phi|\psi\rangle = 0$ 矛盾.

or

$\langle\phi|\psi\rangle = 1$

(是同一个态，这不可能)

No 'U' exist.

Thm3. No perfect copy of $|\phi\rangle$ to distinguish two non-orthogonal states without disturbing them.

(区分而不干扰)

Proof: $|\phi\rangle, |\psi\rangle$. $\langle\phi|\psi\rangle \neq 0$

Suppose U exist. $U(|\phi\rangle\otimes|0\rangle) = |\phi\rangle\otimes|e\rangle$ without disturbing

$U(|\psi\rangle\otimes|0\rangle) = |\psi\rangle\otimes|f\rangle$

即：想要区分它们，就必须改变它们.

where $\langle e|f\rangle \neq 1$ → distinguish

$\langle 0|\otimes\langle\phi|U^{\dagger}U|\psi\rangle\otimes|0\rangle = \langle\phi|\psi\rangle\langle e|f\rangle$

$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle\langle e|f\rangle \Rightarrow \langle\phi|\psi\rangle = 0$ or $\langle e|f\rangle = 1$

Remark: No-cloning theorem → fundamental principle

→ Application → Information security (good!)

Alice $\longrightarrow \bigcirc \longrightarrow$ Bob

$\bigcirc$: Eve

1. Alice want to transmit information to Bob. $|0\rangle, |1\rangle, \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

2. which is unknown to Eve (spy) → wants to know the information (distinguish)

→ wants that this attack is not $\underbrace{\text{realized by}}_{\text{(disturb)}}$ A & B.

$|0\rangle, \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ → 不改变它们，而区分它们 → 不可能

非正交的

$|1\rangle, \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

(secure in principle)

• Study Quantum key distribution online lecture notes

量子传输 {定义：protocal，传输，sharing，two-bit; 结果 (No-cloning, S.R.); 实现：sharing, prepare, 测量, 经典通信, 幺正变换}

Lec. VIII Quantum Teleportation

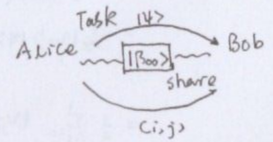
"Teleportation" appears in movie "Star Trek", In movie, how to teleport a human being from this star to another star.

In QM, How to teleport a qubit from one site to another site?

Def. Quantum Teleportation: is an information protocal of Alice transmitting an [1]unknown qubit $|\psi\rangle$ to Bob far away from her. by [2]sharing a maximally entangling state such as the Bell state $|\beta_{00}\rangle$ with Bob and [3]sending a classical two-bit message to Bob.

• 未知，任意的

- 传输 quantum
 必须 经 classical



VIII 1.a. The Algebraic description (The Standard description)

Task: Alice and Bob are in distinct locations, Alice wants to transmit unknown (前提 no-cloning thm.) qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$

Alice　　　Bob

before teleportation　　after teleportation

$|\psi\rangle_A \longrightarrow |\psi\rangle_B$

Remark: No-cloning theorem, $|\psi\rangle_A \to |\psi\rangle_A \otimes |\psi\rangle_B$ forbidden

After teleportation, $|\psi\rangle_A$ must be destructed.

Remark: Special relativity: no-faster-than-light Communication, $v \leq c$

classical communication must be involved.

Quantum communication → Collapse, jump

→ dangerous, 违反 S.R.

(16)

- step I: State preparation

  Alice & Bob shares $|\beta_{00}\rangle$

  the prepared state: $\underset{\substack{\uparrow \\ \text{to be} \\ \text{transmitted}}}{|\psi\rangle_A} \otimes \underset{\substack{\uparrow \\ \text{entangled state}}}{|\beta_{00}\rangle_{AB}}$

Thm: $\underset{\substack{\uparrow \\ \text{Alice has } |\psi\rangle}}{|\psi\rangle_A} \otimes |\beta_{00}\rangle_{AB} = \frac{1}{2}\sum_{i,j=0}^{1} |\beta_{ij}\rangle_{AA} \otimes \underset{\substack{\uparrow \\ \text{Bob has } |\psi\rangle}}{X^j Z^i |\psi\rangle_B}$ (important)

j: parity bit

i: phase bit

$|00\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{00}\rangle + |\beta_{10}\rangle\right)$ $\quad |01\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{01}\rangle + |\beta_{11}\rangle\right)$

$|11\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{00}\rangle - |\beta_{10}\rangle\right)$ $\quad |10\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{01}\rangle - |\beta_{11}\rangle\right)$

proof: $LHS = \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(\alpha|\underset{\sim}{00}0\rangle + \alpha|\underset{\sim}{01}1\rangle + \beta|\underset{\sim}{10}0\rangle + \beta|\underset{\sim}{11}1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\Big(\frac{\alpha}{\sqrt{2}}\left(|\beta_{00}\rangle + |\beta_{10}\rangle\right)|0\rangle + \frac{\alpha}{\sqrt{2}}\left(|\beta_{01}\rangle + |\beta_{11}\rangle\right)|1\rangle + \frac{\beta}{\sqrt{2}}\left(|\beta_{01}\rangle - |\beta_{11}\rangle\right)|0\rangle + \frac{\beta}{\sqrt{2}}\left(|\beta_{00}\rangle - |\beta_{10}\rangle\right)|1\rangle\Big)$$

$$= \frac{1}{2}|\beta_{00}\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}|\beta_{01}\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) + \frac{1}{2}|\beta_{10}\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + \frac{1}{2}|\beta_{11}\rangle\left(\alpha|1\rangle - \beta|0\rangle\right)$$

$$= \frac{1}{2}|\beta_{00}\rangle|\psi\rangle + \frac{1}{2}|\beta_{01}\rangle X|\psi\rangle + \frac{1}{2}|\beta_{10}\rangle Z|\psi\rangle + \frac{1}{2}|\beta_{11}\rangle XZ|\psi\rangle$$

RHS = LHS obviously

$$= \frac{1}{2}\sum_{i,j=0}^{1} |\beta_{ij}\rangle\, X^j Z^i |\psi\rangle$$

- step II: Bell measurement
  - ↗ observables ✓ $\quad X\otimes X$ phase-bit operator
  - ↘ projectors $\quad Z\otimes Z$ parity-bit operator

	$X\otimes X$	$Z\otimes Z$	two bit → $(i,j)_A$
$\lvert\beta_{00}\rangle_{AA}$	1	1	(0 0)
$\lvert\beta_{10}\rangle_{AA}$	−1	1	(1 0)
$\lvert\beta_{01}\rangle_{AA}$	1	−1	(0 1)
$\lvert\beta_{11}\rangle_{AA}$	−1	−1	(1 1)

- step III: With classical communication channel/approach (email, phone) ..., $U \leq C$

  Alice informs Bob (i,j)

Remark: $U \leq C$ $\quad I_2 \otimes I_2 \otimes Z^i X^j$

- step IV: with (i,j), Bob performs local unitary operation on his state

$$\left(I_2 \otimes I_2 \otimes Z^i X^j\right)\frac{1}{2}|\beta_{ij}\rangle_{AA} \otimes X^j Z^i |\psi\rangle_B = \frac{1}{2}\underset{\substack{\uparrow \\ \text{destroyed (no-cloning)} \\ U \leq C}}{|\beta_{ij}\rangle_{AA}} \otimes |\psi\rangle_B$$

2016.04.12

Lec VIII Quantum Teleportation

Any questions, Email, encourage

Yong-zhang @ whu.edu.cn $\quad$ instructor

kun-zhang @ ... $\quad$ 3-rd graduate

Wukonglong @ $\quad$ 4-th undergraduate

Homework 3rd & 4th $\quad$ Due April 26, 2016

VIII.1 Algebraic description of QT

standard description of QT

quantum teleportation (QT) 的 标准描述 (代数)

- 备态
- Alice测量
- 经典信号(i,j)传输
- Bob做之结果

Def: Alice $\sim |\beta_{00}\rangle \sim$ Bob, with Aim: $|\psi\rangle$ and (i,j) Two-bit

Aim: transmission of an unknow qubit

Approach: maximalled entangled state

Two-bit message

A qubit (without quantum channel, e.g. without fibre) = Two-bit message (classical channel, eg. phone) + entangled states (quantum entanglement)

Def1: The Bell basis

$$|\beta_{ij}\rangle = \frac{1}{\sqrt{2}}\left(|0j\rangle + (-1)^i |1\bar{j}\rangle\right) = \left(I_2 \otimes X^j Z^i\right)|\beta_{00}\rangle$$

Def2: basis) unitary transf. from the product basis to Bell basis

$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ $\quad |\beta_{10}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$

$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$ $\quad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

Def3: Basis transf. from Bell basis to product basis

$|00\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{00}\rangle + |\beta_{10}\rangle\right)$

$|11\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{00}\rangle - |\beta_{10}\rangle\right)$

$|01\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{01}\rangle + |\beta_{11}\rangle\right)$

$|10\rangle = \frac{1}{\sqrt{2}}\left(|\beta_{01}\rangle - |\beta_{11}\rangle\right)$

- 喜欢 diagonal mat. 不喜欢 transform.
- 如何选择这样的量子力学 observables.

Def4: parity-bit operate $Z\otimes Z$ (j - parity bit) $\quad Z\otimes Z|\beta_{ij}\rangle = (-1)^j|\beta_{ij}\rangle$ ($j=0$ spin-aligned)

phase-bit operate (i - phase bit) $\quad X\otimes X|\beta_{ij}\rangle = (-1)^i|\beta_{ij}\rangle$ ($i=0$. relative phase + $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\beta_{00}\rangle$)

⑰

Def. Teleportation equation (equality):

$|\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{2}\sum_{ij} |\beta_{ij}\rangle \otimes x^j z^i |\psi\rangle$

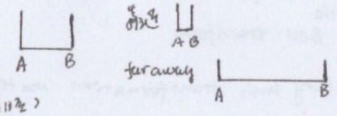
只是存在纠缠态
· Alice 含有 $|\psi\rangle$
自然地，Bob 含有 $|\psi\rangle$ (即 $|\psi\rangle$ 被传送了?)

e.g. Husband ⊗ Wife ⊗ Son = H⊗W⊗S
Entangled (child)

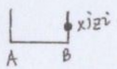
Money
Emotion

(Topological)
VII. 2. Diagrammatical representation of QT

VII. 2. a Diagrammatical representation of Bell states.

Def: $|\beta_{00}\rangle_{AB} =$ 

(topology 不变)

$|\beta_{ij}\rangle_{AB} =$ 

QT 的 topological / diagrammatical 表示
· Bell 态，矩阵，内积，外积，转置 等 的图示法
· QT 的图示：态 → 演化 (ij) (transfer operator)

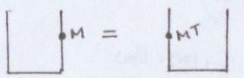
Thm: for Any 2×2 matrix "M":
(这里就是?)

$(I_2 \otimes M)|\beta_{00}\rangle = (M^T \otimes I_2)|\beta_{00}\rangle$

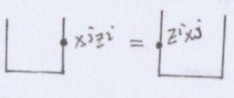
· 在 Diagrammatical 表示 中 此定理是直观的

左右 ↔ 内积 ⇒ inner product, operator, observables

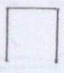
M^T: transpose

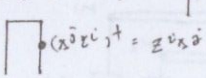


e.g.

$|\beta_{ij}\rangle =$ 

Def: Complex conjugation

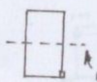
$|\beta_{00}\rangle$: $\langle\beta_{00}| = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) =$ 

$\langle\beta_{ij}| =$  $(x^j z^i)^\dagger = z^i x^j$

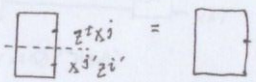
$|\beta_{ij}\rangle = (I_2 \otimes x^j z^i)|\beta_{00}\rangle$
⇕
$\langle\beta_{ij}| = \langle\beta_{00}|(I_2 \otimes (x^j z^i)^\dagger)$

Def: Inner product

$\langle\beta_{00}|\beta_{00}\rangle = 1$ (以前面的方法，对吗?)

$=$  $= \frac{1}{2}\mathrm{tr}(I_2)$

not necessary.

$\langle\beta_{ij}|\beta_{i'j'}\rangle = \delta_{ii'}\delta_{jj'} =$  $= z^i x^j x^{j'} z^{i'} = z^i x^j x^{j'} z^{i'}$

$= \frac{1}{2}\mathrm{tr}(z^i x^j x^{j'} z^{i'}) = \frac{1}{2}\mathrm{tr}(x^{j+j'} z^{i+i'})$

·注意和 P 有关 $= \delta_{jj'}\delta_{ii'}$

 $= \frac{1}{2}\mathrm{tr}(M)$

$= \langle\beta_{00}|(I_2 \otimes M)|\beta_{00}\rangle$

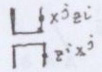
· Observables 就此 表示
operator 就此 表示

· 而的 2 对 2×2 mats
$\langle\beta_{00}|M \otimes N|\beta_{00}\rangle = \frac{1}{2}\sum_{i,j=0}^{1} M_{ij}N_{ij}$

Def: projection operators

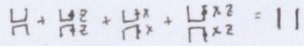
projective measurement

Bell measurement

(projector)
$P_{ij} = |\beta_{ij}\rangle\langle\beta_{ij}| =$ 

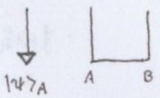
· projector < 正交的 / 这样的性质

Completeness relation

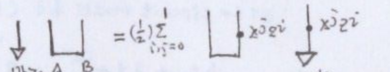
$I_2 \otimes I_2 = I_4 = \sum_{i,j=0}^{1} |\beta_{ij}\rangle\langle\beta_{ij}| =$ 

VII. 2. b Topological representation of QT

Step I: state preparation



Def. Teleportation eg.

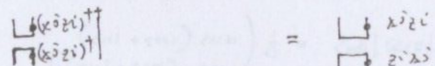


· 如何 看出 这等价?
1. 代数上 证明
2. 图示上 表示

Step II: Bell measurement using projective measurement

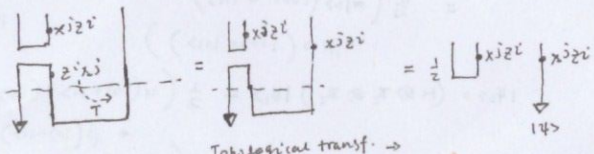
$E_{ij} = |\beta_{ij}\rangle\langle\beta_{ij}| \otimes I_2$

(Bob doing nothing)



Identity operation

Bell measurement



Topological transf. →

eg. $i=0, j=0$ $(|\beta_{00}\rangle\langle\beta_{00}| \otimes I_2)(|\psi\rangle \otimes |\beta_{00}\rangle) = \frac{1}{2}|\beta_{00}\rangle \otimes x^j z^i|\psi\rangle$

(18)

Def. Transfer operator

$$T_{A\to B} = \sum_{i=0}^{1} \overset{\text{gain}}{|i\rangle_B} \otimes \overset{\text{destroy}}{\langle i|_A}$$

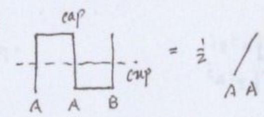
· 用 Transfer operator 描述 同步

e.g. $T_{A\to B}|\psi\rangle_A = |\psi\rangle_B$ （多: 每一次输入 $T_{A\to B}$ 描写）

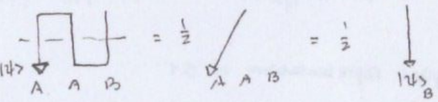
此为 Teleportation 的关键

$$\left(\langle\beta_{00}|_{AA} \otimes I_{2B}\right)\left(I_{2A} \otimes |\beta_{00}\rangle_{AB}\right) = \frac{1}{2} T_{A\to B}$$

proof (using diagramatical approach):



(transfer operator)



Proof: $(|\beta_{00}\rangle\langle\beta_{00}| \otimes I_2)(|\psi\rangle \otimes |\beta_{00}\rangle) = \frac{1}{2}|\beta_{00}\rangle \otimes X^j Z^i|\psi\rangle$ $(i=j=0)$

$= (|\beta_{00}\rangle \otimes I_2)(\langle\beta_{00}| \otimes I_2)(I_2 \otimes |\beta_{00}\rangle)(|\psi\rangle \otimes I_2 \otimes I_2)$

$(\frac{1}{2}T_{A\to B}) \quad \frac{1}{2}|\psi\rangle_B$

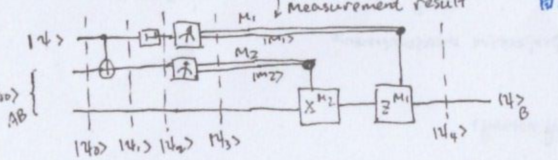
$= \frac{1}{2}|\beta_{00}\rangle \otimes |\psi\rangle$

· 量子计算的实质描述为 Teleportation

VIII.3 Quantum Circuit model of QT

QT 的 circuit model 表示（用的是 product basis 的测量，而非 Bell 测量）

前面 → 逆变换 C_H^{-1} → 测量 (product) $\xrightarrow{(i,j)}$ 传送 Bob



· 注意 $|\beta_{ij}\rangle = CNOT_{12}(H \otimes I_2)\,|i\rangle|j\rangle$ 反其逆

State preparation: $|\psi_0\rangle = |\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle(|00\rangle+|11\rangle) + \beta|1\rangle(|00\rangle+|11\rangle)\right)$

Bell measurement: (not clear)

$|\psi_1\rangle = (CNOT \otimes I_2)|\psi_0\rangle$

$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle(|00\rangle+|11\rangle) + \beta|1\rangle(|10\rangle+|01\rangle)\right)$ $\quad H = \frac{1}{\sqrt{2}}(X+Z)$

$|\psi_2\rangle = (H \otimes I_2 \otimes I_2)|\psi_1\rangle = \frac{1}{2}\left(\alpha(|0\rangle+|1\rangle)(|00\rangle+|11\rangle) + \beta(|0\rangle-|1\rangle)(|10\rangle+|01\rangle)\right)$

$= \frac{1}{2}\left(\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle\right)$

$= \frac{1}{2}\left(|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\right)$

注意 XZ 的顺序，不对易

$= \frac{1}{2}\left(|00\rangle|\psi\rangle + |01\rangle X|\psi\rangle + |10\rangle Z|\psi\rangle + |11\rangle XZ|\psi\rangle\right)$

$= \frac{1}{2}\sum_{i,j=0}^{1}|ij\rangle X^j Z^i|\psi\rangle$

$= \frac{1}{2}\sum_{M_1,M_2=0}^{1}|M_1M_2\rangle X^{M_2}Z^{M_1}|\psi\rangle$

Homework: 实验情形下 $\boxed{用}$ 的证明

it is product-basis's measurement!

$|M_1\rangle|M_2\rangle \quad X^{M_2}Z^{M_1}|\psi\rangle$

(M_1, M_2) classical communicate

Bob → perform unitary correction: $\left(I_2 \otimes I_2 \otimes Z^{M_1}X^{M_2}\right)$ to cancel $X^{M_2}Z^{M_1}$

· QT 的 Bell transform 实现方法（用的是 product basis 的测量）

· Bell transform $(C_H, B, \cdots)$

· 给定 (Bell) → 逆变换 $Bell^{-1}$ → 测量 $\xrightarrow[\text{the}]{(i',j')}$ Bob 得到

· 有了这些东西，还有研究空间吗？还有其他的理论方法吗？在于法

VIII.4 Quantum Teleportation using Bell transform

Def. The Bell transform is a unitary basis transformation matrix from the product basis to the Bell basis in which Bell states can be swapped with each other and have phase factors.

e.g. for the Bell transform: The C_H gate $= CNOT \cdot (H \otimes I_2)$

· this transf 是唯一的 fixed 的吗

· 从 $|ij\rangle$ 到 $|\beta_{ij}\rangle$ 的变换

$$= \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

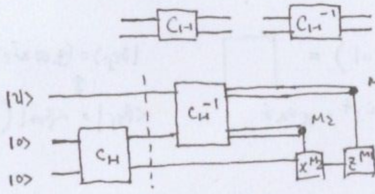
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$C_H|00\rangle = |\beta_{00}\rangle \quad C_H|01\rangle = |\beta_{01}\rangle$

$C_H|10\rangle = |\beta_{10}\rangle \quad C_H|11\rangle = |\beta_{10}\rangle$

The inverse of the C_H gate

$C_H^{-1} = (H \otimes I_2)\,CNOT$

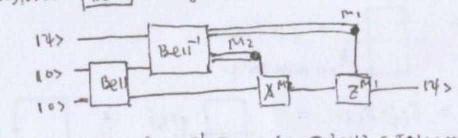


此即为前面 VIII.3 的 $|\psi_0\rangle$

注意:

$|\psi\rangle|\beta_{ij}\rangle = \frac{1}{2}\sum_{i',j'=0}^{1}|\beta_{i'j'}\rangle W_{i'j'}W_{ij}|\psi\rangle$ ①

Bell transform $\boxed{Bell}$ in general case



$(Bell^{-1} \otimes I_2)(I_2 \otimes Bell) =$ Teleportation operator

注意：由于传输方程不够明显，比如① $Bell^{-1}$ 作用的系统等价于在后，因而 Alice 测出的 M_1, M_2 应传给 Bob

(19)

Homework 3rd & 4th

The Yang - Baxter gate $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & 1 \\ & 1 & -1 & \\ & 1 & 1 & \\ -1 & & & 1 \end{pmatrix}$ is eg for Bell transform

$B|00\rangle = |\beta_{10}\rangle$ $B|01\rangle = |\beta_{11}\rangle$ $B|10\rangle = |\beta_{01}\rangle$ $B|11\rangle = |\beta_{00}\rangle$

Lec. IX Density matrix, operator → finite or infinite $\mathcal{H}$

finite dimension $\mathcal{H}$

• Density 矩阵和算符是一回事

但 matrix → finite dimension

算符 → 可以无穷维

Homework 3rd & 4th Due April 8 Tuesday 2015

1. 用密度矩阵 描述 2. 态区别的含义 3. global phase factor

IX.1 Motivation

引入density operator with (ordinary, especially)

Density matrix is an ordinary language of $^{1.}$describing quantum system, $^{2.}$especially quantum (crucial, important) open systems or subsystems of a larger system, which is $^{3.}$equivalent to the language of the state vector.

• 态矢 ↕ density matrix

1) 是另一种语言 (natural)

称为重要, 实际的语言中的

(尤其是 open system)

2) 相对简洁, 见 eg4

eg1. Quantum expectation value

State: $|\psi\rangle$ Observable: A, $\bar{A} = \langle\psi|A|\psi\rangle = tr(\rho A)$ $\rho = |\psi\rangle\langle\psi|$

对于无限维 Hilbert空间 不能, 但trace 不能

eg2. Projective measurement

(using observables 等价 / Projector 研究等价的)

The spectral theorem

$\hat{A} = \sum_n a_n P_n$

$\{P_n\}$ 是 projectors, a_n 是 eigenvalue

$p_n = Prob(n) = \langle\psi|P_n|\psi\rangle = tr(\rho P_n)$

$\langle A\rangle = \sum_n a_n p_n = tr(A\rho)$

Note: Bell measurement (用于 diagramatical 看 tr&T)

$X\otimes X = \sum_{i,j=0}^{1} (-1)^{j}|\beta_{ij}\rangle\langle\beta_{ij}|$

$Z\otimes Z = \sum_{i,j=0}^{1} (-1)^{i}|\beta_{ij}\rangle\langle\beta_{ij}|$

eg3. The post-measurement state (jump)

usual language: $|\psi\rangle$ ----- $\hat{A} = \sum_n a_n|\psi_n\rangle\langle\psi_n| = \sum_n a_n|n\rangle\langle n|$

↓ Jump/collapse

$|n\rangle$

$|\psi\rangle\langle\psi|$ ... $|n\rangle\langle n|$

$p_n = Prob(n) = |\langle n|\psi\rangle|^2 = tr(\rho P_n)$

Quantum system is state $|\psi\rangle$

post-measurement state $|n\rangle$ with p_n

mathematica language:

$\rho = \sum_n p_n|n\rangle\langle n|$

$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_n a_n|\langle n|\psi\rangle|^2 = \sum_n a_n p_n = \sum_n \langle n|A|n\rangle p_n$

$= \sum_n tr(A|n\rangle\langle n|)\, p_n = tr(\sum_n p_n|n\rangle\langle n|A) = tr(\rho A)$

$\rho = |\psi\rangle\langle\psi|$

↓ collapse (not clear)

$\rho = \sum_n p_n|n\rangle\langle n|$

eg4. The global phase has no physical meaning

(relative phase → 有意义)

$|\psi\rangle$ ⇕ 等价, strange (表示方式不同) ⟶ $\rho = |\psi\rangle\langle\psi|$

$e^{i\alpha}|\psi\rangle$ ⟶ $\rho = e^{i\alpha}|\psi\rangle\langle\psi|e^{-i\alpha} = |\psi\rangle\langle\psi|$!

而 density matrix 与 global phase 无关.

Remark: The same density matrix corresponds to ∞ number of state vectors. (difficult to understand)

密度矩阵的意义 1. 态综 $\begin{cases} \text{纯态 } \rho^2=\rho, \ Tr\rho^2=1 \\ \text{混合态} \end{cases}$

2. 随时间的演化 $\begin{cases} \text{Z.态关于时间的演化} \\ \text{以及力学量的演化 (Heisenberg)} \end{cases}$

IX.2. Def of density matrix using ensembles of state vectors (collections)

Def: Density matrix ρ is an ensemble of quantum states $|\psi_i\rangle$, $i = 1, 2, \cdots, n$, with probability

(在同时, 描述一个量子态 (同时同地) 需要一系列 $|\psi_i\rangle$)

distribution $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, and has the form

$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$

(是前面 no motivation 的推广)

• density matrix 是一个定来知道, 即 $\rho^\dagger = \rho$

它是可以对角化, 对角元是本征值, 都是正定 (≥ 0) 的.

Def: Pure state:

$n = 1$, $p_1 = 1$ ($p_2 = p_3 = \cdots = 0$)

$\rho = |\psi_1\rangle\langle\psi_1|$

State vector formalism (态矢是描述纯态的工具)

state vector is exactly specified

一般的情况, 需要 density matrix

Def: Mixed state:

$n \neq 1$, $p_1 \neq 0$, $p_2 \neq 0$, ...

$|\psi_1\rangle$ $|\psi_2\rangle$

quantum systems are specified partially.

Thm: ρ is a pure state. $\Leftrightarrow$ $\rho^2 = \rho$ (or, $tr(\rho^2) = 1$)

• why not check $tr(\rho) = 1$

(如 $tr\rho = 1$)

Note: $\rho = |\psi\rangle\langle\psi|$ for pure state

that means: one dimensional projector $\Rightarrow tr(\rho^2) = 1$.

Thm: ρ is mixed stated $\Leftrightarrow \rho^2 \neq \rho \Leftrightarrow tr(\rho^2) \neq 1$ • 判定常为混合系的方法

Thm: Time evolution

$$\rho(t) = \sum_i P_i |\psi_i(t)\rangle\langle\psi_i(t)|$$

态的 evolution → $\rho(t)$ 的 evolution

$$= \sum_i P_i\, \mathcal{U}(t)|\psi_i(0)\rangle\langle\psi_i(0)|\mathcal{U}^+(t)$$

$\mathcal{U}(t)$ 由 Schrödinger 方程 给出的 时间演化算符

Time-independent Hamiltonian

$$\mathcal{U}(t) = e^{-iHt}$$

• 比不含时时，Schrödinger 方程 形式 $|\psi\rangle = H|\psi\rangle$

$$\rho(t) = \sum_i P_i\, e^{-iHt}|\psi_i(0)\rangle\langle\psi_i(0)|e^{-iHt}$$

S $|\psi\rangle = \mathcal{U}(t)|\psi(0)\rangle$ 备用的 ... 是 $\mathcal{U}(t) = e^{-\frac{i}{\hbar}Ht}$

$$= e^{-iHt}\rho(0)\, e^{-iHt}$$

该问题比是否算符是 ...

$$= \mathcal{U}(t)\rho(0)\,\mathcal{U}^+(t) \quad (\longrightarrow \text{了} \quad |\psi(0)\rangle \to \mathcal{U}(t)|\psi(0)\rangle,\ \rho(0) \to \mathcal{U}(t)\rho(0)\mathcal{U}(t)^+)$$

$$\frac{d\rho(t)}{dt} = \dot{\mathcal{U}}(t)\rho(0)\,\mathcal{U}^+(t) + \mathcal{U}(t)\rho(0)\,\dot{\mathcal{U}}^+(t)$$

$$\text{该}\ i\frac{d}{dt}\rho(t) = [H, \rho(t)] \quad (\text{与 S 方程 相像})$$

Remark: Heisenberg EoM (equation of motion) $\neq$ 上面的方程

$$\text{该}\ i\frac{d}{dt}\hat{A}(t) = [\hat{A}(t), H] = -[H, \hat{A}(t)]$$

差一负号，所带意义

有负号 → observable

无负号 → density matrix

S 表象 picture → S 方程.

H picture → 态不演化 $\hat{A}(t)$ 的演化

density matrix $\neq$ observable

operator, state

Ix.3. The operator def of density matrix

密度矩阵的定义 2: 算符 $\begin{cases} \text{自共轭算符} \\ \text{迹 1}\ (\sum\lambda_i = 1) \\ \text{半正定}\ (\lambda_i \geq 0 \Leftrightarrow \langle\phi|\rho|\phi\rangle \geq 0) \end{cases}$

$\Updownarrow$

态综定义 (类比经典的概率分布)

指定的 $\rho \Leftrightarrow$ 无穷多种态综，通过 U 连接

Def: The density matrix ρ is an operator satisfying • 如何区别 ρ 与 $\hat{A}(t)$

1) $tr(\rho) = 1$. Unital trace (数学)

total probability is 1 (物理)

2) ρ is a non-negative operator

probability distribution $P_i \geq 0$

Note: $\rho \geq 0 \Leftrightarrow \lambda_i$, eigenvalues of ρ (不为负的) $\lambda_i \geq 0$

$\rho \geq 0 \Leftrightarrow \forall |\phi\rangle \in \mathcal{H}, \langle\phi|\rho|\phi\rangle \geq 0$

Thm: The state ensemble def of ρ is equivalent to the operator def of ρ.

proof: State ensemble $\Rightarrow$ operator

$$\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|, \quad P_i \geq 0, \quad \sum_{i=1}^{n} P_i = 1$$

• 量子 computer，必须由人操控. 因此必是 open system. 因此必由 density matrix 描述

check $tr(\rho) = \sum_{i=1}^{n} P_i = 1$

$\langle\psi_i|\psi_i\rangle = 1$

$$\forall\ |\phi\rangle \in \mathcal{H},\quad \langle\phi|\rho|\phi\rangle = \sum_i P_i |\langle\phi|\psi_i\rangle|^2 \geq 0$$

operator $\Rightarrow$ state ensemble

The spectral theorem : $\rho = \sum_i \lambda_i P_i = \sum_j \lambda_j |j\rangle\langle j|$

(重要: 请记住这定理 for any operator)

ρ density matrix $\Rightarrow \sum_j \lambda_j = 1,\ \lambda_j \geq 0$

$\Rightarrow \{\lambda_j, |j\rangle\}$ is state ensemble

Thm: A given ρ corresponds to ∞ number of state ensembles (可以通过 phase 等来实现), which are related to each other by unitary transformation.

• 指定密度矩阵，可以通过许多的 ensembles (态可以差 phase) 因此，怎样判定不同的 ensemble 对应相同的 matrix

Proof: See Nielsen & Chuang P103–104. (simple proof)

Remark: $\rho = \frac{1}{2}I_2$, $tr(\rho) = \frac{1}{2}tr(I_2) = 1$.

• 在量子统计内，必须引入密度矩阵

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$$

$H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$

$$= \frac{1}{2}(|\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|)$$

$$\rho = \{(\tfrac{1}{2}, |\vec{n}\rangle), (\tfrac{1}{2}, |\vec{n}\rangle)\}$$

$|0\rangle\ |1\rangle$: $Z|i\rangle = (-1)^i|i\rangle$

$|+\rangle\ |-\rangle$: $X|\pm\rangle = \pm|\pm\rangle$ 基的选取不同，所以

$|\vec{n}\rangle\ |-\vec{n}\rangle$: $\sigma_n|\pm\vec{n}\rangle = \pm|\pm\vec{n}\rangle$ 同样一密度矩阵对应不同的态综

$\sigma_n = \vec{\sigma}\cdot\vec{n}$

April 26. Homework 1st and 2nd.

Density matrix is quantum analogue of classical probability distribution.

$$(i, P_i) \rightarrow (|\psi_i\rangle, P_i)$$

Thermodynamics, Statistics $\rightarrow$ quantum statistics

$\langle$qubit 的态的刻画: $\rho=\frac{1}{2}(\mathbb{1}_2+\vec{p})$, $\begin{cases}\vec{p}\text{ 的意义, 取值范围}\\ \text{纯态, 混态}\end{cases}$
定理: ρ 是 density matrix
ρ 下 σ_n 的测量与取值的期望值 $\vec{p}\cdot\vec{n}$ 以及分布

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IX.4. Mixed state of a qubit (一个量子比特的混合态)

Def: The mixed state of a qubit has the form $\rho(\vec{p}) = \frac{1}{2}\left(I_2 + \overset{\text{泡利matrix}}{\vec{\sigma}}\cdot\vec{p}\right)$

① $\vec{p}$: polarization vector

② 3-dimensional, real vectors, $\vec{p}=(p_1,p_2,p_3)$

③ $|\vec{p}|\le 1$

④ when $|\vec{p}|=1$, called Bloch sphere (pure state) $\longrightarrow |\psi_n\rangle$

$|\vec{p}|\le 1$, called Bloch ball (mixed state)

$|\vec{p}|=0$, The origin of Bloch ball

·上面定义的 $\rho(\vec{p})$ 是 density 吗?

Thm: $\rho(\vec{p})$ is a density matrix

Proof: $\mathrm{tr}(\rho(\vec{p}))=1 \iff \mathrm{tr}\left(\frac{1}{2}I_2+\frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)=\frac{1}{2}\mathrm{tr}(I_2)+\frac{1}{2}\mathrm{tr}(\vec{\sigma}\cdot\vec{p})$

泡利 mat 是无迹的

$\rho(\vec{p})\ge 0 \iff \lambda_1,\lambda_2$ are eigenvalues of $\rho(\vec{p})$, $\rho=\frac{1}{2}\begin{pmatrix}1+p_3 & p_1+ip_2\\ p_1-ip_2 & 1-p_3\end{pmatrix}$

$\mathrm{tr}(\rho)=1 \Rightarrow \mathrm{tr}(\underbrace{U\rho U^\dagger}_{\text{对角化}})=1 \Rightarrow \lambda_1+\lambda_2=1.$

$\mathrm{Det}(\rho)=\mathrm{Det}(U\rho U^\dagger)=\lambda_1\lambda_2\ge 0$

$\| \quad \frac{1}{4}(1-|\vec{p}|^2)$

$\Downarrow$

$\lambda_1,\lambda_2\ge 0$

·矩阵的两个不变量 Det tr 是在变换下不变量

Lemma: $\langle\vec{\sigma}\cdot\vec{n}\rangle=\mathrm{tr}(\vec{\sigma}\cdot\vec{n}\,\rho)=\vec{n}\cdot\vec{p}$

proof: $\mathrm{tr}(\rho\,\vec{\sigma}\cdot\vec{n})=\mathrm{tr}\left(\left(\frac{1}{2}I_2+\vec{\sigma}\cdot\vec{p}\right)\sigma_n\right)=\underset{0}{\mathrm{tr}\left(\frac{1}{2}\sigma_n\right)}+\mathrm{tr}(\sigma_n(\vec{\sigma}\cdot\vec{p}))\cdot\frac{1}{2}=\vec{p}\cdot\vec{n}$

$\cdot\ \sigma_p\sigma_n=\vec{p}\cdot\vec{n}\,\mathbb{1}_2 + i\vec{\sigma}\cdot(\vec{p}\times\vec{n})$

Remark: $\langle\sigma_n\rangle=\cos\theta$

$\begin{cases}\vec{n}\parallel\vec{p} & \langle\sigma_p\rangle=1\\ \vec{n}\perp\vec{p} & \langle\sigma_n\rangle=0\end{cases}$ $\quad\vec{p}=0,\ \forall\vec{n},\ \langle\sigma_n\rangle=0$

Home work 3rd and 4th Due next Tuesday (May 3rd)

* Solve two (or more) problems
* Focus on two problems
* Solve them completely
* Write down solutions step by step (on the lecture notes)
* Quote your references carefully

Lec. XI Density matrix (II)

引入 density matrix 的意义

用迹描述 $\begin{cases}\text{物理量}\\ \text{几率}\end{cases}$

描述测量 $\begin{cases}\text{测前态}\\ \text{测后态}\end{cases}$

global phase

大量粒子 $\hat{\rho}=e^{-\beta\hat{H}}$

子系统 $\rho_{AB}\to\rho_A=\mathrm{Tr}_B(\rho_{AB})$

Homework 3rd & 4th Due today

collected class by class

XI Overviewing of density matrix

Density operator is a language of describing quantum systems, especially open systems or sub systems, which is equivalent to the state vector.

Why introduce density matrix?

① Describe physical quantities via Trace

物理量 / 几率 > 均可用 trace 表述 $\quad\langle A\rangle=\mathrm{Tr}(\rho A)$

$\mathrm{Prob}(i)=\mathrm{Tr}(\rho P_i)$, $P_i^2=P_i$

② Global phase factor has no physical meaning

$\overset{|\psi\rangle}{|\psi'\rangle}=|\psi\rangle e^{i\alpha}$ > 等价 $\quad\rho=|\psi\rangle\langle\psi|=|\psi'\rangle\langle\psi'|$

③ How to describe quantum measurement process or the decoherence.

$|\psi\rangle=\sum_i c_i|i\rangle$

$\rho=|\psi\rangle\langle\psi| = \sum_{ij}c_ic_j^*|i\rangle\langle j|$ $\quad$ initial: ρ

final: ρ' $\quad$ Collapse (decoherence)

相干项消失

$\rho'=\sum_i|c_i|^2|i\rangle\langle i|$

④ 怎么描述大量的粒子?

How to describe huge number of quantum 'particles'? ← density matrix only choice

classical $\qquad$ quantum

$\left(\begin{array}{l}\text{Boltzman distribution}\\ \text{i}: P_i=e^{-\beta\varepsilon_i}\end{array}\right)\ \longrightarrow\ \begin{array}{l}\hat{\rho}=e^{-\beta\hat{H}}\\ \langle i|\hat{\rho}|i\rangle=e^{-\beta\varepsilon_i}\end{array}$ (求和的形式即得 Boltzmann分布)

⑤ How to describe a subsystem of a larger system

Ⓐ～Ⓑ environment noise $\quad$ 前面介绍 → 对 close system 而言 是 special case

一般系统必须考虑环境

Larger system: ρ_{AB}

Subsystem A: ρ_A $\quad > \quad \rho_A=\mathrm{Tr}_B(\rho_{AB})$ (partial trace)

reduced density matrix

(22)

XI.2 Density matrix of a qubit
(mixed state)

qubit的 density matrix ρ

总结 $\begin{cases} \rho \text{是 density (pure} \Leftrightarrow |\vec{p}|=1) \ \ |\vec{n}\rangle \\ \text{若}\rho\text{下. } \sigma_n \text{测量...} \\ \text{convex 集合} \end{cases}$

$$\rho(\vec{p}) = \frac{1}{2}(\mathbb{1}_2 + \vec{\sigma}\cdot\vec{p})$$

$|\vec{p}| \leq 1$. polarization vector

Thm1: $\rho(\vec{p})$ density matrix $\begin{cases} \rho(\vec{p}) \geq 0 \\ Tr(\vec{p}) = 1 \end{cases}$

Thm2: $\langle \vec{n}\cdot\vec{\sigma}\rangle = \vec{n}\cdot\vec{p} = \|\vec{p}\|\cos\theta$ (期望值)

$|\vec{n}|=1$, $|\vec{p}|\leq 1$

Thm3: $prob(|\vec{n}\rangle) = Tr(|\vec{n}\rangle\langle\vec{n}|\rho) = \langle\vec{n}|\rho|\vec{n}\rangle \rightarrow$ 记得: $Tr(|\vec{n}\rangle\langle\vec{n}|\rho)$ (投影算符)

$= Tr\left(\frac{1+\sigma_n}{2}\rho\right)$

$= \frac{1}{2} + \frac{1}{2}Tr(\sigma_n\rho)$

$= \frac{1}{2} + \frac{1}{2}\vec{n}\cdot\vec{p}$

$$prob(|-\vec{n}\rangle) = Tr(|-\vec{n}\rangle\langle-\vec{n}|\rho) = \frac{1}{2}(1-\vec{n}\cdot\vec{p})$$

Thm4: $\rho(\vec{p})$, pure state $\Leftrightarrow |\vec{p}|=1 \Leftrightarrow \rho^2=\rho \Leftrightarrow Tr(\rho^2)=1$

Proof: $\rho^2 = \frac{1}{2}(\mathbb{1}_2+\vec{\sigma}\cdot\vec{p})\cdot\frac{1}{2}(\mathbb{1}_2+\vec{\sigma}\cdot\vec{p})$ $\qquad \rho = \frac{1}{4}(\mathbb{1}_2 + 2\vec{p}\cdot\vec{\sigma} + \mathbb{1}_2)$

$= \frac{1}{4}(\mathbb{1}_2 + 2\vec{p}\cdot\vec{\sigma} + (\vec{p}\cdot\vec{\sigma})^2)$

$\qquad\qquad \|\vec{p}\|^2 \quad (\sigma_p\sigma_q: \ \vec{p}\cdot\vec{q}\,\mathbb{1}_2 + i\vec{\sigma}\cdot(\vec{p}\times\vec{q}))$

$\Rightarrow \|\vec{p}\| = 1 \quad \Leftrightarrow$ pure states

Bloch sphere

pure state:

$$\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{p}\cdot\vec{\sigma}), \quad \|\vec{p}\|=1$$

如何得到对应的 state vector? $\qquad \vec{\sigma}\cdot\vec{p}\,|\vec{p}\rangle = |\vec{p}\rangle$, $\vec{\sigma}\cdot\vec{p}\,|-\vec{p}\rangle = -|-\vec{p}\rangle$

observables

$$|\vec{p}\rangle\langle\vec{p}| = \frac{\mathbb{1}_2+\vec{\sigma}\cdot\vec{p}}{2} = \rho(\vec{p}) \leftarrow \begin{cases} \vec{\sigma}\cdot\vec{p} = |\vec{p}\rangle\langle\vec{p}| - |-\vec{p}\rangle\langle-\vec{p}| \\ \hat{\mathbb{1}}_2 = |\vec{p}\rangle\langle\vec{p}| + |\vec{p}\rangle\langle\vec{p}| \end{cases}$$

所以 $|\vec{p}\rangle$ 就是 $\rho(\vec{p})$ 对应的量子态.

e.g.

$\rho(\vec{e}_z) = |\vec{e}_z\rangle\langle\vec{e}_z| = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\rho(\vec{e}_x) = |\vec{e}_x\rangle\langle\vec{e}_x| = |+\rangle\langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(1 \ \ 1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

i) 任何的 state vector, 都能写出 density matrix

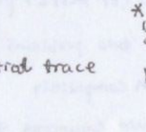
ii) density matrix 比 $|\psi\rangle$ 更广(和多於上)

Thm5. $\rho(\vec{p})$

$\vec{p}$: polarization vector

$\|\vec{p}\|=1$ Block Sphere

$\|\vec{p}\|<1$ physical meaning? $\rightarrow$ qubit state 在 ... 的物理图像



$\vec{p}$ 是 $\vec{n}_1$, $\vec{n}_2$ 的混合

For any vector $\vec{p}$, define $\vec{p} = \lambda\vec{n}_1 + (1-\lambda)\vec{n}_2$ (三点共线条件), $\|\vec{n}_1\| = \|\vec{n}_2\| = 1$

Proof: $\rho(\vec{p}) = \frac{1}{2}(\mathbb{1}_2 + \vec{\sigma}\cdot\underbrace{(\lambda\vec{n}_1+(1-\lambda)\vec{n}_2)}) = \frac{1}{2}(\lambda\mathbb{1}_2 + \lambda\vec{n}_1\cdot\vec{\sigma}) + \frac{1}{2}((1-\lambda)\mathbb{1}_2 + (1-\lambda)\vec{n}_2\cdot\vec{\sigma})$

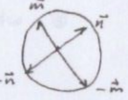
Convex combination always possible

$= \lambda\rho(\vec{n}_1) + (1-\lambda)\rho(\vec{n}_2)$

e.g. $\vec{n}_1 = -\vec{n}_2$, $\vec{n}_1 = \vec{n}$, $\vec{n}_2 = -\vec{n}$

$\rho(c) = \frac{1}{2}\left(|\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|\right) = \frac{1}{2}I_2$

the origin of Bloch ball



Every polarization direction possible! $\Leftrightarrow$ Information is completely Hidden

Remark: The convex set $\{\vec{n}_1, \vec{n}_2\}$ (凸集) $\Leftrightarrow$ a ball is convex set and its boundary such as sphere, is a collection of its extremal points.

Thm: The density matrix can form a convex set, and its pure states form the set of extremal points of this convex set.

(density matrix 有明确的 geometric and topological picture)

XI.3 Reduced density matrix

Reduced density matrix → 描述子系统的方法

(描述 open system / subsystem)

1. ρ_{AB} [Tr_B(求和, 求迹)] ρ_A (约化density)

2. $|\beta_{ij}\rangle$ are Reduced maximal entangled → $\rho_A = \frac{1}{2}\mathbb{1}_2$, 完全未知

It is a best tool of describing subsystems of a composite system.

Composite system $A\otimes B$: A & B

we know ρ_{AB}. and want to study system A

subsystem A: $M_A \otimes I_d$

↳ (注意 A.B 之相互影响)

$$\langle M_A\rangle_A = \langle M_A\otimes I_d\rangle_{AB} = tr_{AB}((M_A\otimes I_d)\rho_{AB})$$

$\|$ 希望引入 ρ_A. 使之成立

$Tr_A(\rho_A M_A)$

what's the relationship between ρ_A & ρ_{AB}

$$Tr_A(\rho_A M_A) = Tr_A(M_A \underset{\substack{\uparrow \\ \text{partial trace}}}{Tr_B}\rho_{AB})$$

所以 $\rho_A := Tr_B\,\rho_{AB}$

Def: ρ_A: reduced density matrix from ρ_{AB} via partial trace

Def: partial trace:

$\mathcal{H}_A\otimes\mathcal{H}_B$

$|a_1\rangle\langle a_2|\otimes|b_1\rangle\langle b_2|$ $\qquad |a_2\rangle\in\mathcal{H}_A$. $|b_2\rangle\in\mathcal{H}_B$

ordinary trace $Tr_{AB}(|a_1\rangle\langle a_2|\otimes|b_1\rangle\langle b_2|) = \langle a_1|a_2\rangle\langle b_1|b_2\rangle$

partial trace $Tr_B(|a_1\rangle\langle a_2|\otimes|b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2|\langle b_1|b_2\rangle$

* 理解

法① Hilbert A ↓ $\mathcal{J}^{\circ}$ $\circ$ 结号. ↑ B 法②

(从结果想象得出的角度考虑)

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$$\mathrm{Tr}_B \rho_{AB} = \sum_\mu {}_B\langle \mu | \rho_{AB} | \mu \rangle_B$$

$\{|\mu\rangle_B\}$: orthogonal basis of $\mathcal{H}_B$

Proof: $\rho_{AB} = \sum_{a_1 a_2 b_1 b_2} |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\, C_{a_1 a_2 b_1 b_2}$ ← 각 tensor의 곱

$\mathrm{Tr}_B \rho_{AB} = \sum_{a_1 a_2 b_1 b_2} C_{a_1 a_2 b_1 b_2} \mathrm{Tr}_B \left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2| \right)$
$|a_1\rangle|b_1\rangle\langle b_2|\langle a_2|$

$= \sum_{a_1 a_2 b_1 b_2} C_{a_1 a_2 b_1 b_2} \langle b_2 | b_1 \rangle |a_1\rangle\langle a_2|$

$$\rho_A = \sum_{|\mu\rangle_B} {}_B\langle \mu | \rho_{AB} | \mu \rangle_B = \sum_{a_1 a_2 b_1 b_2} C_{a_1 a_2 b_1 b_2} |a_1\rangle\langle a_2| \underbrace{\sum_\mu \langle \mu | b_1 \rangle\langle b_2 | \mu \rangle}_{= \langle b_2 | b_1 \rangle}$$

$$= \sum_{a_1 a_2 b_1 b_2} C_{a_1 a_2 b_1 b_2} \langle b_2 | b_1 \rangle |a_1\rangle\langle a_2|$$

Thm: ρ_{AB} is a density matrix

$\Rightarrow$ $\rho_A = \mathrm{Tr}_B \rho_{AB}$ is a density matrix

$(\rho_A \geq 0,\ \mathrm{tr}(\rho_A) = 1)$

Proof: for special case (pure states) ---→ 특수한 경우를 통해 general case

$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ ← $\mathcal{H}_A \otimes \mathcal{H}_B$

$\{|i\rangle\}_A$ orthogonal basis

$\{|\mu\rangle\}_B$ orthogonal basis

i) $|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu} |i\rangle_A |\mu\rangle_B$

${}_{AB}\langle\psi|\psi\rangle_{AB} = 1 \iff \sum_{i,\mu} |a_{i\mu}|^2 = 1 \quad \cdots ①$

$$\rho_{AB} = \sum_{i,\mu}\sum_{j,\nu} a_{i\mu} a_{j\nu}^* |i\rangle_A |\mu\rangle_B {}_B\langle\nu| {}_A\langle j|$$

$\curvearrowright$ $|i\rangle\langle j| \otimes |\mu\rangle\langle\nu|$

$$= \sum_{i,\mu,j,\nu} a_{i\mu} a_{j\nu}^* |i\rangle_A\langle j| \otimes |\mu\rangle_B\langle\nu|$$

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = \sum_{\mu'} {}_B\langle\mu'|\rho_{AB}|\mu'\rangle_B$$

$$= \sum_{i,\mu}\sum_{j,\nu} a_{i\mu} a_{j\nu}^* |i\rangle_A\langle j|\, \langle\nu|\mu\rangle$$

$$= \sum_{i,j}\left(\sum_\mu a_{i\mu} a_{j\mu}^*\right)|i\rangle\langle j|$$

$$\mathrm{Tr}_A \rho_A = \sum_i \sum_\mu a_{i\mu} a_{i\mu}^* = \langle\psi|\psi\rangle \underset{①}{=} 1$$

ii) proof for $\rho_A \geq 0$:

$\iff \forall |\phi\rangle_A \in \mathcal{H}_A$: $\langle\phi_A|\rho_A|\phi\rangle_A \geq 0$ $\qquad |\phi_A\rangle = \sum_i b_i |i\rangle_A$

$${}_A\langle\phi|\rho_A|\phi\rangle_A = \sum_{i,j,\mu} a_{i\mu} a_{j\mu}^* \langle\phi|i\rangle\langle j|\phi\rangle$$

$$= \sum_{i,j,\mu} a_{i\mu} a_{j\mu}^* b_i^* b_j$$

$$= \sum_{i,j,\mu} a_{i\mu} b_i^* a_{j\mu}^* b_j = \sum_{\substack{\mu \\ i,j}} |a_{i\mu} b_i^*|^2 = \sum_\mu \left|\sum_i a_{i\mu} b_i^*\right|^2 \geq 0$$

e.g. $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$= \frac{1}{2}\left(|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + |0\rangle_A\langle 1| \otimes |0\rangle_B\langle 1| + |1\rangle_A\langle 0| \otimes |1\rangle_B\langle 0| + |1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1| \right)$$

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = {}_B\langle 0|\rho_{AB}|0\rangle_B + {}_B\langle 1|\rho_{AB}|1\rangle_B$$

$$= \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)$$

$$= \frac{1}{2}\mathbb{1}_2$$

(pure state)

The entire system is exactly known, yet the parts of system are completely unknown.

That is quantum entanglement.

When subsystem is completely hidden, that's maximal quantum entanglement.

Lec XII Quantum Entanglement in biparticle system (比较简单)

Homework 5th - 6th, Due May 31, 2016

Nielsen & Chuang's text book (quantum circuit model)

Page 182-183	Ex. 4.24	Toffoli gate construction
	Ex. 4.25	Fredkin gate construction
	Ex. 4.26	Toffoli gate construction (different ways)
Page 188	Ex. 4.34	Measuring an operator
	Ex. 4.35	Measurement Commutes with Controls
Page 197	Ex. 4.41	Verify Figure 4.17

XII. 1 Introduction

최대한 간단한 quantum entanglement
Biparticle system

- Quant. Entang. is the most important resource widely used in QIC.
- Quant. Entang. is the deepest difference between classical physics and quantum physics
- Harnessing quant. entang. is the key to realizing a large-scale quantum computer. (small-scale available today)

Capable of solving hard problems (which classical computer can't solve)

㉔

One particle system : $\mathcal{H}_A$

Biparticle system: $\mathcal{H}_A \otimes \mathcal{H}_B$ Triparticle system: $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ ...

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{multiparticle system}}$

Statement **Theorem** Quantum entanglement in biparticle system is the key to understanding quant. entang. in multiparticle system.

纠缠态的定义: separability / entanglement

判定是否entangled: Schmidt 数 = 1 $\Leftrightarrow$ separable, ≥ 2 $\Leftrightarrow$ entangled; ρ_A (ρ_B) 是 pure state $\Leftrightarrow$ separable

相关定理: Schmidt 分解 $\Rightarrow \rho_A \rho_B$; $U_A \otimes V_B |\chi\rangle_{AB}$ 与 $|\chi\rangle_{AB}$

Sch数的定义 / Sch系数定义

两比特的Bell态 最大纠缠的态

XII.2 Quantum Entanglement in biparticle pure state

[illegible] quant. entang. 纠缠的定义 ([illegible])

Def. of Quant. separability :

A biparticle pure state $|\chi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if it can be expressed as a tensor product of $|\varphi_A\rangle \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$, namely

$$|\chi\rangle_{AB} = |\varphi\rangle_A \otimes |\psi\rangle_B$$

Def. of Quant. entang. (indirect way) :

A biparticle pure state $|\chi\rangle_{AB}$ is entangled if & only if it is NOT separable.

如何判断给定的态是否 entangled?

Thm (Schmidt decomposition) :

In a biparticle Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ ($n = \dim \mathcal{H}_A$, $m = \dim \mathcal{H}_B$, $n \leq m$) for convenience

A biparticle pure state $|\chi\rangle_{AB}$ has the Schmidt decomposition as

$$|\chi\rangle_{AB} = \sum_{i=1}^{m(\text{or } n)} \sqrt{p_i}\, |i\rangle_A \otimes |i'\rangle_B \quad i' = f(i) \text{ [illegible]}$$

相当于 power 的定义.

where $p_1, p_2, \cdots, p_n \geq 0$, $p_{n+1}, \cdots p_m = 0$; $\sum_{i=1}^{n} p_i = \sum_{i=1}^{m} p_i = 1$

$\{|i\rangle_A\}$ orthonormal basis of $\mathcal{H}_A$

$\{|i'\rangle_B\}$ orthonormal basis of $\mathcal{H}_B$

[illegible] $|i'\rangle = \frac{1}{\sqrt{p_i}} \langle i|\chi\rangle_{AB}$

p_i 是 ρ_A 的本征值

Proof: See online lec. notes □

Remark: The reduced density matrix ρ_A :

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \mathrm{Tr}_B\left(|\chi\rangle_{AB}\langle\chi|\right) = \sum_{i'} {}_B\langle i'|\rho_{AB}|i'\rangle = \sum_i p_i |i\rangle_A\langle i|$$

$$= \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}_{n\times n} = \{(p_i, |i\rangle_A)\}$$

$\sum_i p_i = 1$ 说明 ρ_A 对应的 density

$$\rho_B = \sum_i {}_A\langle i|\rho_{AB}|i\rangle_A = \sum_i p_i |i'\rangle_B\langle i'| = \begin{pmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & \ddots & & & \\ & & & p_n & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}_{m\times m}$$

$$= \{(p_i, |i'\rangle_B)\}$$

subtle

eg 1. The Schmidt decomposition of $|\beta_{ij}\rangle$

$$|\beta_{00}\rangle = \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \sum_{i=1}^{1} \tfrac{1}{\sqrt{2}} |ii\rangle, \quad p_0 = p_1 = \tfrac{1}{2}$$

$$|\beta_{01}\rangle = \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = (\mathbb{1}_2 \otimes X)|\beta_{00}\rangle = \sum_{i=0}^{1} \tfrac{1}{\sqrt{2}} |ii'\rangle, \quad |i'\rangle = X|i\rangle$$

$p_0 = p_1 = \tfrac{1}{2}$

$$|\beta_{ij}\rangle = \tfrac{1}{\sqrt{2}}\left(\mathbb{1}_2 \otimes X^j Z^i\right)|\beta_{00}\rangle = \sum_{k=0}^{1} \tfrac{1}{\sqrt{2}} |kk'\rangle, \quad |k'\rangle = X^j Z^i |k\rangle$$

$p_0 = p_1 = \tfrac{1}{2}$

$$\rho_A = \sum_{k=0}^{1} {}_B\langle k'|\rho_{AB}|k'\rangle_B = \tfrac{1}{2}(|0\rangle\langle 0|) + \tfrac{1}{2}(|1\rangle\langle 1|) = \tfrac{1}{2}\mathbb{1}_2$$

$$\rho_B = \sum_{k=0}^{1} {}_A\langle k|\rho_{AB}|k\rangle_A = \tfrac{1}{2}\mathbb{1}_2$$

Thm : $|\chi\rangle_{AB}$ and $(U \otimes V)|\chi\rangle_{AB}$, U and V are single-qubit gates, have the same Schmidt decomposition.

eg 2. $\dim \mathcal{H}_A = 2$, $\dim \mathcal{H}_B = 4$

$$|\chi\rangle_{AB} = \sum_{i=1}^{2} \sqrt{p_i}\, |i\rangle_A \otimes |i'\rangle_B$$

for $\mathcal{H}_A = \mathrm{span}\{|i\rangle_A\} = \{|1\rangle_A, |2\rangle_A\}$

$\mathcal{H}_B = \mathrm{span}\{|i'\rangle_B\} = \{|1'\rangle_B, |2'\rangle_B, |3'\rangle_B, |4'\rangle_B\}$

$$\rho_A = \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} \qquad \rho_B = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Def. p_i are called Schmidt coefficient defining probability distribution in ρ_A and ρ_B

Def. The number of non-vanishing p_i is called the Schmidt number, $\mathrm{Sch}(|\chi\rangle_{AB})$

e.g. Bell state $|\beta_{ij}\rangle$ $p_0 = \tfrac{1}{2} = p_1$, $\mathrm{Sch}(|\beta_{ij}\rangle) = 2$

e.g. $p_1 \neq 0$, $p_2 \neq 0$, $p_3 = p_4 = 0$, $\mathrm{Sch}(|\chi_{ij}\rangle) = 2$

eg. $|\chi\rangle_{AB} = |\varphi\rangle_A \otimes |\psi\rangle_B = |i\rangle_A \otimes |i'\rangle_B$, $p_1 = 1$, $p_2 = p_3 = \cdots = 0$, $\mathrm{Sch}(|\chi\rangle_{AB}) = 1$.

Thm: when $\mathrm{Sch}(|\chi\rangle_{AB}) = 1$, Separable

when $\mathrm{Sch}(|\chi\rangle_{AB}) \geq 2$, entangled

Corollary : $\mathrm{Sch}(|\chi\rangle_{AB}) = \mathrm{Sch}\left((U \otimes V)|\chi\rangle_{AB}\right)$

$|\chi\rangle_{AB}$ and $(U \otimes V)|\chi\rangle_{AB}$ have the same entanglement property.

这样，确定一个态 $|\chi\rangle_{AB}$ 的纠缠度

就可以确定一系列态 $U \otimes V|\chi\rangle_{AB}$ 的纠缠度

Thm : $\rho_{AB} = |\chi\rangle_{AB}\langle\chi|$, $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$, $\rho_B = \mathrm{Tr}_A\, \rho_{AB}$

1°) ρ_{AB} separable $\Leftrightarrow$ ρ_A and ρ_B are pure states

2°) ρ_{AB} entangled $\Leftrightarrow$ ρ_A and ρ_B are mixed states

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Special case $\rho_A = \rho_B = \frac{1}{2}\mathbb{1}_2$. Information Maximally Hidden

Our maximal ignorance of ρ_A ($tr(\sigma_i) = 0$)

Def: $\rho_A = \frac{1}{d}\mathbb{1}_d$, $d = \dim(\mathcal{H}_A)$. $\Leftrightarrow$ ρ_{AB} entangled maximally   security $\xleftarrow{\text{good for}}$ Information Hidden

eg. Higher dimensional Bell states

$$|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle \qquad (U\otimes V \to 其它 \text{ Bell states})$$

Schmidt decomposition, $p_0 = p_1 = \cdots = p_{d-1} = \frac{1}{d}$, $\rho_A = \rho_B = \frac{1}{d}\mathbb{1}_{d\times d}$ (maximally entangled)

纠缠度的纠缠量 → Von Neumann entropy
定义 $S = S_A = S_B = -Tr(\rho_A \log\rho_A)$
定理 1) 2) 3) 4)

问: entangled 的程度怎么判定?

How much entangled?
↓
Von Neumann entropy

XII.3 Entanglement

Entangle measures in biparticle pure state

XII.3a) The Von Neumann entropy

(great theoretical physist)

It is distinguishable entangle measure of biparticle pure state defined as

$$S_A = -Tr(\rho_A \log\rho_A) = -Tr(\rho_B \log\rho_B) = S_B = S$$

Remark: In Thermodynamics (i, p_i) $\leftrightarrow$ Quantum Statistics $(p_i, |\psi_i\rangle) = \rho$

$$S = -\sum_i p_i \ln p_i \qquad S = -Tr(\rho\ln\rho)$$

second law $\leftrightarrow$ entanglement

Thm: 1) $0 \leq S(\rho_A) \leq \ln d$, $d = \dim\mathcal{H}_A$

2) $S(\rho_A) = 0 \Leftrightarrow |\chi\rangle_{AB}$ separable

3) $S(\rho_A) = \ln d \Leftrightarrow |\chi\rangle_{AB}$ maximally entangled

4) $S(\rho_A) \neq 0 \Leftrightarrow |\chi\rangle_{AB}$ entangled

· 不仅可以判断是否 entangled 还可以判断纠缠度

eg 1. Bell state $|\beta_{ij}\rangle$ are maximally entangled, (Higher dimensional $|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$ is the same.)

check above Thm:

$$\rho_A = \rho_B = \frac{1}{2}\mathbb{1}_2 \qquad \rho_A = \rho_B = \frac{1}{d}\mathbb{1}_{d\times d} = \begin{pmatrix} \frac{1}{d} & & \\ & \ddots & \\ & & \frac{1}{d}\end{pmatrix}$$

$$S(\rho_A) = -Tr\left[\begin{pmatrix} \frac{1}{d} & & \\ & \ddots & \\ & & \frac{1}{d}\end{pmatrix}\begin{pmatrix} \ln\frac{1}{d} & & \\ & \ddots & \\ & & \ln\frac{1}{d}\end{pmatrix}\right] = \ln d, \text{ maximally entangled}$$

eg 2. $|\chi\rangle_{AB} = |\varphi\rangle_A \otimes |\psi\rangle_B$, separable, check:

$$\rho_A = |\varphi\rangle_A\langle\varphi| = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0\end{pmatrix} \qquad 0\ln 0 := \lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = 0$$

$$S(\rho_A) = -Tr(\rho_A\ln\rho_A) = -Tr\left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots\ 0\end{pmatrix}\ln\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots\ 0\end{pmatrix}\right) = -Tr\begin{pmatrix} 1\ln 1 & & \\ & 0\ln 0 & \\ & & \ddots\ 0\ln 0\end{pmatrix}$$

$$= 0$$

eg 3. Two-qubit system

two-qubit pure state ρ_{AB}: $\rho_A = \frac{1}{2}(\mathbb{1}_2 + \vec{p}\cdot\vec{\sigma})$

$$= \lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1| \quad \underline{\text{Schmidt}} \text{ coefficient } \lambda_1, \lambda_2$$

$$= \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix} \qquad \lambda_1 + \lambda_2 = 1, \ \lambda_1, \lambda_2 \geq 0$$

$$S = -Tr\left(\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\ln\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\right)$$

$$= -\lambda_1\ln\lambda_1 - \lambda_2\ln\lambda_2 = -\lambda_1\ln\lambda_1 - (1-\lambda_1)\ln(1-\lambda_1)$$

1) ρ_A pure state $\lambda_1 = 1, \lambda_2 = 0$ or $\lambda_1 = 0, \lambda_2 = 1$ $\Big\}$ $S(\rho_A) = -1\ln 1 - 0\ln 0 = 0$

2) ρ_A maximally entangled $\lambda_1 = \lambda_2 = 1/2$. $S(\rho_A) = \ln 2$ take the maximal

XII.3b) Concurrence (并发度)

Two-qubit biparticle pure state $|\chi\rangle_{AB} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$= \begin{pmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{pmatrix}$$

$$\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_y\otimes\sigma_y = \begin{pmatrix} & & & -1\\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$|\tilde{\chi}\rangle = (\sigma_y\otimes\sigma_y)|\chi^*\rangle \qquad |\chi^*\rangle = \begin{pmatrix}\alpha_{00}^*\\ \alpha_{01}^*\\ \alpha_{10}^*\\ \alpha_{11}^*\end{pmatrix}, \quad |\tilde{\chi}\rangle = \begin{pmatrix}-\alpha_{11}^*\\ \alpha_{10}^*\\ +\alpha_{01}^*\\ -\alpha_{00}^*\end{pmatrix}$$

$$\text{Concurrence}(|\chi\rangle_{AB}) = \left|{}_{AB}\langle\chi|\tilde{\chi}\rangle_{AB}\right|, \quad {}_{AB}\langle\chi|\tilde{\chi}\rangle_{AB} = 2(\alpha_{01}^*\alpha_{10}^* - \alpha_{00}^*\alpha_{11}^*)$$

$$C(|\chi\rangle_{AB}) = 2\left|\alpha_{01}^*\alpha_{10}^* - \alpha_{00}^*\alpha_{11}^*\right|$$

$$= 2\left|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}\right|$$

eg.1. $|\chi\rangle_{AB} = |00\rangle_{AB}$, $\alpha_{00} = 1$, $\alpha_{其它} = 0$, $C(|\chi\rangle_{AB}) = 0$

eg.2. $|\chi\rangle_{AB} = |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow C(|\chi\rangle_{AB}) = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1$

$\alpha_{00} = \alpha_{11} = 1/\sqrt{2}$

$\alpha_{其它} = 0$

2016.05.17

Final homework, due May 31

5th–6th

Focus on one or two problems in 3 weeks

XII.4 Quantum entanglement in biparticle mixed state

· no people understand quantum entanglement today

Def: Separability

A biparticle mixed state ρ_{AB} is called separable if & only if it can be expressed as $\rho_{AB} = \sum_{ij} p_{ij}\ \rho_{A,i}\otimes\rho_{B,j}$

$$p_{ij} \geq 0, \ \sum_{ij} p_{ij} = 1, \ \rho_{A,i} \geq 0, \ \rho_{B,j} \geq 0, \ tr\rho_{A,i} = 1, \ tr\rho_{B,j} = 1$$

eg 1: $p_{i,j} = p_i p_j$. $p_i \ge 0$, $p_j \ge 0$ $\rho_{A,i} = |\alpha_i\rangle\langle\alpha_i|$ $\rho_{B,j} = |\beta_j\rangle\langle\beta_j|$

$\Rightarrow$

$$\rho_{AB} = \sum_{i,j} p_{ij} |\alpha_i\beta_j\rangle\langle\alpha_i\beta_j| = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \otimes \sum_j p_j |\beta_j\rangle\langle\beta_j|$$

$$= \rho_A \otimes \rho_B$$

这是十分好的诠释 no saparability

eg 2. $\rho_1 = \frac{1}{4} I_2 \otimes I_2$, $\rho_2 = \frac{1}{2}\sum_{i=0}^{1} |i\rangle\langle i| \otimes \frac{1}{2}\sum_{j=0}^{1} |j\rangle\langle j| = \frac{1}{4}\sum_{i,j=0}^{1} |ij\rangle\langle ij|$

$\rho_1 = \rho_2$ 好诠释 纠缠诠释

$\rho_3 = \frac{1}{4}\sum_{i,j=0}^{1} |\beta_{ij}\rangle\langle\beta_{ij}|$

$\rho_1 = \rho_2 = \rho_3$

坏纠缠诠释

Remark: ρ_1 separable (easy!)
ρ_3 separable (NOT easy!)

Def: Entanglement

If ρ_{AB} is not separable, then it is entangled. · 定义 entangled 很难. 因此等价问题转换

Remark: Quant. Entanglement is difficult and open

Reason: A given ρ allows ∞ number of state ensemble descriptions. each of which defines its own quantum entanglement.

· 有的区分好诠释, 有的则难诠释

XII 4b) PPT criterion (partial-positive-transpose)

Thm: If ρ_{AB} is separable, (it can we expressed as)

$$\rho_{AB} = \sum_{i,j} p_{ij}\, \rho_{A,i} \otimes \rho_{B,j}$$

Then

$$(\rho_{AB})^{PT} \equiv \sum_{i,j} p_{ij}\, \rho_{A,i} \otimes \rho_{B,j}^{T} \ge 0$$

partial transpose

· PT 也有其他的定义方法, 如 $\rho_{A,i}^{T} \otimes \rho_{B,j}$

Proof: Transpose does not change eigenvalues · 注意转置不改变 eigenvalue

$$\rho_{AB} \ge 0 \Rightarrow \begin{matrix}\rho_{A,i} \ge 0 \\ \rho_{B,j} \ge 0\end{matrix} \Rightarrow \begin{matrix}\rho_{A,i} \ge 0 \\ \rho_{B,j}^{T} \ge 0\end{matrix} \Rightarrow \rho_{AB}^{PT} \ge 0$$

另外: 正定的张量积 必须正定吗?
正定的和 必须正定吗

Collary: If ρ_{AB}^{PT} is not Non-negative, then ρ_{AB} is entangled. 另外: 转置与坐标系有关

Remark: Transpose $\rightarrow$ linear operator 另外 $\begin{pmatrix} a_{11}A^T & a_{12}A^T \\ a_{21}A^T & a_{22}A^T \end{pmatrix}$

$(|i\rangle\langle j|)^T = |j\rangle\langle i|$

即 $A^T{}_{ij} = A_{ji}$

eg. The Werner State (see Homework 1st and 2nd) in $d=4$

$$\rho_W(p) = \frac{1}{4}(1-p) I_4 + p |\beta_{00}\rangle\langle\beta_{00}|$$

in which range of p $\rho_W(p)$ is separable
in which range of p $\rho_W(p)$ is entangled

① $\rho_W(p)$ is density matrix $\Leftrightarrow$ $-\frac{1}{3} \le p \le 1$

that means: $tr\, \rho_W(p) = 1$
$\rho_W(p) \ge 0$

$|\beta_{00}\rangle$ is its eigenstate, $\lambda = \frac{1}{4}(1-p) + p = \frac{1+3p}{4} \ge 0$

$|\beta_{01}\rangle$ $|\beta_{10}\rangle$ $|\beta_{11}\rangle$, $\lambda = \frac{1-p}{4} \ge 0$

② Compute PT of $\rho_W(p)$, $\rho_W^{PT}(p) \ge 0 \Leftrightarrow p = ?$

$\Uparrow$

separable

$$\rho_W(p)^{PT} = \frac{1}{4}(1-p) I_2 \otimes I_2 + p(|\beta_{00}\rangle\langle\beta_{00}|)^{PT}$$

$$(|\beta_{00}\rangle\langle\beta_{00}|)^{PT} = \frac{1}{2}\left(\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)^{PT} = \frac{1}{2}\sum_{i,j=0}^{1} |i\rangle\langle j| \otimes |j\rangle\langle i|$$

$\frac{1}{\sqrt{2}}\sum_{i=0}^{1}|ii\rangle$

$$= \frac{1}{2}\sum_{i,j=0}^{1} |ij\rangle\langle ji| = \frac{1}{2} SWAP$$

Lemma: SWAP gate = permutation gate

$SWAP = \sum_{i,j=0}^{1} |ij\rangle\langle ji| \rightarrow$ eigenstate $|\beta_{00}\rangle$ $+1$

$|\beta_{01}\rangle$ $+1$

? $|\beta_{10}\rangle$ $+1$ (但自旋本征态和自旋的本征态

$|\beta_{11}\rangle$ -1 Bell 态的 是 = 电子态)

$$(\rho_W(p))^{PT} = \frac{1}{4}(1-p) + \frac{1}{2}p\, SWAP$$

$\left.\begin{matrix}|\beta_{00}\rangle \\ |\beta_{01}\rangle \\ |\beta_{10}\rangle\end{matrix}\right\}$ $\xrightarrow{\text{eigenvalue}}$ $\frac{1}{4}(1-p) + \frac{1}{2}p = \frac{1}{4}(1+p) \ge 0$

$|\beta_{11}\rangle \rightarrow \frac{1}{4}(1-p) - \frac{1}{2}p = \frac{1}{4}(1-3p) \ge 0$

$\rho_W(p)^{PT} \ge 0 \Leftrightarrow -1 \le p \le \frac{1}{3}$

when $-\frac{1}{3} \le p \le \frac{1}{3}$ $\rho_W(p)$ Separable ?

$\frac{1}{3} < p \le 1$ $\rho_W(p)$ entangled

· 问: 对 bipartile mixed state 如何 entangle measure?
没有很好的定义

XII 4c) Entanglement measure

(just for feeling)

Def. Entanglement of Formation (EOF) is a typical entang. measure.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

↑ [illegible]

$$EOF = \text{Inf}\left\{ \sum_i p_i E\left(|\psi_i\rangle\langle\psi_i|\right) \right\}$$

$E\left(|\psi_i\rangle\langle\psi_i|\right)$: Von Neumann entropy

Inf: Infimum = Great lowest bound

- quantum entanglement

essential $\langle$ pure states, Von Neumann entropy

crucial $\langle$ [illegible] entropy

eg. two-qubit mixed state ρ

$$EOF^{(\rho)} = S\left(\frac{1 + \sqrt{1 - C^2(\rho)}}{2} \right)$$

- [illegible]

$S(x) = -x\log x - (1-x)\log(1-x)$ [illegible] von Neumann entropy

$C(\rho)$ = Concurrence of ρ

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)$$

eigenvalues of $\rho\tilde{\rho} = \left\{ \lambda_1^2\ \lambda_2^2\ \lambda_3^2\ \lambda_4^2 \right\}$, $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq \lambda_4^2$

$$C(\rho) = \max\left\{ 1, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \right\}$$

XIII. Bell inequalities

XIII.1. Review on modern/present quantum mechanics

- Bell inequalities

[illegible]

Def. Weaker locality

- Causality in S.R. (invented by Einstein)
- $v \leq c$

no faster-than-light transmission of classical formation

Def. Stronger locality

- Einstein's locality (also invented by Einstein)
- Measurement on System A cannot modify System B far away from System A.

Def. Hidden variables

- pre-existing and hidden variable

eg. statistics thermodynamics

↓

is hidden variable theory of thermodynamics

- Einstein: ?(some theory) is HV of QM.

Def: Deterministic versus probabilistic

- Einstein: the God does not play the dice

Def: Complete versus incomplete

- Einstein: QM is incomplete

(HV must be existed)

Def: Reality

| (realistic) versus unrealistic

classical reality

quantum reality

- Einstein: the physical quantity cannot be changed by measurement
- Bohr: The physical quantity can be ...

The moon does not exist if we don't look at it

(this is right)

Statement 1: Standard QM in text book

from copenhagen's people $\left\{ \begin{array}{l} \text{Bohr} \\ \text{Heisenberg} \\ \text{Schrödinger} \end{array} \right.$

- No hidden-variables
- Complete
- weaker locality $v \leq c$ is obeyed

stronger locality is violated

(Due to quantum correlation)

Standard QM = Local non-hidden variable theory

Statement 2: Bohm's quantum mechanics

- [illegible] QM [illegible]

- Hidden-variable theory
- locality is violated (S.R. is not possible)

Bohm's QM = Non-local hidden variable theory

Statement 3: Einstein's Quantum Mechanics

- Hidden-variable
- locality

Local hidden-variable theory

Remark: Einstein's QM is denied by Bell inequalities (Einstein is wrong)

- [illegible] local + Non-HV

[illegible] non-local + HV [illegible]

XIII.2. The CHSH inequality

① it is typical example for Bell inequalities

② simple

③ often tested in experiments

principle underlying CHSH inequality

Standard QM : ① The Heisenberg uncertainty relation tells that Non-commuting [Commutative] observables can not be assigned values simultaneously.

Einstein (LHV) :

② all the observables are determined by hidden variables before measurement and can be determined simultaneously.

亦即，决定了的设置

·同时不相容

Experiment setup

Alice ~~~ $\boxed{|\beta_{00}\rangle}$ ~~~ Bob

Observables: Q^A, R^A (Pauli matrix); $q=\pm1$, $r=\pm1$

Observables: S^B, T^B; $s=\pm1$, $t=\pm1$

$q+r=0 \Rightarrow q-r=2q=\pm2$

$q-r=0 \Rightarrow q+r=2q=\pm2$

define $M=(Q+R)S+(Q-R)T$

Note: In Nielsen & Chuang's book $M'=(Q+R)S+(R-Q)S$ does not matter

① Einstein's LHV theory $M=(Q+R)S+(Q-R)T$

$p(q,r;s,t)$ = probability distribution

$$\langle M\rangle=\sum_{q,r,s,t} p(q,r;s,t)\big((q+r)s+(q-r)t\big)$$

$(q+r)s+(q-r)t=\pm2$

so: $\langle M\rangle \leqslant \sum_{qrst} p(q,r;s,t)\cdot 2 = 2$

$|\langle M\rangle| \leqslant 2$: CHSH inequality

Lemma: $\langle M\rangle=\langle QS\rangle+\langle RS\rangle+\langle QT\rangle-\langle RT\rangle$

CHSH inequality : $|\langle QS\rangle+\langle RS\rangle+\langle QT\rangle-\langle RT\rangle| \leqslant 2$

② Specify Q, S, R, T

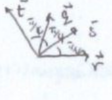
$Q=\vec{\sigma}^A\cdot\vec{q}$, $|\vec{q}|=1$

$S=\vec{\sigma}^B\cdot\vec{s}$

$R=\vec{\sigma}^A\cdot\vec{r}$

$T=\vec{\sigma}^B\cdot\vec{t}$

$\langle QS\rangle=\langle\beta_{11}|\left(\vec{\sigma}^A\cdot\vec{q}\otimes\vec{\sigma}^B\cdot\vec{s}\right)|\beta_{11}\rangle=-\vec{q}\cdot\vec{s}$



$Q=Z^A$, $R=X^A$

$S=\frac{1}{\sqrt2}(Z^B+X^B)$

$T=\frac{1}{\sqrt2}(Z^B-X^B)$

$\langle QS\rangle=-\cos\frac{\pi}{4}=-1/\sqrt2=\langle RS\rangle=\langle QT\rangle$

$\langle RT\rangle=-\cos\frac{3\pi}{4}=\frac{1}{\sqrt2}$

$|\langle M\rangle|=\left|-1/\sqrt2-1/\sqrt2-1/\sqrt2-1/\sqrt2\right|=2\sqrt2 \geqslant 2$

CHSH inequality is violated (已被实验证实 ~1980)

LHV (Einstein) wrong!

locality + Hidden variable = wrong

local Non-Hidden variable (Standard QM)

Non-local Hidden variable (Bohm's QM)

see online lecture notes, at least 5 Bell inequalities

2016.05.24

Lec XIV Deutsch's Algorithm and Deutsch-Jozsa's Algorithm

(Deutsch-Jozsa是Deutsch的一般化推广)

Final homework, Due next Tuesday May 31. Focus on One or Two problems.
5th-6th

XIV. 1a) Introduction to Algorithms

Def. Quantum (Classical) Algorithm is a well-defined procedure (program) with a finite description for realizing a quantum (classical) information processing task to solve a computational problem.

Remark: Quantum Algorithm can solve hard problems beyond the reach of classical Algorithm.

speed up	example
polynomial speed up	quantum search algorithm vs. classical search
relative exponential	Simon's algorithm (online lec. notes)
exponential speed up	Shor's factoring algorithm vs. ...

XIV. 1b) "Oracle" model (Black Box model)

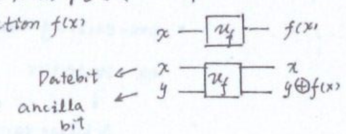
A very special type of Algorithm.

Def. Oracle model is a subroutine (small program) to evalue a function with specified input and output (black box) but how this function is performed is not concerned. (or unknown)

(big simplification, just for mathematica)

(实际情况，则需考虑如何计算)

example 1. A classical black-box (such as) to evaluate a function $f(x)$



$U_f:\ (x,y)\mapsto(x,\ y+f(x))$

U_f : a reversible gate

$U_f\circ U_f=\mathbb{I}_d$ (that's why 引入这一个比特 in gate)

$(x,y)\xrightarrow{U_f}(x,\ y\oplus f(x))\xrightarrow{U_f}(x,\ y\oplus f(x)\oplus f(x))=(x,y)$

example 2. A quantum black-box model

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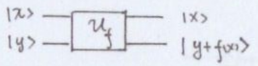


· 只有如此定义, u_f 才是 unitary transformation

$u_f^+ u_f = \mathbb{I}d$

Remark: Oracle model $\begin{cases} \text{Deutsch's Algorithm} \\ \text{Simon's Algorithm} \\ \text{Quantum Search algorithm} \\ \text{Factoring Algorithm} \end{cases}$

· 这样一个简单的 Oracle model 可以包含许多具体的 model

XIV Query complexity

· 为比较经典与量子算法, 利用 query complexity

Def. A query is performed once if A black-box is performed once.

Def. Query complexity of an oracle model (model) (both classical and quantum) is the minimal number of performing the oracle if the time of performing such the oracle is far larger than time of performing other gates.

eg. Time(u_f) = 1 year   Time(other gates) = 1 sec.

只有此时, query complexity 才有意义
(即 u_f 是主要花销)

XIV 2. Deutsch's problem

Deutsch's algorithm (1989) $\left\{ \begin{array}{l} \text{the first quantum Algorithm (QA)} \\ \text{the simplest QA (useless)} \\ \text{the first QA which is Oracle} \end{array} \right\}$ QIC

Def. Constant function

A function $f(x)$ is constant if and only if (iff) $f(0)=f(1)$

$(f(x): \{0,1\} \mapsto \{0,1\})$

eg. $f_1(0)=f_1(1)=0$

$f_2(0)=f_2(1)=1$

Def. Balanced (Balance) function

A function $f(x): \mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathbb{Z}_2$: 两个整数) is called balanced if $f(0)\oplus f(1)=1$, or $\#\{x|f(x)=0\} = \#\{x|f(x)=1\}$

eg. f_3: $f_3(0)=0$, $f_3(1)=1$

f_4: $f_4(0)=1$, $f_4(1)=0$

· balanced 意思是

$\#\{x|f(x)=0\}=1$

$\#\{x|f(x)=1\}=1$

对 n bit 的情况, 这样定义才有意义

Deutsch's problem:

Input: A classical black-box u_f

(A quantum black-box U_f)

promise [Condition]: $f(x)$ either constant or balanced

Question: What's the minimal number of computing u_f (U_f) to determine f is constant or balanced.

toy model / 实际无用, 但对设计 algorithm 有益

Output (Answer):

classical algorithm $\# u_f = 2$

quantum algorithm $\# U_f = 1$ (advantage)

XIV. 3. classical Algorithm of solving Deutsch's problem

Step 1: Compute $f(0)$ with u_f

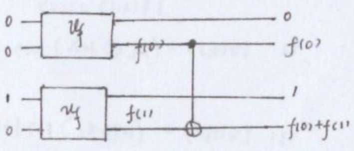
Step 2: Compute $f(1)$ with u_f

Step 3: $f(0)\oplus f(1)$ with XOR
(CNOT 的经典情况)

Step 4: when

$f(0)\oplus f(1) = 0$. constant

$f(0)\oplus f(1) = 1$. balanced



条件是 u_f 时间比 ⊕ 时间长

Remark: $\# U_f = 1$

Quantum parallelism: $f(0)$ and $f(1)$ can be computed simultaneously because of superposition principle || Decoherence (measurement)
↳ 不能同时知道

XIV. 4. The phase kick-back technique

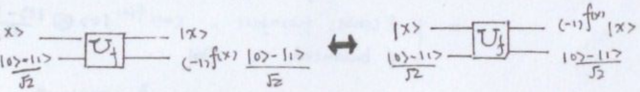
Lemma: $U_f |x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}} = (-1)^{f(x)} |x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

Remark: $f(x) \to$ global phase factor

Remark: $U_f |x\rangle = (-1)^{f(x)} |x\rangle$
(隐去了第2个qubit)

· 仅有 $|0\rangle, |1\rangle$ 可以同时计算.
但 Decoherence 会消失同时计算的结果.
只有 phase factor 可以留存下来

Remark:



control qubit (data qubit)

target qubit (ancilla qubit)

The phase factor is kicked back from the target qubit to the control qubit.

Proof: $U_f |x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}} = U_f |x\rangle \frac{|0\rangle}{\sqrt{2}} - U_f |x\rangle \frac{|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|x\rangle \frac{|f(x)\rangle}{\sqrt{2}} - |x\rangle \frac{|f(x)\oplus 1\rangle}{\sqrt{2}}\right)$

$= \begin{cases} f(x)=0, & \frac{1}{\sqrt{2}}(|x\rangle|0\rangle - |x\rangle|1\rangle) = |x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}} \\ f(x)=1, & \frac{1}{\sqrt{2}}(|x\rangle|1\rangle - |x\rangle|0\rangle) = |x\rangle \frac{|1\rangle-|0\rangle}{\sqrt{2}} \end{cases}$

$= (-1)^{f(x)} \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ □

Remark: Such technique is applied in $\begin{cases} \text{Deutsch's algorithm} \\ \text{Search's algorithm} \\ \dots \end{cases}$

XIV. 5 Deutsch's algorithm

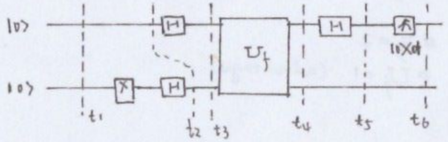
$\# U_f = 1$ $\begin{cases} \text{quantum parallelism} \\ \text{Global phase to overcome the decoherence by quantum measurement} \end{cases}$

$(-1)^{f(x)}(|A\rangle + |B\rangle)$

$(-1)^{f(x)}|A\rangle$ $\quad$ $(-1)^{f(x)}|B\rangle$

• 量子力学多道，希望同时计算，又希望信息存在 phase 内

如果 f(0) 和 f(1) 成为 global phase，则与测量无关，可以留存下来



t_1: state preparation

$|\psi(t_1)\rangle = |00\rangle$

t_2: $|\psi(t_2)\rangle = (\mathbb{1}_2 \otimes HX)|00\rangle = |0\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ $\quad |-\rangle$

t_3: $|\psi(t_3)\rangle = (H \otimes \mathbb{1}_2)|0\rangle|-\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ ，这时所有量子比特都相干

t_4: $|\psi(t_4)\rangle = U_f|\psi(t_3)\rangle = \frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right) \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

phase kick-back

f(0), f(1) 成为 phase，这样不被 measure 消除

$= \frac{1}{\sqrt{2}}\sum_{i=0}^{1}(-1)^{f(i)}|i\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

t_5: $|\psi(t_5)\rangle = (H \otimes \mathbb{1}_2)|\psi(t_4)\rangle$ ，QFT^{-1}，消去相位 结构下来

$= \frac{1}{2}\sum_{i,j=0}^{1}(-1)^{f(i)\oplus ij}|j\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

• $H|i\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1}(-1)^{i \cdot j}|j\rangle$

t_6: quantum measurement

$j=0$ $\quad |\psi(t_6)\rangle = \frac{1}{2}\sum_{i=0}^{1}(-1)^{f(i)}|0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$= \frac{1}{2}\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

global phase

$= \begin{cases} f \text{ const}, \ f(0)=f(1), & (-1)^{f(0)}|0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} \Rightarrow \text{一次测出} \\ f \text{ balanced} & 0 \text{ 项} \end{cases}$

Obtain $|0\rangle$ with probability = 1. f constant

Obtain $|0\rangle$ with probability = 0. f balanced

$\Vert$

Obtain $|1\rangle$ with prob = 1. f balanced

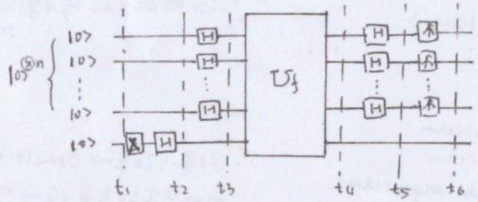
• 如何可以看出来计算的量子? Jozsa 作出了贡献!

XIV. 6 Deutsch - Jozsa's algorithm

n-bit function $f(x): \mathbb{Z}_2^n \to \mathbb{Z}_2^{m=1}$ $\quad f(x_1 \cdots x_n) = 0$ or 1

$\# U_f = 2^{n-1}+1$

$\# U_f = 1$



$|\psi(t_1)\rangle = |0\rangle^{\otimes n} \otimes |0\rangle$

$|\psi(t_2)\rangle = |0\rangle^{\otimes n} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$|\psi(t_3)\rangle = H^{\otimes n}|0\rangle^{\otimes n} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} = H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$= \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \cdots \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$= \frac{1}{2^{n/2}}\sum_{x_1 \cdots x_n = 0}^{1}|x_1 x_2 \cdots x_n\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

t_4: $|\psi(t_4)\rangle = U_f|\psi(t_3)\rangle = \frac{1}{2^{n/2}}\sum_{x_1 \cdots x_n=0}^{1}(-1)^{f(x_1 \cdots x_n)}|x_1 \cdots x_n\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

t_5: $|\psi(t_5)\rangle = (H^{\otimes n} \otimes \mathbb{1}_2)|\psi(t_4)\rangle = \frac{1}{2^n}\sum_{x_1 \cdots x_n=0}^{1}\sum_{X_1 \cdots X_n=0}^{1}(-1)^{f(x_1 \cdots x_n) \oplus (x_1 y_1 \oplus x_2 y_2 \oplus \cdots \oplus x_n y_n)}|x_1 \cdots x_n\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

t_6: quant. measurement $|0\rangle^{\otimes n}\langle 0| \otimes \mathbb{1}_2$

$y_1 = y_2 = \cdots = y_n = 0$, $|0\rangle^{\otimes n}$

$|\psi(t_6)\rangle = \frac{1}{2^n}\sum_{x_1 \cdots x_n=0}^{1}(-1)^{f(x_1 \cdots x_n)}|0\rangle^{\otimes n} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

If $f(x_1 \cdots x_n)$ is constant, $(-1)^{f(x_1 \cdots x_n)}|0\rangle^{\otimes n} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

probability = 1.

If $f(x_1 \cdots x_n)$ is balanced,

$\#\{(x_1 \cdots x_n) \mid f(x_1 \cdots x_n) = 0\} = \#\{(x_1, \cdots x_n) \mid f(x_1 \cdots x_n) = 0\}$

$y_1, y_2, \cdots, y_n \neq 0$. $\quad |\psi(t_6)\rangle = 0$, prob = 0! $\quad \square$

量子傅里叶变换：

定义：DFT. QFT $\quad$ 优点：$[DFT_n] \sim 2^n \cdot n$ $\quad O(QFT_n) \sim n^2$ $\quad$ 意义：$QFT_n \to$ phase encoding, $QFT_n^{-1} \to$ phase decoding

量子傅里叶变换实现的电路：$n + (n-1) + \cdots + 1 + [\frac{n}{2}] \times 3$ gates ; binary fraction. CR_n

eg. QFT_1, QFT_2 and QFT_2^{-1}. $QFT_3 \cdots$ (矩阵, 电路)

next lec. chapter 6 in N and C's

Lec XV. Quantum fourier transform and Its inverse transform

Final homework, due today, collected class by class

XV. I. Definition

Def. Quantum Fourier transform (QFT)

is quantum analogue of discrete fourier transform (DFT).

It is widely used in quantum computation and quantum information, especially quantum algorithm based on QFT, including Deutsch-Jozsa's algorithm, factoring algorithm and order-finding algorithm, phase estimation algorithm, Hidden subgroup problem.

(量子力学中的问题)

Quant. fourier transform $\leftrightarrow$ Quant. field theory

Def. $\boxed{\text{DFT}}$ is N维的一组复数 → N维的一组复数

$(x_0, x_1, \cdots, x_{N-1}) \overset{\text{DFT}}{\mapsto} (y_0, y_1, \cdots, y_{N-1})$

DFT 是连续 Fourier transf. 的类比

$$y_k = \frac{1}{\sqrt N}\sum_{j=0}^{N-1} \exp i\frac{2\pi}{N} j\cdot k \; x_j$$

$\downarrow \; \omega = \exp i\frac{2\pi}{N}$ (exponential function)

$$y_k = \frac{1}{\sqrt N}\sum_{j=0}^{N-1} \omega^{j\cdot k}\, x_j \quad \sim$$

可以看作. 对基矢 $|j\rangle$ 的 Fourier Trans.

$\downarrow$ DFT

$\frac{1}{\sqrt N}\sum_k \omega^{k\cdot j}|k\rangle$, 给出基矢的叠加.

[illegible]

(类比连续的情况, 给出 $\delta(x-x_0)$ 的变换就行)

Def. $\boxed{\text{QFT}}$ $N = 2^n$,

$|x_1\rangle|x_2\rangle\cdots|x_n\rangle \xmapsto{\text{QFT}_n} |y_1\rangle|y_2\rangle\cdots|y_n\rangle$

or

$|j_1\rangle|j_2\rangle\cdots|j_n\rangle \xmapsto{\text{QFT}_n} |k_1\rangle|k_2\rangle\cdots|k_n\rangle$

or

$$|j_1\rangle|j_2\rangle\cdots|j_n\rangle \xmapsto{\text{QFT}} \frac{1}{\sqrt{2^n}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1} e^{i\frac{2\pi}{N}j\cdot k}|k_1\rangle|k_2\rangle\cdots|k_n\rangle$$

$j\cdot k \to$ Decimal multiplication + 进制 [illegible]

(not binary multiplication)

$j = j_1 2^{n-1} + j_2\cdot 2^{n-2} + \cdots + j_n$

$k = k_1 2^{n-1} + k_2\cdot 2^{n-2} + \cdots + k_n$

(这里无所谓是否量 [illegible])

Decimal number — binary number

introduce notation: $|j\rangle \equiv |j_1\rangle|j_2\rangle\cdots|j_n\rangle$, $|k\rangle \equiv |k_1\rangle|k_2\rangle\cdots|k_n\rangle$

$$|j\rangle \xmapsto{\text{QFT}_n} \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}\omega^{j\cdot k}|k\rangle$$

• 记号简单, [illegible] $\omega^{j\cdot k}$, [illegible]

or: $$\text{QFT}_n = \frac{1}{\sqrt{2^n}}\sum_{k=0,\,j=0}^{2^n-1}\omega^{j\cdot k}|k\rangle\langle j|$$

n-qubit space

$$\text{QFT}_n^{-1} = \frac{1}{\sqrt{2^n}}\sum_{k=0,\,j=0}^{2^n-1}\omega^{-j\cdot k}|k\rangle\langle j|$$

classical Computation	DFT_n	$= O(\log N_N)$ $O(n\cdot 2^n)$ elementary gates are needed → useful but expensive
quantum Computation	QFT_n	$O(n^2)$ elementary gates are needed

$n^2 \lll n\cdot 2^n$, $n \to$ large

XV.2 $n=1$. QFT and QFT^{-1}

$$|j_1\rangle \xmapsto{\text{QFT}} \frac{1}{\sqrt2}\sum_{k_1=0}^{1}\omega^{j_1 k_1}|k_1\rangle = \frac{1}{\sqrt2}\left(|0\rangle + \omega^{j_1}|1\rangle\right) \underset{\omega = e^{i\frac{2\pi}{2}} = (-1)}{=\!=\!=} \frac{1}{\sqrt2}\left(|0\rangle + (-1)^{j_1}|1\rangle\right)$$

this means: $\text{QFT}_1 = \text{Hadamard}$

$|j_1\rangle \xrightarrow{H} H|j_1\rangle = \frac{1}{\sqrt2}\left(|0\rangle + (-1)^{j_1}|1\rangle\right)$

$\text{QFT}_1^{-1} = H$ $\quad\downarrow H$

$|j_1\rangle$

$H\,\frac{1}{\sqrt2}\left(|0\rangle + (-1)^{j_1}|1\rangle\right) = |j_1\rangle$

The circuit model: $\text{QFT}_1 = -\boxed{H}-$

$\text{QFT}_1^{-1} = -\boxed{H}-$

Note: $H|j_1\rangle = \frac{1}{\sqrt2}\left(|0\rangle + (-1)^{j_1}|1\rangle\right)$

$\downarrow$ qubit $\quad\uparrow$ phase

very important: phase 信息被储存在 [illegible]

QFT: phase encoded

phase: 量子力学和经典 [illegible] 方法 [illegible]

phase encoding and phase decoding:

$$\text{QFT}^{-1}: \; H\left(\frac{1}{\sqrt2}\left(|0\rangle + (-1)^{j_1}|1\rangle\right)\right) = |j_1\rangle \xrightarrow{\text{measurement}} |j_1\rangle$$

量子力学利用 phase 编码

XV.3(a) QFT_2 and QFT_2^{-1}

$n=2$, $N=4$. $$|j_1 j_2\rangle \xmapsto{\text{QFT}_2} \frac12\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\omega^{j\cdot k}|k_1 k_2\rangle$$

$\omega = e^{i\cdot\frac{2\pi}{4}} = i = \sqrt{-1}$

$j = 2j_1 + j_2$ $\quad$ 进制和指标 [illegible]

$k = 2k_1 + k_2$

$\omega^{j\cdot k} = \omega^{j\cdot(2k_1+k_2)} = \omega^{j\cdot 2k_1}\omega^{jk_2}$

or

$$|j_1 j_2\rangle \xmapsto{\text{QFT}_2} \frac12\sum_{k_1=0}^{1}\omega^{j\cdot 2k_1}|k_1\rangle\sum_{k_2=0}^{1}\omega^{jk_2}|k_2\rangle$$

k_1, k_2 seperating

k_1, k_2 [illegible]

$$= \frac12\left(|0\rangle + \omega^{2j}|1\rangle\right)\left(|0\rangle + \omega^{j}|1\rangle\right)$$

or

$$|j_1 j_2\rangle \xrightarrow{\text{QFT}_2} \frac12\left[|00\rangle + \omega^{j}|01\rangle + \omega^{2j}|10\rangle + \omega^{3j}|11\rangle\right]$$

or

$$\text{QFT}_2 = \frac12\sum_{j_1,j_2=0}^{1}\left(|00\rangle + \omega^{j}|01\rangle + \omega^{2j}|10\rangle + \omega^{3j}|11\rangle\right)\langle j_1 j_2|$$

$$\text{QFT}_2 = \left(\begin{pmatrix}1\\ \omega^j\\ \omega^{2j}\\ \omega^{3j}\end{pmatrix}_{j_1,j_2=0}\; \begin{pmatrix}1\\ \omega^j\\ \omega^{2j}\\ \omega^{3j}\end{pmatrix}_{j_1=0,j_2=1}\; \begin{pmatrix}\omega^j\\ \omega^{2j}\\ \omega^{3j}\end{pmatrix}_{j_1=1,j_2=0}\; \begin{pmatrix}\omega^j\\ \omega^{2j}\\ \omega^{3j}\end{pmatrix}_{j_1=1,j_2=1}\right)$$

$j_1 = j_2 = 0$, $j = 2j_1 + j_2 = 0$. $\omega^j = \omega^{2j} = \omega^{3j} = 1$

$j_1 = 0, j_2 = 1$. $j = 1$, $(\omega^j, \omega^{2j}, \omega^{3j})^T = (\omega, \omega^2, \omega^3)^T$

$j_1 = 1$. $j_2 = 0$. $j = 2$. $(1, \omega^2, \omega^4, \omega^6)^T = (1, \omega^2, 1, \omega^2)^T$

$j_1 = 1$, $j_2 = 1$. $j = 3$ $\quad (1, \omega^3, \omega^6, \omega^9)^T = (1, \omega^3, \omega^2, \omega)^T$

$$\text{QFT}_2 = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & \omega & \omega^2 & \omega^3\\ 1 & \omega^2 & 1 & \omega^2\\ 1 & \omega^3 & \omega^2 & \omega\end{pmatrix}$$

<u>very very strange</u>!

what's the quantum circuit model?

XV.3(b) Quantum Circuit model of QFT_2

$$|j_1 j_2\rangle \xmapsto{\text{QFT}} \frac12\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\omega^{jk}|k_1 k_2\rangle = \frac12\sum_{k_1=0}^{1}\omega^{2jk_1}|k_1\rangle\sum_{k_2=0}^{1}\omega^{jk_2}|k_2\rangle = \frac12\left(|0\rangle + \underbrace{\omega^{2j}}_{与 j_1 无关}|1\rangle\right)\left(|0\rangle + \omega^{j}|1\rangle\right)$$

$\omega^{jk} = \omega^{2jk_1}\cdot\omega^{jk_2}$

$\omega = e^{i\frac{2\pi}{4}}$

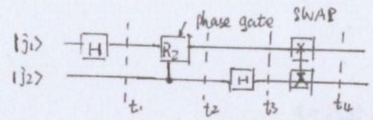
$$\omega^{2j} = \exp i\cdot 2\pi\cdot \tfrac{j}{2} = \exp i\cdot 2\pi\cdot\tfrac12(j_1 2 + j_2) = \exp i\cdot 2\pi\left(j_1 + \tfrac{j_2}{2}\right) = \exp i\cdot 2\pi\cdot\tfrac{j_2}{2} = \exp^{i\cdot 2\pi(0.j_2)}$$

computer science 中的 [illegible] $\left\langle \begin{array}{l}\frac{j_2}{2}: \text{binary fraction}\\ .j_2 = \frac{j_2}{2}\end{array}\right.$

$$\omega^j = \exp i\cdot 2\pi\tfrac{j}{4} = \exp i\tfrac{2\pi}{4}(2j_1 + j_2) = \exp i2\pi\left(\tfrac12 j_1 + \tfrac14 j_2\right) = \exp i\cdot 2\pi\, 0.j_1 j_2$$

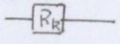
$$|j_1 j_2\rangle \xmapsto{\text{QFT}_2} \frac12\left(|0\rangle + \exp i\cdot 2\pi\, 0.j_2|1\rangle\right)\left(|0\rangle + \exp i\cdot 2\pi\, 0.j_1 j_2|1\rangle\right) = \frac12 H|j_2\rangle\left(|0\rangle + \exp^{i\cdot 2\pi\cdot 0.j_1 j_2}|1\rangle\right)$$

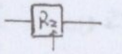
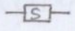
swap



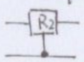
encode / decode phase

$|\psi(t_1)\rangle = H|j_1\rangle\otimes|j_2\rangle = QFT_1|j_1\rangle\otimes|j_2\rangle = \frac{1}{\sqrt2}(|0\rangle + (-1)^{j_1}|1\rangle)|j_2\rangle$

 $= \begin{pmatrix}1 & 0\\ 0 & e^{i\frac{2\pi}{2^{k}}}\end{pmatrix}$ 相位

 $= \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} =$  phase gate

2-qubit QFT

 Controlled-phase gate $\quad CR_2 = \mathbb{1}\otimes|0\rangle\langle 0| + R_2\otimes|1\rangle\langle 1|$

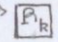
$|\psi(t_2)\rangle = CR_2|\psi(t_1)\rangle = \left(\mathbb{1}_2\otimes|0\rangle\langle 0| + R_2|1\rangle\langle 1|\right)\left(|0\rangle+(-1)^{j_1}|1\rangle\right)|j_2\rangle/\sqrt2$

• factoring algorithm 従核 QFT

$j_2=0:\quad (|0\rangle+(-1)^{j_1}|1\rangle)|j_2=0\rangle/\sqrt2$

$= (|0\rangle + \exp i\cdot 2\pi\cdot 0.j_1\,|1\rangle)|j_2=0\rangle/\sqrt2$

$= (|0\rangle + \exp i\cdot 2\pi\, 0.j_1j_2|1\rangle)|j_2=0\rangle/\sqrt2$

• binary fraction $\Longleftrightarrow$  controlled phase gate

$|j_2=1\rangle\quad \left(|0\rangle + \exp i\cdot 2\pi\, 0.j_1\,\exp i\cdot 2\pi\frac14\,|1\rangle\right)|j_2\rangle/\sqrt2$

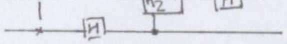
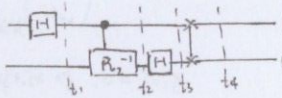
$R_2\cdot|0\rangle = |0\rangle$
$R_2\cdot|1\rangle = e^{i2\pi/4}|1\rangle$

$= \left(|0\rangle + \exp i2\pi\, 0.j_1j_2\,|1\rangle\right)|j_2\rangle/\sqrt2$

$= (|0\rangle + e^{i\cdot 2\pi\cdot 0.j_1j_2}|1\rangle)|j_2\rangle/\sqrt2$

$|\psi(t_3)\rangle = (\mathbb{1}_2\otimes H)|\psi(t_2)\rangle = (|0\rangle + e^{i\cdot 2\pi\, 0.j_1j_2}|1\rangle)(|0\rangle + e^{i\cdot 2\pi\cdot 0.j_2}|1\rangle)$

$|\psi(t_4)\rangle = SWAP|\psi(t_3)\rangle = (|0\rangle + e^{i\cdot 2\pi\, 0.j_2}|1\rangle)(|0\rangle + e^{i\cdot 2\pi\, 0.j_1j_2}|1\rangle) = QFT_2|j_1j_2\rangle$

QFT_2^{-1}:  (exercise) equivalent $=$ 

• 两者就是一位对调SWAP的位置不同

XV. 4. QFT_n & QFT_n^{-1}

$|j_1\rangle|j_2\rangle\cdots|j_n\rangle \mapsto \frac{1}{\sqrt{2^n}}\sum_{k_1,k_2\cdots k_n=0}^{1}\omega^{j\cdot k}|k_1\rangle|k_2\rangle\cdots|k_n\rangle \qquad$ 由于每个态能拆开 $\omega^{jk} = (\ )^{k_1}\cdots(\ )^{k_n}$

$\omega^{j\cdot k} = \omega^{j\cdot(2^{n-1}k_1 + 2^{n-2}k_2 + \cdots + 2^0 k_n)}$

$= \omega^{j\cdot k_1\cdot 2^{n-1}}\,\omega^{j\cdot k_2\cdot 2^{n-2}}\cdots\omega^{j k_n}$

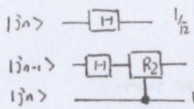
$\omega^{j k_1\cdot 2^{n-1}} = \exp i\frac{2\pi}{2^n}j\cdot k_1\cdot 2^{n-1} = \exp i\cdot 2\pi\, j\, k_1/2 = \exp i\cdot 2\pi\, \frac{j}{2}k_1$

$\exp i\cdot 2\pi\cdot j/2 = \exp i\cdot 2\pi\cdot\frac12(2^{n-1}j_1 + 2^{n-2}j_2 + \cdots + 2j_{n-1} + j_n) = \exp i\cdot 2\pi\frac{j_n}{2} = \exp i\cdot 2\pi\, 0.j_n$

$\begin{cases}\omega^{j k_1\cdot 2^{n-1}} = \exp i\cdot 2\pi(0.j_n)k_1\\ \omega^{j k_2\cdot 2^{n-2}} = \exp i\cdot 2\pi(0.j_{n-1}j_n)k_2\\ \vdots\\ \omega^{j k_n} = \exp i\cdot 2\pi(0.j_1j_2\cdots j_n)k_n\end{cases}$

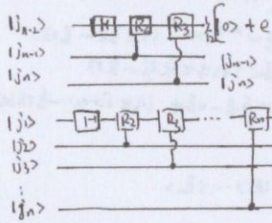
$|j_1j_2\cdots j_n\rangle \overset{QFT_n}{\longmapsto} \frac{1}{\sqrt{2^n}}\left(|0\rangle + \exp i\cdot 2\pi(0.j_n)|1\rangle\right)\left(|0\rangle + \exp i\cdot 2\pi(0.j_{n-1}j_n)|1\rangle\right)\cdots\left(|0\rangle + \exp i\cdot 2\pi(0.j_1j_2\cdots j_n)|1\rangle\right)$

quant. circuit model of it ?



1 Hadamard = 1 gates

1 Hadamard + 1 Controlled-phase = 2 gates



1 Hadamard + $(n-1)$ Controlled-phase = n gates

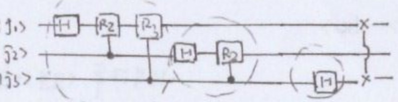
$\downarrow$

$1+2+3+\cdots+n = \frac{(1+n)n}{2}$ gates

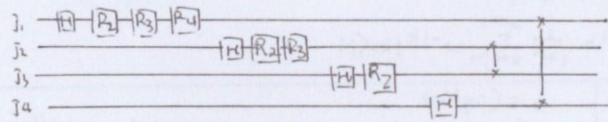
$+\ [\frac{n}{2}]\times 3\quad \underbrace{\text{SWAP gate}\times 3}_{\text{Cnot gates}}$

$= \frac{(1+n)n}{2} + \frac{3n}{2} \propto O(n^2)$

QFT_3: $\quad 1+2+3+[\frac32]\times 3 = 6+3 = 9$ gates



QFT_4:



2016. 6. 14

Final exam schedule

Time : June 28 (Tues.) place : Room 101, at old building of school of physics

2:30 pm ~ 4:30 pm

Remark : It is open test, only open to ur own classnotes & homeworks → understand. copy

During exam, you are not allowed to discuss with ANYONE.

Research projects: I. Topological features in Measurement-based QIC

(teleportation-based QIC is just a special example of measurement-based QIC)

($\neq$ Quantum circuit model)

II. Topological features in Fault-tolerant QC

Quantum circuit model
teleportation based QC
measurement based QC
Fault-tolerant QC

} 四种 QC

III. Quant computation using Yang-Baxter gates

Topological QC ~ almost done.
Integrable QC ~ don't know.
} special case of fault-tolerant QC

* Classical Simulation of Yang-Baxter gates (great example)

* Topological invariants versus Topological entanglement measures

(多粒子的纠缠, measure 仍未被定义?)

★ IV. Bell transform and GHZ transform

Bell states, GHZ states > useful and interesting

★ V. Physical realization of quantum computers. (implementation)

VI. Other topic in QIC * No-cloning theorem

* Bell's inequalities

* Quantum entanglement measures

VII. Research topics between QIC and QFT (quantum field theory)

* qubit field theory

* quantum simulation of Quantum Field Theories on QC

(高能物理很昂贵, 高能模拟再好不过)

* The Calculation of partition function (in statistical physics in QFT using quantum circuit model.

* The calculation of scattering matrix on QC.

* More topics (Next semester)

Reference: Download pdf file in instructor's home page : z-ol-research-teleportation ...

GHZ transform (I): Bell transform and Quant. Teleportation.

↳ new concepts

Research Project IV (just ideas, not detail)

Def. Bell transform

product basis: $|kl\rangle$

Bell basis: $|\psi(kl)\rangle$

Bell transform is defined as a unitary basis transformation matrix from the product basis $|k'l'\rangle$ to the Bell basis $e^{i\phi_{kl}}|\psi(kl)\rangle$ with the phase factor $e^{i\phi_{kl}}$; where k and l are bijective functions of k' and l', namely $k=k(k',l')$, $l=l(k',l')$, $k,l,k',l'=0,1$, respectively.

So that the Bell transform **Bell** is bijective mapping between $|k',l'\rangle$ and $e^{i\phi_{kl}}|\psi(k,l)\rangle$,

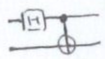
and $Bell|k'l'\rangle = e^{i\phi_{kl}}|\psi(kl)\rangle$

$$Bell = \sum_{k',l'=0}^{1} e^{i\phi_{kl}} |\psi(k,l)\rangle\langle k',l'|$$

↑ 这里的 k, l 应该作为 k', l' 的函数

· Bell transform 的一般定义 一但找出, 就会有多大的好处! 这篇人们一直在研究特别的 Bell state.

e.g. C_H gate

$C_H =$ 

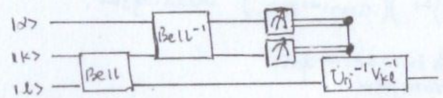
$$C_H = CNOT \otimes (H \otimes \mathbb{1}_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$C_H|k,l\rangle = |\psi(k,l)\rangle$ 没有 phase factor (β_{kl})?

e.g. Yang-Baxter gate in Home work 3rd and 4th.

B

Thm: The teleportation operator $(Bell^{-1} \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes Bell)$ (← interesting) (is capable of QT) is a meaningful algebraic operator describing quantum teleportation.



Teleportation eq. $(Bell^{-1} \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes Bell)|\alpha\rangle|kl\rangle = \frac{1}{2}\sum_{i,j=0}^{1} |ij\rangle V_{kl} U_{ij} |\alpha\rangle$

e.g. $Bell = C_H$, $U_{ij} = x^j z^i$. $V_{kl} = x^k z^l$

Question: what about high-dimensional generalization of Bell transform.

Answer 1: GHZ transform

Def. GHZ states of n-qubit

phase-bit operators $x_1 x_2 x_3 \cdots x_n |G(j_1 j_2 \cdots j_n)\rangle = (-1)^{j_1}|G(j_1 j_2 \cdots j_n)\rangle$

↑ 这是 乘积?

1st parity-bit operator (也可以是 2nd)

$$z_1 z_2 |G(j_1 j_2 \cdots j_n)\rangle = (-1)^{j_2}|G(j_1 j_2 \cdots j_n)\rangle$$

2nd parity-bit operator

$$z_2 z_3 |G(j_1 j_2 \cdots j_n)\rangle = (-1)^{j_2+j_3}|G(j_1 j_2 \cdots j_n)\rangle$$

⋮

i-th parity-bit operator ($i \geq 3$)

$$z_{i-1} z_i |G(j_1 \cdots j_n)\rangle = (-1)^{j_{i-1}+j_i}|G(j_1 \cdots j_n)\rangle$$

· n-qubit GHZ states are eigen states of $x_1 x_2 \cdots x_n$, $z_1 z_2$, $z_2 z_3$, $\cdots$ $z_{n-1} z_n$

$$|G(j_1 j_2 \cdots j_n)\rangle = \frac{1}{\sqrt{2}}\left(|0 j_2 \cdots j_n\rangle + (-1)^{j_1}|1 \bar{j}_2 \cdots \bar{j}_n\rangle\right)$$

eg1. Bell states

eg2. three-qubit GHZ states ($2^3=8$ 个线性无关的 GHZ states) — linear independent

$|G_1\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$ $\quad |G_2\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle + |110\rangle\right)$

$\vdots$

$|G_4\rangle = \frac{1}{\sqrt{2}}\left(|011\rangle + |100\rangle\right)$ $\quad |G_8\rangle = \frac{1}{\sqrt{2}}\left(|011\rangle - |100\rangle\right)$

• Def GHZ basis

$|G_j(j_1 \cdots j_2)\rangle$

$j = 2^{n-1}j_1 + 2^{n-2}j_2 + \cdots + j_n + 1$, $1 \le j \le 2^n \Rightarrow \dim(\mathcal{H}_2^{\otimes n}) = 2^n$

可以证明 orthonormal relation
completeness relation

• Def GHZ transform

$GHZ^{(n)} = \sum_{j_1 \cdots j_n = 0} e^{i\phi_{k_1 k_2 \cdots k_n}} |a(k_1 k_2 \cdots k_n)\rangle\langle j_1 j_2 \cdots j_n|$

$GHZ^{(n)} = C_H^{(n)} P^{(n)} E^{(n)}$

$C_H^{(n)} = CNOT_{1,n}\, CNOT_{1,n-1} \cdots CNOT_{1,2}\, H$

$P^{(n)} = \sum_{j_1 j_2 \cdots j_n = 0}^{1} |k_1 \cdots k_n\rangle\langle j_1 j_2 \cdots j_n|$ — permutation gate

$E^{(n)} = \sum_{j_1 j_2 \cdots j_n = 0} e^{i\phi_{k_1 \cdots k_n}} |j_1 j_2 \cdots j_n\rangle\langle j_1 \cdots j_n|$ — phase gate

Question: 1) Quantum Simulation of GHZ transform

2) teleportation operator $\left(\mathbb{I}_2 \otimes (GHZ^{(n)})^{-1}\right)\left(GHZ^{(n)} \otimes \mathbb{I}_2\right)$ meaningful?

3) Quantum Algorithms using QFT (注意和 1D QFT 的等价)

Quantum Algorithm using GHZ transform

Research Project V Physical Implementation of QC

V.1. Guiding principles · How to build a quantum computer?

VIa. Robust representation of qubit · 如何定义 qubit
可在实验中实现的 qubit

superposition

Quantum (QC) Computer ⇌ environment − noise
open system ⇌ observer

· 量子计算机 一定与外界有相互作用
一定受到人为控制 } ⇒ open system

⇓

decoherence
(superposition is killed)

量子计算机的优势 ⇒ superposition
劣势 ⇒ decoherence

Decoherence is unavoidable!

Decoherence time: τ_Q

Operation time: τ_{op} (before decoherence, what we can do)

$\frac{\tau_Q}{\tau_{op}} =$ number (operation)

对 $n \gg 1$. qubit is robust against noise

eg. qubit = electron spin , $\tau_Q = 10^{-3}$ sec
$\tau_{op} = 10^{-7}$ sec
$n = \frac{\tau_Q}{\tau_{op}} = 10^4$

• 对 electron spin (natural candidates of qubits)

(before decoherence, we can do many many thing)

VI.b. perform a universal quantum gate set

eg1. {CNOT gate, single qubit gates}

eg2. {CNOT gate, Hadamard, T}

• 在时间 τ_Q 之内 perform gate?
如何 realize quantum gate in QC

T gate $= \frac{\pi}{8}$ gate $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{8}} \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix}$

⇒ All single-qubit gates can be generalized by {H & T}

Physical Realization of Quantum Gates

approach I: Dynamical evolution of system
controlled by Hamiltonian

• 从薛定谔 equation ⇒ evolution of qubit must be unitary

approach II: Quantum measurement!
(Measurement-based QC) ← 改变了我们对量子力学的认识.
与传统的 QC 不同
传统: 认为 measurement useful
但 Measurement-based QC:
可以利用 measurement realize unitary transform.

VI.c. (Input) preparation of initial states

usually ground state $|0\rangle^{\otimes n} = |0\rangle|0\rangle \cdots |0\rangle$ (how to prepare?)
有时很难实现!

VI.d. (Output) Quantum measurement of final quantum states.

VI.e. Classical Computation: Classical Communication

建立 quantum computer 时,
也能建 classical computer (special case of quantum computation)

eg1. Harmonic oscillator QC

simple Harmonic oscillator ⇒ Hamiltonian of SHO
SHO
(有两种, 一种是坐标表象的表示, 一种是 Dirac 表示)

$H = \hbar\omega\left(\frac{1}{2} + a^{+}a\right)$ ← Dirac approach
$[a, a^{\dagger}] = 1$

ground state $a|0\rangle = 0$

$|n\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}}|0\rangle$

$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$

• 谐振子的优点: 量子力学 / 量子场论 / black hole 的 / 最后留下的东西

define single qubit ($\mathcal{H}_2$): $|0\rangle_L = |0\rangle$
$|1\rangle_L = |1\rangle$ (第一激发态)

two qubit system ($\mathcal{H}_4$):

$|00\rangle_L = |0\rangle$
$|01\rangle_L = |1\rangle$
$|10\rangle_L = |2\rangle$
$|11\rangle_L = |3\rangle$

or

$|00\rangle_L = |0\rangle$
$|01\rangle_L = |2\rangle$
$|10\rangle_L = (|4\rangle + |1\rangle)/\sqrt{2}$
$|11\rangle_L = (|4\rangle - |1\rangle)/\sqrt{2}$

... look strange

N-qubit System

$(\mathcal{H}_2^{\otimes n})$ $\dim(\mathcal{H}_2^{\otimes n}) = 2^n$.

$|0\rangle_L = |0\rangle$
$|1\rangle_L = |1\rangle$
$\vdots$
$|2^n\rangle_L = |2^n-1\rangle$

energy $H|2^n\rangle_L = \hbar\omega(2^n-1+\frac{1}{2})|2^n\rangle_L \Rightarrow 2^n\hbar\omega$. too high. expensive

n-qubit = collection of single qubit system

energy $\Rightarrow n\hbar\omega$, relative low.

Next semester • Quantum Field Theory $\begin{cases} \text{Special relativity} \to \text{G.R.} \\ \text{Quantum mechanics} \end{cases}$ (non-linear combination)

change → QIC

QIC + S.R. = ?

QIC + G.R. = ? "Quantum gravity", big challenge (string? loop gravity?)

• Course Introduction 2016 in instructor's home page

PDF for lec. notes & main reference

6.21 of 2016.

Harmonic oscillator quantum computer → toy model
but concept is important

$H = (a^\dagger a + \frac{1}{2})\hbar\omega$

$a|0\rangle = 0$ $\quad [a, a^\dagger] = 1$

$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$

$H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$

Qubit : $\mathcal{H}_2$ $\quad |0\rangle_L = |0\rangle$, $|1\rangle_L = |1\rangle = a^\dagger|0\rangle$

$\mathcal{H}_2 = \{|0\rangle, a^\dagger|0\rangle\}$

2-qubit : $\mathcal{H}_4 = \text{span}\left\{|0\rangle, a^\dagger|0\rangle, \frac{(a^\dagger)^2}{\sqrt{2!}}|0\rangle, \frac{(a^\dagger)^3}{\sqrt{3!}}|0\rangle\right\}$

n-qubit $= \mathcal{H}_{2^n} = \text{span}\{|0\rangle, |1\rangle, \cdots, |2^n-1\rangle\} \Rightarrow H|2^n-1\rangle = \hbar\omega(2^n-1+\frac{1}{2})|2^n-1\rangle$

"one harmonic oscillator" $\quad$ need $2^n\hbar\omega$ energy

n-qubit = 1 qubit + 1 qubit + ... + 1 qubit

"n simple Harmonic oscillator" $\quad$ need $n\hbar\omega$ energy

• qubit 的 design 方法千变万化

(但不同的方法，优缺不同)

Quantum gates: unitary transformation

↙ unitary evolution By Hamiltonian

↘ Measurement-based QC

(decoherence → strange)

(仍然不能避免 collapse，但可以用其实现去变换)

$U(t) = e^{-\frac{i}{\hbar}Ht}$, $H = \hbar\omega(a^\dagger a + \frac{1}{2})$

$|n(t)\rangle = e^{-\frac{i}{\hbar}Ht}|n\rangle = \underbrace{e^{-i\frac{\omega t}{2}}} e^{-in\omega t}|n\rangle$ $\qquad |m(t)\rangle + |n(t)\rangle = \underbrace{e^{-i\frac{\omega}{2}t}}(e^{-in\omega t}|n\rangle + e^{-im\omega t}|m\rangle)$

与n无关，不用考虑，$\frac{1}{2}\hbar\omega$ zero point energy

不重要 in QIC / QM, but crucial in QFT

Realize of CNOT gate : $\begin{cases} |00\rangle_L \to |00\rangle_L \\ |01\rangle_L \to |01\rangle_L \\ |10\rangle_L \to |11\rangle_L \\ |11\rangle_L \to |10\rangle_L \end{cases}$

Two qubit $\begin{cases} |00\rangle_L = |0\rangle \\ |01\rangle_L = |1\rangle \\ |10\rangle_L = |2\rangle \\ |11\rangle_L = |3\rangle \end{cases}$

THEN:
$|0\rangle \to |0\rangle$
$|1\rangle \to |1\rangle$
$|2\rangle \to |3\rangle$
$|3\rangle \to |2\rangle$
difficult!

define qubit in another way:

Two qubit

$|00\rangle_L = |0\rangle$

$|01\rangle_L = |2\rangle$

$|10\rangle_L = \frac{|4\rangle + |1\rangle}{\sqrt{2}}$

$|11\rangle_L = \frac{|4\rangle - |1\rangle}{\sqrt{2}}$

time evolution:

$|n\rangle \to e^{-i\frac{\omega}{2}t}e^{-in\omega t}|n\rangle$

define $t = \frac{\pi}{\omega}$

$|n\rangle \to e^{-i\frac{\pi}{2}} e^{-in\pi}|n\rangle$

$= e^{-i\frac{\pi}{2}}(-1)^n|n\rangle$

Thus we have:

$|0\rangle \to |0\rangle$ $\quad |2\rangle \to |2\rangle$

$|1\rangle \to -|1\rangle$ $\quad |3\rangle \to -|3\rangle$

$|4\rangle \to |4\rangle$

we realize CNOT:

$t = \frac{\pi}{\omega}$ $\begin{cases} |00\rangle_L \to |00\rangle_L \\ |01\rangle_L \to |01\rangle_L \\ |10\rangle_L = \frac{|4\rangle + |1\rangle}{\sqrt{2}} \to \frac{|4\rangle - |1\rangle}{\sqrt{2}} = |11\rangle_L \\ |11\rangle_L = \frac{|4\rangle - |1\rangle}{\sqrt{2}} \to \frac{|4\rangle + |1\rangle}{\sqrt{2}} = |10\rangle_L \end{cases}$