Introduction to Quantum Field Theory (Fall, 2016)

Download Lecture Notes (homeworks) in the instructor's English homepage:
http://physics.whu.edu.cn/en/?q=node/112

1 Schedule, Instructor and Teaching Assistant

Time and Place: Tuesday, Class 6-8, I-3-404
Instructor: Yong Zhang.

- Office: 5-516, at the new building of the School of Physics and Technology;
- Email: yong_zhang@whu.edu.cn.

Teaching Assistant: (None).

2 Lecture Notes

* Yong Zhang (Wuhan): the fourth version, online lecture notes on QFT.
The teaching plan of this course is made in accordance with the main reference.

3 Main Reference

* Michael Luke (Toronto): online lecture notes on QFT.

4 References

* Peskin & Schroeder (Stanford): An Introduction to Quantum Field Theory;

* David Tong (Cambridge): online lecture notes on QFT;

* Sidney Coleman (Harvard): online lecture notes;

* Mandl & Shaw: Quantum Field Theory (Second Edition).

5 Homeworks

Peskin & Schroeder's Problem Sets or Luke's Problem Sets or Tong's Problem Sets.

6 Research Projects

Ask the instructor for it.

7 Evaluation

Homeworks(50%) + Final Exam(50%).

Final Exam → Open test.
→ Open to Classnotes and Homeworks.

Remark: do homwork=do research

$$\left\{\begin{array}{l}\text{Perspective: Focus on a small problem;}\\ \text{Persistence: Solve your problem completely;}\\ \text{Patience: Write down your solution step by step;}\\ \text{Power: Quote your references carefully.}\end{array}\right.$$

① 20160906 3-404 Zhang Yong

lec 1. Introduction to QFT
lec 2. Introduction to Feynman diagram (key point of this course)
lec 3. Classical Field Theory

What about the final exam in Quantum Information & Computation?
open test.
Results are terrible.

P1 – P2
1. Modern Theoretical Physics, branches
2. CPM $\xrightarrow{①}$ QFT Continuum limit; $\xrightarrow{②}$ Quantization

Definition of Quantum Field Theory
① SR + QM
② particle
③ particle-wave
④ CFT

I.1. About the Course (4_00-QFT-Course-Introduction-2016.pdf)

I1A Class meeting: Tues. class 6-8, I-3-404

I1B Instructor: Yong Zhang; office: 5-516 New building; Email address Yong-zhang@whu.edu.cn
homepage (English): http://physics.whu.edu.cn//en//?q=node//112
No teaching assistant.

I1C lec. notes, the fourth version lec. notes on QFT;

I1d Main reference: Michael Luke (Toronto), online notes on QFT; (simple)

I1e Reference: Peskin & Schroeder (Standford), Introduction to QFT;
pdf files can be downloaded in Instructor's homepage

I1f Homeworks: 3 homeworks
taken from Luke's problem sets; Tong's problem sets (Cambridge); Peskin & Schroeder's;

Do homework = Do re-search
perspective: focus on a small problem
persistence; solve your problem completely
patience; write down your solution step by step (check your solution's best way)
power; self-confidence, quote your reference carefully

I1g Evaluation: 3 homeworks ($3\times3=9$ problems) → 50%
final exam → 50% open test
open to class notes and homeworks

Remark: Homeworks & Final exams will be much simpler than Homeworks & Final exam in QIC.

Remark: All the solutions to final exam will be discussed in detail in class.
Taking classnotes and doing homeworks are crucial for doing Final exam

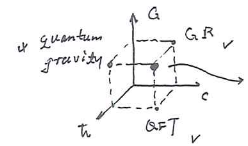
I1h Research projects: Ask instructor for it.

I.2. Overview of modern Theoretical physics
what's the place (position) of QFT in MTP. Historical Conceptual Development of QFT.

particle mechanics (newtonian mechanics) \ classical mechanics ……
field theory /
gravity, electromagnetic field

quantum mechanics, special relativity } quantum field theory
special relativity, gravity } general relativity
quantum mechanics } quantum gravity
} unification theory (M theory) (string theory)

(gravity) G → gravitational constant
$\hbar$ → Planck constant (quantum mechanics)
c (light speed) (special relativity)



→ unification theory

classical particle mechanics $\xrightarrow[\text{limit}]{\text{continuum}}$ classical field theory (Approach II)
↓ quantization (Approach I) ↘ (Approach III) ↓ quantization
quantum mechanics $\xrightarrow[\text{limit}]{\text{continuum}}$ quantum field theory

what is quantization procedure

Quantization: Canonical quantization (we know it but don't know its name)
Integral Path quantization (we know its name but don't know it)

I.3 definition of QFT (Note: QFT (always) means Relativistic Quantum Field Theory)
Note: Relativistic Quantum Mechanics (wrong name) = QFT

Def 1. QFT = Special relativity + quantum mechanics
(use models) (unification of two)

Def 2. QFT = quantum many-particle system with unfixed particle number (about particles)
= Quantum infinity number of particles system
(informed by mass-energy relation)

Remark: At high energies, with special relativity, particle can be created and annihilated (destroyed), so the particle number is not conserved, which is described by relativistic QFT;

Remark: At low energies, for example, Condensed matter physics, particle number is a conserved quantity, which is described by non-relativistic QFT.

Non-relativistic QFT
= Quantum many-particle theory with fixed particle number.
= Quantum many-particle theory using key techniques of QFT such as Feynman diagrams.
(迄今为止，在particle, condensed (about particles ↕ field) 里面都可以用同样的技术)

Def 3. QFT = quantization (relativistic) of classical field theory = quantization of classical field theory satisfying the constrains from special relativity
(about field)
e.g.1. quantum Electrodynamics (QED) = 3. Relativistic 2. quantization of 1. Electrodynamics.
e.g.2. Quantum Optics = Non-relativistic quantization of Electrodynamics.
(never talk about relativistic) (Non-relativistic QFT in condensed matter physics not in High energy)

Def 4. QFT ⇒ is a natural language of describing the particle-wave duality; which states that
(we don't understand the duality, (the nature, the root)
but use QFT, we can described the duality)
each fundamental particle is a derived object of quantization of the associated (classical?) field.
(i.e. quark, electron, photon ...)

② e.g.

particle	duality	field (wave)
Electron	$\Longleftrightarrow$	Dirac field (Electron field; bispinor field)
quark	is result of	quark field
photon		Electromagnetic field (Maxwell field)
Higgs particle		Higgs field

Strange but supported by all the experiment

(Another definition:
QFT is the language of describing the human being's present understanding of the nature; i.e., a language of describing the particle-wave duality)

theoretical QFT $\leftrightarrow$ experimental particle physics

Remark: particle-wave duality
particle with four-momentum $p_\mu = (E, \vec{p})$
wave (field) with wave frequency $\nu(\omega)$ and wave length $\lambda(k)$

$E = \hbar\omega = h\nu$
$|\vec{p}| = h/\lambda = \hbar k$

Remark: The same type of particles are indistinguishable
is assumption in QM
is derived conclusion in QFT

is a principle in QM
but can have a natural interpretation in QFT, because identical particles are quanta of the same quantum field.

I.4. Research Project in QFT

Conventional project $\to$ Doing High Energy Experiment
$\to$ Phenomenology in Experiment = particle physics
$\to$ Mathematical aspect of QFT (pure theoretical stuff)

New Project QFT = SR + QM $\to$ Quantum I & C

SR + QIC? Gravity + QIC?
QFT + QIC? Quantum Gravity + QIC?

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Lec II. Introduction to Feynman diagrams

(这Quantum Field Theory 变得simple的关键)

II.1 Perturbative QFT (PQFT) ← 这种情况下，可以做费曼图

Def. PQFT = QFT with weak coupling constant allowing Taylor expansion, This semester

= Set of Feynman diagrams = Tree FDs + FDs with loops = Tree FDs + One-loop FDs + Two-loop FDs + ⋯ + Multiloop FDs
(hope to study)

Note: Perturbative QFT (Feynman diagrams) are enough to explain most experiments in High energy physics.

Def. Non-perturbative QFT = QFT with strong coupling constant NOT allowing Taylor expansion

Note: NPQFT explain Quark confinement,
Soliton solution (Monopole, Instanton);

This course this semester focus on FDs.

Remarks: We talk about three models in this course

right ⟨ pseudo nucleon

Model I: The Pseudo-Yukawa interaction using pseduo-nucleon and meson

Note: Pseduo-Nucleon, proton-like, neutron-like spin-0 particles
neucleon: proton, neutron $\to$ spin-$\frac{1}{2}$ particles

Model II: The Yukawa Interaction using neucleon and mesons

Note: Yukawa Interaction
(In the Standard model for particle physics)

Model III: Quantum Electrodynamics using electron and photon.

II.2. The Pseudo-Yukawa Interaction

		"nucleon"	"Anti-Nucleon"	meson
Particle description	Symbols	"N"	"$\bar{N}$"	ϕ
	spin	0	0	0
	charge	+1	-1	0
	mass	m	m	μ
	physics	ideal particle	ideal particle	π^0 meson
Field description		Complex scalar fields $\psi(x)$	$\psi^*(x)$	real scalar field $\phi(x) = \phi^*(x)$

Note: The pseudo-Yukawa interaction is a toy model to understand Yukawa interaction and QED. (simple but useful)

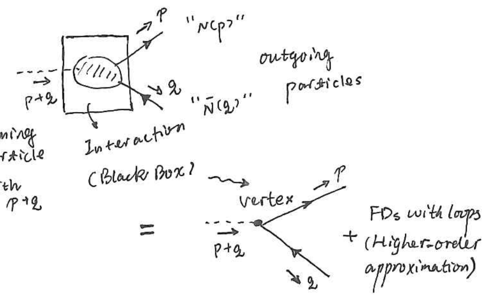
eg1. The decay of a meson

$\phi(p+q) \to$ "$N(\vec{p})$" + "$\bar{N}(\vec{q})$"

meson particle with four momentum $p+q$

$p = (E_p, \vec{p})$ $q = (E_q, \vec{q})$

FDs

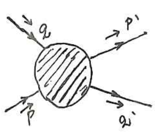


= First-order approximation (Tree Feynman diagrams) + FDs with loops (Higher-order approximation)

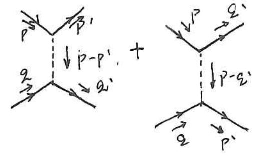
Tree Approximation + Loop Approximation + ⋯

eg2. pseudo-nucleon "Nucleon" - "Nucleon" scattering

"N"(p) + "N"(q) $\to$ "N"(p') + "N"(q'), momentum conservation $p+q = p'+q'$



= Tree FDs (Second-order) + FDs with loops (High-order) =



+ FDs with loops

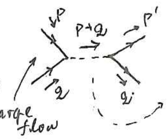
($p' \leftrightarrow q'$)

• Two "Nucleon" interacts by interchange of a virtual meson ϕ with $p-p'$ or $p-q'$

virtual: unobserved in Black-Box

• 第二个FD是第一个FD的两个粒子不变，而动量交换的结果；原因是粒子不可区分

NOT can also be drawn as: (not conserve q)

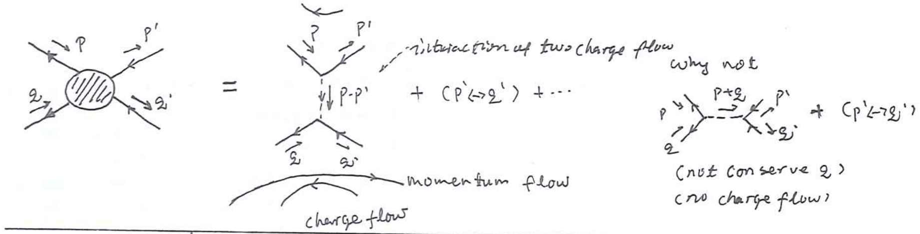


interchange of virtual particle (meson) + ($p' \leftrightarrow q'$) (?)

"N"(p), "N"(q) are identical particles with different p, q

③

e.g. 3. Two Anti-"Nucleon" scattering $``\bar{N}"(p) + ``\bar{N}"(q) \to ``\bar{N}"(p') + ``\bar{N}"(q')$, $p+q = p'+q'$



	same direction	diff. direction / opposite
charge flow momentum flow	"Nucleon"	"Anti-Nucleon"

In this sense, QFT is very simple; we just draw the FDs without understanding the QFT.

eg 4. "Nucleon" - Anti-"Nucleon" scattering $``N"(p) + ``\bar{N}"(q) \to ``N"(p') + ``\bar{N}"(q')$

Refer to Online lec. notes to QFT V4. 20160920 Zhang Yong

Lec III Introduction to FDs

Lec. IV Introduction to classical Field theory (later)

Review:

Models — Particle (Field)

The pseudo-Yukawa interaction

$p = (p_\mu) = (p_0, \vec{p}) = (E_{\vec{p}}, \vec{p})$ T, S : spin $\left\{ \begin{array}{l} = \frac{1}{2} \uparrow \\ = -\frac{1}{2} \downarrow \end{array} \right.$

ε : polarization vector (4-dim) $\varepsilon = (\varepsilon^\mu) = (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3)$

pseudo-nucleon anti-pseudo nucleon meson

$``N"(p)$  $``\bar{N}"(q)$ $\phi(q)$

complex scalar field $\psi(x), \psi^*(x)$ real scalar field $\phi^*(x) = \phi(x)$

The Yukawa interaction

nucleon anti-nucleon meson

$N(p, \tau)$ $\bar{N}(q, s)$ $\phi(q)$

spin spin

Dirac Field : $\psi(x)$

Dirac conjugation : $\bar{\psi}(x)$

$\phi^*(x) = \phi(x)$ (just complex conjugation)

Quantum Electrodynamics (QED)

electron positron photon

$e^-(p, \tau)$ $e^+(q, s)$ $\gamma(k, \varepsilon)$

Dirac field $\psi(x)$, $\bar{\psi}(x)$

maxwell field $(\vec{E}, \vec{B})$

gauge potential $A_\mu(x)$

Review:

$\overset{p+q}{\dashrightarrow}$ $\Leftrightarrow$ $\phi(p+q)$

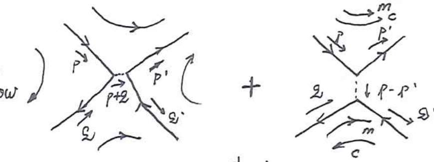
$\to$ momentum arrow (flow)

$\to$ charge arrow (flow) } same direction $\Leftrightarrow$ $``N(p)"$

$\overset{q}{\to}$ $\Leftrightarrow$ $``\bar{N}"(q)$

III. 1. The pseudo-Yukawa Interaction

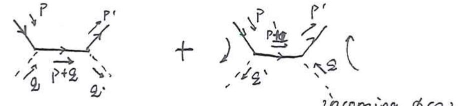
e.g. 4 $``N"(p)$ and $``\bar{N}"(q)$ scattering : $``N"(p) + ``\bar{N}"(q) \to ``N"(p') + ``\bar{N}"(q')$, $p+q = p'+q'$



{ one way : calculation $\xrightarrow{\text{derive}}$ FDs
  another way : intuition

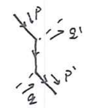
有向射分法 to charge flow 方向

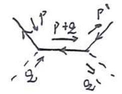
e.g.5 $``N"(p)$ and $\phi(q)$ scattering : $``N"(p) + \phi(q) \to ``N"(p') + \phi(q')$



e.g.6. $``\bar{N}"(p)$ and $\phi(q)$ scattering

why not:

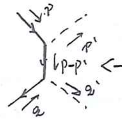




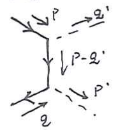
$+ (q \leftrightarrow q')$

e.g.7. $``N"(p)$ and $``\bar{N}"(q)$ annihilation : $``N"(p) + ``\bar{N}"(q) \to \phi(p') + \phi(q')$

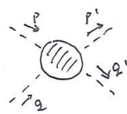
- particles are changed !
- That is QFT, with unfixed number of particles which can not be described using QM.



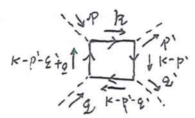
⟵ virtual "N" pseudo-nucleon +



e.g. 8 meson-meson scattering : $\phi(p) + \phi(q) \to \phi(p') + \phi(q')$

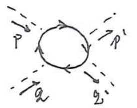


No tree FDs + one-loop FD (= one-box FD)



virtual pseudo-nucleon ("Nucleon") flow

⇕ topological

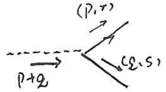


④

III. 2. The Yukawa Interaction

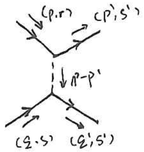
e.g.1. The Decay of meson : $\phi(p+q) \to$ "N"(p) + "$\bar{N}$"(q) "pseudo-nucleon"

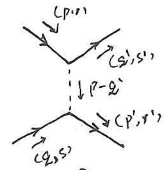
$\phi(p+q) \to N(p,r) + \bar{N}(q,s)$ nucleon



$r.s = \begin{cases} \frac{1}{2} & \text{spin-up} \\ -\frac{1}{2} & \text{spin-down} \end{cases}$

e.g.2 nucleon-nucleon interaction $N(p,r) + N(q,s) \to N(p',r') + N(q',s')$

1) 

\+ 2) 

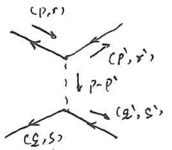
$\Updownarrow$

$(p',r') \leftrightarrow (q',s')$

The difference of 1) and 2):

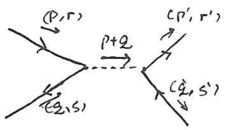
在计算时T最终结果完全不同

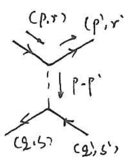
e.g.3 anti-nucleon anti-nucleon scattering $\bar{N}(p,r) + \bar{N}(q,s) \to \bar{N}(p',r') + \bar{N}(q',s')$



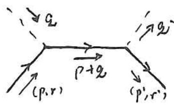
\+ $(p',r') \leftrightarrow (q',s')$

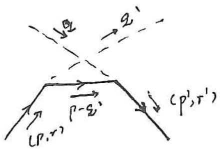
e.g.4 $N(p,r) + \bar{N}(q,s) \to N(p',r') + \bar{N}(q',s')$



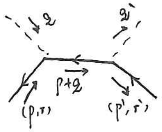
\+ 

e.g.5 $N(p,r) + \phi(q) \to N(p',r') + \phi(q')$



\+ 

e.g.6. $\bar{N}(p,r) + \phi(q) \to \bar{N}(p',r') + \phi(q')$



\+ $(q \leftrightarrow q')$

note: We can exchange momentum q, q'
or, equivalently, can change external line / \

$P_2 - P_1$

Introduction to FDs

1. Def of pQFT / FDs = ① = ②
   Non-p QFT = ①

Three model $\begin{cases} ① \text{ Pseudo Yukawa} \\ ② \text{ Yukawa} \\ ③ \text{ QED} \end{cases}$

Concept { mass, charge, field, spin — Fermion / Boson — Statistical theorem — indistinguished

virtual propagator — internal line; tree / loops; external line — incoming / outgoing, real particle; vertex — Black Box

Conserved — momentum (4) = $\gamma \to e^+e^-$ forbid; charge; angular (total)

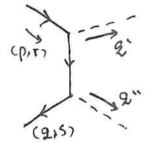
advanced concept {
particle, anti-particle - charge ...
(about Q=0 particle.)
part. — $\overrightarrow{p}$
anti-part — p, ε^μ mediator } line/vertex [illegible]
[illegible]
e.g. one loop
$\gamma+\gamma \to \gamma+\gamma$
$\gamma \to \gamma$
virtual particle ⇒ Causality violation
⇓
$P_\gamma^2 \neq 0$
}

CPT invariance

2. Pseudo-Yukawa Int. { "N" (ψ, ψ^*), "$\bar{N}$", ϕ

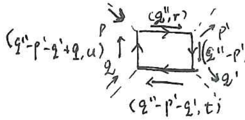
① Decay of ϕ
② "N" "N"
③ "$\bar{N}$" "$\bar{N}$"
④ "N" "$\bar{N}$"
⑤ "N" ϕ
⑥ "$\bar{N}$" ϕ
} Scattering
⑦ "N" "$\bar{N}$" anihilation
⑧ ϕ ϕ Scattering

e.g.7 $N(p,r) + \bar{N}(q,s) \to \phi(q') + \phi(q'')$



\+ $(q' \leftrightarrow q'')$

e.g.8 $\phi(p) + \phi(q) \to \phi(p') + \phi(q')$



3. Yukawa Int. $\begin{cases} N \\ \bar{N} \\ \phi \end{cases}$ } 自旋, Dirac场

{ ① Decay of ϕ
② N, N ③ $\bar{N}$, $\bar{N}$ ④ N, $\bar{N}$ ⑤ N ϕ ⑥ $\bar{N}$ ϕ Scattering
⑦ N $\bar{N}$ anihilation
⑧ ϕ ϕ Scattering

4. QED $\begin{cases} e^+, e^- > \text{自旋. Dirac场} \\ \gamma - \text{自旋. Maxwell场} \\ \quad P, \varepsilon^\mu \text{ (2 DOF)} \end{cases}$

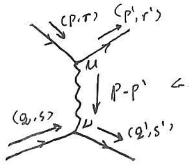
① e^-e^- (moller)
② e^+e^+
③ e^+e^- (Bhabha)
④ γe^- (Compton)
⑤ γe^+
} Scattering
⑥ $e^+e^- \to 2\gamma$ ⑦ $\gamma+\gamma \to \gamma+\gamma$
⑧ $\gamma \to \gamma$

Three parts of QFT $\begin{cases} \text{Classical} \to \text{QED} \\ \text{Standard model} \to \text{Yang-Mills} \\ \text{String} \end{cases}$

III. 3. Quantum Electrodynamics (QED)

e.g.1. photon decay (?)

e.g.2. Moller Scattering (Two-electron Scattering) $e^-(p,r) + e^-(q,s) \to e^-(p',r') + e^-(q',s')$

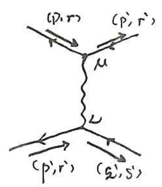


<-- two electrons interact by interchanging of a virtual photon

μ, ν: polarization

\+ $(p',r') \leftrightarrow (q',s')$

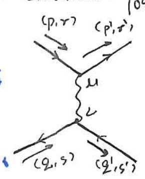
e.g.3. Two-Positron Scattering $e^+(p,r) + e^+(q,s) \to e^+(p',r') + e^+(q',s')$

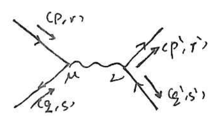


\+ $(p',r') \leftrightarrow (q',s')$

interchange momentum assignment
spin

e.g.4 electron - positron scattering (Bhabha Scattering) $e^-(p,r) + e^+(q,s) \to e^-(p',r') + e^+(q',s')$

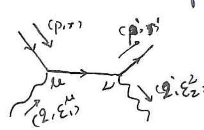


\+ 

two way to draw the charge flow

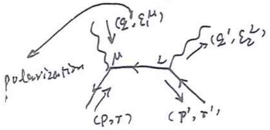
e.g.5. Compton Scattering (electron-photon scattering) $e^+(p,r) + \gamma(q,\varepsilon_1) \to e^-(p',r') + \gamma(q',\varepsilon_2)$

polarization (ε_1)



\+ $(q, \varepsilon_1^\mu) \leftrightarrow (q', \varepsilon_2^\nu)$

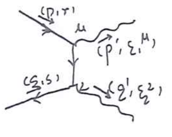
e.g.6 Positron - Photon Scattering (Compton Scattering) $e^+(p,r) + \gamma(q,\varepsilon_1) \to e^+(p',r') + \gamma(q',\varepsilon_2)$



$+\ (q,\varepsilon_1^\mu) \leftrightarrow (q',\varepsilon_2^\nu)$

e.g.7. Pair annihilation

$e^-(p,r) + e^+(q,s) \to \gamma(p',\varepsilon_1) + \gamma(q',\varepsilon_2)$



$+\ (p',\varepsilon_1) \leftrightarrow (q',\varepsilon_2)$

Lec IV Classical Field Theory　　　　20160927 Yong Zhang

Lec V Symmetries & Conservation laws

Recall Lec II & Lec III Feynman diagrams

II & III 1. perturbative QFT
2. pseudo - Yukawa interaction
3. Yukawa interaction
4. QED (Quantum electrodynamics)

- Nucleon, Electron are spin-½, Fermions — Indentical particles
  Meson (spinless), photon (spin-1) are Bosons

e.g.1. Decay of Boson into Fermion and Anti-Fermion
e.g.2. Fermion - Fermion Scattering
e.g.3. Anti-Fermion — Anti-Fermion scattering
e.g.4. Fermion — Anti-Fermion scattering
e.g.5. Fermion — Boson Scattering
e.g.6 Anti-Fermion — Boson scattering
e.g.7 Fermion — Anti-Fermion Annihilation
e.g.8 Boson and Boson scattering

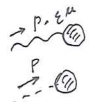
- Spin-statistics theorem :
  Interchange two Fermions : "−"
  Interchange two Bosons : "+"

Remark 1 : Charge Q, distinguish particle and anti-particle

$e^-: Q=-1$　　$e^+: Q=+1$　　| meson π^0 : $Q=0$
(same spin/mass)
proton : $Q=1$　　anti-proton : $Q=-1$　　| photon γ : $Q=0$
Neutron : $Q=0$　　anti-neutron : $Q=0$
(special example, how to distinguish them ?)

positive particle
$e^-: Q=-1$

anti-particle
$e^+: Q=+1$

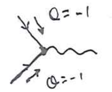


$\gamma: Q=0$
$\phi: Q=0$

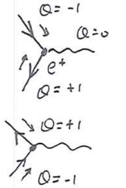
Feynman Diagram are able to represent particle and anti-particle

Remark 2: How to draw FDs → Charge Conservation in (and) FDs

At every interaction vertex, total charge $\Delta Q = 0$ (guide)

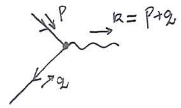


$\Delta Q = -2$　wrong diagram

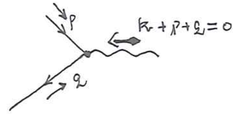


$\Delta Q = 0$
$\Delta Q = 0$　} right FDs

Remark 3 : energy - momentum conservation

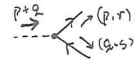


$\Delta p = 0$

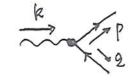


- FD内，p的方向并不重要，不同的方向将有不同的等式

$\phi(p+q) \to N(p,r) + \bar N(q,s)$ is allowed,　but $\gamma \to e^- + e^+$ isn't allowed



meson is massive



$k = p+q$
$k^2 \neq (p+q)^2$
$k^2 = 0$ (photon is massless)
$(p+q)^2 = p^2 + q^2 = m_{e^-}^2 + m_{e^+}^2 + \left(\frac{E_1E_2}{c^2} - \vec p_1\cdot\vec p_2\right)$

Remark 4 : Total angular - momentum conservation

$J = L + S$

$e^-(p,r) + e^-(q,s) \to e^-(p',r') + e^-(q',s')$　　$p+q = p'+q'$

$(L_r + r) + (L_s + s) = (L_{r'} + r') + (L_{s'} + s')$

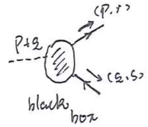
↑必须计及轨道，它的角动量守恒性

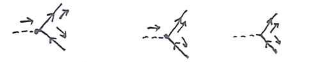
Remark 5 : How to draw FDs

$\phi(p+q) \to N(p,r) + \bar N(q,s)$

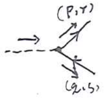
Step 1. draw incoming lines and outgoing lines

Step 2. draw interaction vertex





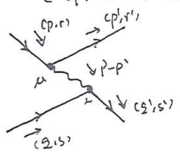
Step 3. Connect incoming / outgoing lines with vertices



(one lowest tree diagram)

Remark 6: Spin - statistics theorem and FDs

$e^-(p,r) + e^-(q,s) \to e^-(p',r') + e^-(q',s')$



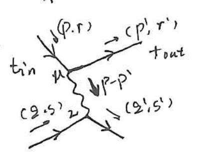
$\pm\ (p',r') \leftrightarrow (q',s')$
interchange two outgoing particles
"+" : FDs
"−" : spin-statistics theorem
Feynman integrals

⑥

Remark 7: Virtual particles and Causality

virtual particles can cause violation of causality, but who knows! (virtual particles cannot be observed)

e.g.

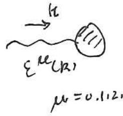


→ virtual particle (hidden nature)

→ real particle

$(p-p')^2 \neq 0 \Rightarrow$ virtual photon is massive, causality is broken down, we get interaction between past and future.

Remark 8: photon = quanta of EM field, photon only allows two polarization degrees of freedom.

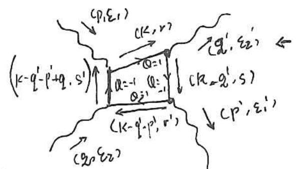


$\mu = 0,1,2,3$

In FDs, photon has 4 polarization degrees of freedom
we must need two constrains to kill two degrees of freedom.
(FDs 在这一点上不完善)

Remark 9: photo-photo scattering

$$\gamma(p,\varepsilon_1) + \gamma(q,\varepsilon_2) \to \gamma(p',\varepsilon_1') + \gamma(s',\varepsilon_2')$$



Remark 10: CPT invariance and FDs (need more knowledge)

IV 1. Four-vector calculation

① Three-vector Calculation $\vec{r} = (x,y,z)$, $\vec{p} = (p_x,p_y,p_z)$, $\vec{D} = (\partial_x,\partial_y,\partial_z)$

$\vec{r}^2 = \vec{r}\cdot\vec{r} = x^2+y^2+z^2$ $\quad \vec{p}^2 = p_x^2+p_y^2+p_z^2$ $\quad D^2 = \partial_x^2+\partial_y^2+\partial_z^2 = \Delta$

② Contra-variant 4-vector

$x^{\mu} = (ct, \vec{r})$ c: Light speed
$= (ct, x, y, z)$

$p^{\mu} \overset{①}{=} (E, \vec{p}c)$
$\overset{②}{=} (E, \vec{p}c)/c$
$\overset{③}{=} (c|E, p_x, p_y, p_z)$

$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

协变导数算符. 就是微分的算符

③ Covariant 4-vector

$x_{\mu} = (ct, -\vec{r})$
$= (ct, -x, -y, -z)$
(有的书上, x^{μ}是协变, x_{μ}是逆变)

$p_{\mu} \overset{①}{=} (E, \vec{p}c)$
$\overset{②}{=} (c|E, -\vec{p})$
$\overset{②}{=} (c|E, -p_x, -p_y, -p_z)$

$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$

Contra-variant

④ Minkowski metric $\eta = (\eta_{\mu\nu})$: associates contra-variant and covariant

Note: $(\eta_{\mu\nu})$ has physical meaning, it means flat space-time without gravity

$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, $\quad x_{\mu} = \eta_{\mu\nu} x^{\nu}$

Note: $\eta^{-1} = (\eta^{\mu\nu}) = \eta = (\eta_{\mu\nu})$, $\quad x^{\mu} = \eta^{\mu\nu} x_{\nu}$

Note: $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_{\mu}{}^{\sigma}$ [preferred] $\quad (\eta \cdot \eta^{-1} = \mathbb{1}_{4\times 4})$

4-momentum $p^{\mu} = (E, \vec{p}c)$
$p^{\mu} = (E/c, \vec{p})$

$x^{\mu} = (ct, \vec{x})$ $\quad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
$x^{\mu} = (\vec{x}, ct)$ $\quad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$
$x^{\mu} = (ict, \vec{x})$ $\quad \delta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$x^{\mu} = (\vec{x}, ict)$ $\quad \delta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Remark: Notation and Convention makes QFT difficult. (different people use different notation)

But spacetime interval is unchanged $c^2t^2 - x^2 - y^2 - z^2$, and mass-energy relation holds.

Note: $p_{\mu} = \eta_{\mu\nu} p^{\nu}$, $p^{\mu} = \eta^{\mu\nu} p_{\nu}$, $\partial_{\mu} = \eta_{\mu\nu}\partial^{\nu}$, $\partial^{\mu} = \eta^{\mu\nu}\partial_{\nu}$ (the same as x^{μ}, x_{ν})

⑤ Inner product (scalar product) and invariant in special relativity

space time interval $\quad x^2 = x^{\mu}x_{\mu} = \eta_{\mu\nu}x^{\mu}x^{\nu}$ (最重要的不变量) $= c^2t^2 - \vec{r}^2$

the definition of x^{μ} is flexible,
but all should make x^2 invariant.

mass-energy relation $\quad p^2 = p^{\mu}p_{\mu} = \eta_{\mu\nu}p^{\mu}p^{\nu} = (p^0)^2 - \vec{p}^2c^2 = E^2 - \vec{p}^2c^2 = (m_0c^2)^2$

simple but abstract

$E^2 = (m_0c^2)^2 + \vec{p}^2c^2$

$E = \sqrt{(m_0c^2)^2 + \vec{p}^2c^2}$ (the second type of invariant)

complex but physical

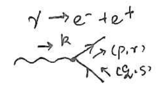
$= mc^2 = \gamma m_0 c^2 \approx m_0c^2 + T$

kenetic energy

$$\Delta E = \Delta m \cdot c^2$$

Remark 1. In high-energy physics, particles can be created or annihilated, number is unfixed $\Rightarrow$ QFT

Remark 2. $p^2 = (m_0c^2)^2$ $\quad \gamma \to e^- + e^+$



$k = p+q \Rightarrow k^2 = (p+q)^2$
$0 = 2m_{e^-}^2 + 2p\cdot q$

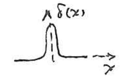
⑥ Fourier transform

$kx = k^0x^0 - \vec{k}\cdot\vec{x}$

def: $f(x) = \int \frac{d^4k}{(2\pi)^4} f(k)\, e^{ikx}$, $\quad f(k) = \int d^4x\, f(x) e^{-ik\cdot x}$ are (inverse) transform. in 4-vector language.

Dirac-delta function (def using integral)

1-dim $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$ $\qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$



4-dim $\delta^4(x) = \delta(t)\delta(x)\delta(y)\delta(z) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}$

Step function

$\theta(t) = \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$

there is natural relationship between $\delta(t)$ and $\theta(t)$
$\delta(t) = \frac{d\theta(t)}{dt}$

Kronnecker-delta function

$\delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$ $\qquad (\delta_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \mathbb{1}_4 = \mathbb{1}_{4\times 4}$

$\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_{\mu}{}^{\sigma}$

Note: not prefered $\delta_{\mu\nu}$,
but prefered $\delta^{\mu}{}_{\nu}$

Summary: Lec IV Remark on FDs
Four-vector Calculation

Lec V (Oct 4, vacation, missing)
Lec VI (Oct 11, classical field theory)
Lorentz transformation
Lorentz group and representation
Ref of classical field theory
Lagrangian & Hamiltonian
examples

Lec VII Symmetries and Conservation laws (important for FDs)
Space time translation invariance and energy-momentum conservation
Space time rotational invariance and angular-momentum conservation
CPT invariance

7

Lec VI Special Relativity

Lec VII Classical Field Theory

Recall what we have done

Topic 1. Conceptual development of QFT, QFT = QM + SR

Topic 2. Defs of QFT = Quantization of CFT = Quantum many-particle with particle number unfixed

Topic 3. pQFT = set of FDs

Topic 4. Example 1. pseudo-Yukawa : "N", "$\bar{N}$", ϕ

Tree FDs, one loop FDs.

Topic 5. Example 2. Yukawa : $N, \bar{N}, \phi$

Topic 6. Example 3. QED : e^-, e^+, γ

Topic 7. How to understand and draw FDs ?

energy-momentum Conservation

charge Conservation

Angular momentum Conservation

Causality, virtual particles

spin-statistical (tics) theorem

Topic 8. Four-vector calculation

P7–P8

Four vec. Calculation

1. Three vec $\vec{r}$ $\vec{\nabla}$, $\vec{r}^2$ $\vec{\nabla}^2 = \Delta$

2. Contra-variant 4-vec $\left\{ x^\mu, p^\mu, \partial^\mu \right.$ > $x^\mu = \eta^{\mu\nu} x_\nu$, $\partial^\mu \partial_\mu = \Box$

covariant 4-vec $\left\{ x_\mu, p_\mu, \partial_\mu = (\partial ct, \partial x, \partial y, \partial z) \right.$

metric $\eta^{\mu\nu} \xrightarrow{-1} \eta_{\mu\nu}$, $\delta^\mu{}_\nu = \eta^\mu{}_\nu$

inner product $x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = X \cdot \eta \cdot Y^T$

mass-energy relation $p \cdot p = m^2 c^4$ > 两种不变量

3. Fourier/Inverse-F transform $f(x) = \frac{1}{(2\pi)^4} \int e^{ik \cdot x} \tilde{f}(k) d^4k$, $\tilde{f}(k) = \int e^{-ik \cdot x} f(x) d^4x$

Dirac δ - $\int_{-\infty}^{\infty} \delta(x) dx = 1$ Step $\delta(t) = \frac{d\theta(t)}{dt}$ Kronecker-δ $\delta_{\mu\nu}$

Remark (about the teaching schedule/plan) : only ⅔ contents of online lecnotes will be taught this semester.

How to draw Feynman diagrams : FD = external lines (Incoming particles, Outgoing particles) + Vertices : Interaction

+ internal Lines :

propagation of a virtual particle(s)

called Feynman propagator

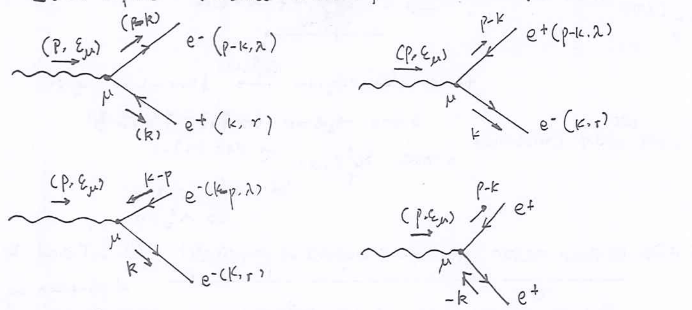
e.g. QED

External lines

photon

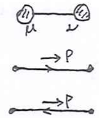
	electron	positron
Incoming	(p, λ) $\lambda = \pm\frac{1}{2}$	(p, r)
outgoing	(p, λ)	(p, r)

Vertices $\Delta Q = 0$ $\Delta p = 0$ the same vertex, four understandings

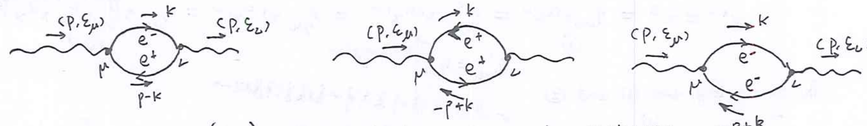


(electron, electron interactions) (positron-positron interactions)

Interlines : photon propagator, electron propagator, positron propagator



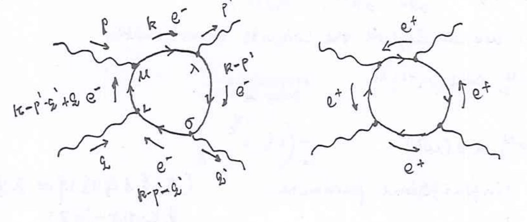
e.g.1 one-loop FDs for photon-polarization   the same loop, but three stories



In one loop, internal(lines) momentum k is unconstrained, $-\infty < k < +\infty$.

virtual particles : (e^-, e^+), (e^-, e^-), (e^+, e^+), can not be observed

e.g.2. Two photon-Scattering



$-\infty < k < +\infty$

the virtual particle

From the FDs, we can understand many many thing, the hidden nature.

IV & VI.1 : Recall Four-vector Calculation

- In SR, space time interval ΔS^2 is invariant under Lorentz transform.

$$\Delta S^2 = c^2t^2 - x^2 - y^2 - z^2 = (ct, x, y, z)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad x^\mu = (ct, x, y, z),\ \eta = \mathrm{diag}(1,-1,-1,-1) \quad x_\mu = \eta_{\mu\nu} x^\nu$$

$= \eta_{\mu\nu} x^\mu x^\nu$ (tensor formulation)

$X = (x^\mu)$, X^T ⟹ $\Delta S^2 = X \eta X^T$ (matrix formulation)

mathematics makes it simple and beautiful

- $\Delta S^2 = -(x, y, z, ct)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$ is another Convention, $\underbrace{(x, y, z, ct)}_{x^\mu}$, $\eta' = -\eta$

$= -X \eta' X^T$

(key point)

but space-time interval is unchanged, but four vector, metric could be changed

$\Delta S^2 = -(x, y, z, ict)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} = -\tilde{X} \delta \tilde{X}^T$ is another convention

- In SR, rest-mass, m_0 (mass-energy relation), is invariant under LT. (Lorentz transform)

$E^2 = \vec{p}^2 c^2 + (m_0 c^2)^2$

$E^2 - \vec{p}^2 c^2 = (m_0 c^2)^2$

$$(E, p_x c, p_y c, p_z c)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = (m_0 c^2)^2, \quad p^\mu = (E, \vec{p}c)$$

$P = (p^\mu)$, $P \eta P^T = (m_0 c^2)^2 = p^\mu p^\nu \eta_{\mu\nu}$

- $(p^\mu) = (\frac{E}{c}, p_x, p_y, p_z) = \tilde{P}$, $\tilde{P} \eta \tilde{P}^T = (m_0 c)^2$, but m_0 is invariant.

⑧ $x^2,\ p^2=(m_0c^2)^2,\ \partial_\mu\partial^\mu = \frac{\partial^2}{c^2\partial t^2} - \nabla^2 := \Box$ is invariant

P8-P9
Lorentz transformation (Def, Constrain, Type) 保度规 10个 $\Lambda\cdot\Lambda^{-1}$ $\begin{cases}\text{boost - 3 param.}\\ \text{rotation - 3 param.}\end{cases}$
Infinitesimal LT (Def, Thm $\begin{cases}\omega_{\mu\nu} = -\omega_{\nu\mu}\\ \omega^i{}_0 = \omega^0{}_i,\ \omega^j{}_i = -\omega^i{}_j\end{cases}$) Inverse
LG $SO(1,3), SO_+(1,3), SO^\uparrow_+(1,3)$ (是否有问题?) 保持手征 保因果律

IV & VI . II . Lorentz transformation (LT)

Def: LT: $\Lambda = (\Lambda^\mu{}_\nu)$ is define as 4x4 matrix making $x^2, p^2, \partial_\mu\partial^\mu$ invariant under $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ (proper det=1; proper-orthochronous det=1, $\Lambda^0{}_0\geq 1$)

$$x'^2 = x^2 = \underset{①}{\eta_{\mu\nu}x^\mu x^\nu} = \eta'_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}x'^\mu x'^\nu = \underset{②}{\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta}$$

$\eta'_{\mu\nu} = \eta_{\mu\nu}$ ← assumption

By comparing the ① and ② ← 说明坐标变换时各个时空坐标是平直的

• 重要结论 $\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$ (1) $\Rightarrow$ $x'^2 = x^2,\ p'^2 = p^2$, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\eta'_{\mu\nu} = \eta_{\mu\nu}$ $\left[\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}\right]$

• $\Lambda^\mu{}_\nu$ 可以是反演, 转动, 伪转 不一定是连续

Inverse of LT: $(\Lambda^{-1})^\rho{}_\sigma = \frac{\partial x^\rho}{\partial x'^\sigma}$

check: $\Lambda^\mu{}_\nu(\Lambda^{-1})^\nu{}_\sigma = \frac{\partial x'^\mu}{\partial x^\nu}\cdot\frac{\partial x^\nu}{\partial x'^\sigma} = \frac{\partial x'^\mu}{\partial x'^\sigma} = \delta^\mu{}_\sigma$

Indeed, we can defined the Lorentz transformation

Infinitesimal transform (Expansion) ation of $\Lambda^\mu{}_\nu$ and $(\Lambda^{-1})^\mu{}_\nu$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + o(\omega^2)$$

↳ infinitesimal parameter

boost along x may be $-\begin{pmatrix} 0 & +\frac{v}{c} & & \\ +\frac{v}{c} & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}$

(当涉及多个坐标变换时, 总是用无穷小变换, 且只讨论一阶量)

$$(\Lambda^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \omega^\mu{}_\nu + o(\omega^2)$$

check1: $\Lambda^\mu{}_\nu(\Lambda^{-1})^\nu{}_\sigma = \delta^\mu{}_\sigma + O(\omega^2)$

check2: $\omega_{\mu\nu} = -\omega_{\nu\mu}$ Anti-symetric tensor (very strong constrain on LT)

• $\eta_{\mu\nu}(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha)(\delta^\nu{}_\beta + \omega^\nu{}_\beta) = \eta_{\alpha\beta}$

$\parallel$

$\eta_{\mu\nu}\delta^\mu{}_\alpha\delta^\nu{}_\beta + \eta_{\mu\nu}\omega^\mu{}_\alpha\delta^\nu{}_\beta + \eta_{\mu\nu}\delta^\mu{}_\alpha\omega^\nu{}_\beta = \eta_{\alpha\beta}$

$\parallel$

$\eta_{\alpha\beta} + \eta_{\mu\beta}\omega^\mu{}_\alpha + \eta_{\alpha\nu}\omega^\nu{}_\beta = \eta_{\alpha\beta}$

$\Rightarrow$ $\omega_{\beta\alpha} + \omega_{\alpha\beta} = 0$

check3: $\omega_{\mu\nu} = -\omega_{\nu\mu}$ ← rank-2 Covariant tensor

$\omega^0{}_i = \omega^i{}_0$, $\omega^i{}_j = -\omega^j{}_i$ ← rank (1,1) tensor

• $\omega_{0i} = -\omega_{i0}$, $\omega^0{}_i = \eta^{00}\omega_{0i} \Rightarrow \omega^i{}_0 = \eta^{ii}\omega_{i0} = -\omega_{i0} \Rightarrow \omega^0{}_i = \omega^i{}_0$
$= -\omega_{i0}$

$\omega_{ij} = -\omega_{ji}$, $\left.\begin{array}{l}\omega^i{}_j = \eta^{ii}\omega_{ij} = -\omega_{ij}\\ \omega^j{}_i = \eta^{jj}\omega_{ji} = -\omega_{ji}\end{array}\right\rangle$ $\omega^i{}_j = -\omega^j{}_i$

e.g.1. Lorentz boost along x-direction

ct, x $\xrightarrow{v_x}$ ct', x'

$$\begin{cases} ct' = \gamma(ct - \beta x)\\ x' = \gamma(-\beta ct + x)\\ y' = y\\ z' = z\end{cases}\qquad \beta = \frac{v_x}{c},\quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$(\Lambda^\mu{}_\nu)_{v_x} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & 0\\ & & 0 & 1\end{pmatrix} \underset{v_x\ \text{small}}{=\!=\!=} \delta^\mu{}_\nu + \begin{pmatrix} 0 & -\beta & & \\ -\beta & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix} \rightsquigarrow \begin{array}{l}\omega_{01} = -\omega_{10}\\ \omega^0{}_1 = \omega^1{}_0\end{array}$$

e.g.2. Lorentz rotation along z-axis about θ

$$\begin{cases} ct' = ct\\ x' = x\cos\theta - y\sin\theta\\ y' = x\sin\theta + y\cos\theta\\ z' = z\end{cases}\qquad (\Lambda^\mu{}_\nu)_{R_z} = \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1\end{pmatrix}$$

$$= \delta^\mu{}_\nu + \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0\end{pmatrix} \rightsquigarrow \begin{array}{l}\omega_{12} = -\omega_{21}\\ \omega^1{}_2 = -\omega^2{}_1\end{array}$$

don't write $\Lambda_{12} = -\Lambda_{21}$
$\Lambda^1{}_2 = -\Lambda^2{}_1$

Def:
Lorentz group = the set of all Lorentz transformations $(\Lambda^\mu{}_\nu)$

Prove that "Lorentz group" is actually a group $\begin{cases}\text{Inverse}\\ \text{identity}\\ \text{closure (for multiplication)}\\ \text{associativity}\end{cases}$

• which means $LG = \{\Lambda^\mu{}_\nu : \Lambda^T\eta\Lambda = \eta\}$
手征, 伪转动, 反演.

this is a very big group. we always talk about its subgroup

Subgroup of Lorentz group $SO(1,3)$ (special, orthogonal, det = ±1; 1 time, 3 space)

① $SO_+(1,3) \subset SO(1,3)$: $\det(\Lambda) = 1$
↳ proper Lorentz group

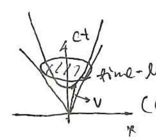
• $d^4x = dt\,dx\,dy\,dz$ is invariant under $SO_+(1,3)$
4-d infinitesimal space-time volume
it is crucial (Jacobi determinant)

$$dt'dx'dy'dz' = \underbrace{\det(\Lambda)}_{\text{Jacobi determinant}}dt\,dx\,dy\,dz = 1\,dt\,dx\,dy\,dz$$

• group theory is crucial for modern QFT

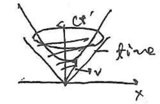
• $SO_+(1,3)$ can include 反演. 但至少是两个反演轴
$SO^\uparrow_+(1,3)$ 不同: 不允许反演, 只能在空间 如反演(至少2个)
↑ 无时间反演, 结果是因果律保持

② Subgroup of $SO_+(1,3)$ with $\Lambda^0{}_0 \geq 0$ (≥ 1) : $SO^\uparrow_+(1,3)$
called ortho-chronus - proper LG
正时向的正LG.



time-like region (crucial for causility)

$\xrightarrow{\Lambda}$



time-like

time-like region $\xrightarrow{SO^\uparrow_+(1,3)}$ time-like region.

In physics, we are only concerned about $SO^\uparrow_+(1,3)$ $\begin{array}{l}\to \det(\Lambda) = 1\\ \to t' > 0 \to t > 0\end{array}$ $\Delta t > 0 \to \Delta t' > 0$ (因果律不遭到破坏)
$\Leftrightarrow \Lambda^0{}_0 > 0$

Oct. 11.
Today's review:

① How to draw FDs: ② Four vector (from the viewpoint of invariant) ③ LT and Inverse-LT expansion $\omega_{\mu\nu} = -\omega_{\nu\mu}$

④ LG $SO(1,3)$, $SO_+(1,3)$, $SO^\uparrow_+(1,3)$

Oct. 18. ① Representation of LG. $SO^\uparrow_+(1,3)$
② Def of Classical Field Theory
③ Lagrangian / Hamiltonian Formulation of CFT
④ Example for CFT: Dirac bispinor field; Electrodynamics; Real Scalar field theory (no charge) → Higgs; Complex Scalar field theory (Charge)

(9)

Lec VII Classical Field Theory

Lec VIII Symmetries & Conservation laws

20161018

Homework 1st, Due Nov. 8

(Three weeks for one problem)

The updated version of Luke's problem set #1, problem 6 (Instructor's home page)

problem 6: Consider the Lagrangian for a real vector field $A^{\mu}(x)$

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\alpha A_\beta(x)\right]\left[\partial^\alpha A^\beta(x)\right] + \frac{1}{2}\left[\partial_\alpha A^\alpha(x)\right]\left[\partial_\beta A^\beta(x)\right] + \frac{1}{2}\mu^2 A_\alpha(x) A^\alpha(x)$$

质量项，μ=0时，则是光子

(a) Derive the equation of motion (Euler-Lagrange equation)

$$\left[\eta_{\alpha\beta}(\Box + \mu^2) - \partial_\alpha \partial_\beta\right] A^\beta(x) = 0$$

← check text books about ED (at least 3)

and verify the Lorentz condition $\partial_\alpha A^\alpha(x) = 0$

(b) $\mu^2 = 0$, $A^\mu = (\varphi, \vec{A})$, with $\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$, Derive Maxwell equations. (from the Lagrange formulation)

(c) With and Without Noether's theorem, derive energy-momentum tensor $T_{\mu\nu}$ for $\mu^2 = 0$.

(just by Maxwell eq → without; by Noether's theorem → with) (Noether's law = Conservation law)

(d) (optional problem) With and Without Noether's thm, Derive angular-momentum tensor $M_{\mu\nu\rho}$ for $\mu^2 = 0$.

Recall Lorentz transf. and groups.

$\det = \pm 1$, 转动 ↑   Eq

$|\Lambda^0{}_0| \geq 1$

{ 转动 $\Lambda^0{}_0 = 1$ ($\Lambda^0{}_k = \Lambda^k{}_0 = 0$); boost $\Lambda^0{}_0 > 1$ }

Lorentz Group $SO(1,3)$ $SO_+(1,3)$ $SO_+^{\uparrow}(1,3)$

LG 的表示 { vector, Weyl Spinor, Dirac bispinor, scalar } ↔ 不一样一套旋律，不同的粒子 $\approx SU(2) \otimes SU(2)$ $|j_1 m_1\rangle$ $|j_2 m_2\rangle$ (j_1, j_2)

In special Relativity, Space-time interval is Lorentz invariant $\Delta s^2 = c^2t^2 - x^2 - y^2 - z^2$. (physics)

Formulate it into matrix formulation $\Delta s^2 = (ct, x, y, z)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \vec{x}^T \eta \vec{x}$, $\vec{x} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

Lorentz transf. $\vec{x}' = \Lambda \vec{x}$, $\vec{x}'^T \eta \vec{x}' = \vec{x}^T \eta \vec{x}$

$\downarrow$

$\vec{x}^T \Lambda^T \eta \Lambda \vec{x} = \vec{x}^T \eta \vec{x}$

$\downarrow$ 一般来说 $\vec{x}^T A \vec{x} = 0 \Rightarrow A$ 是反对称的，但这里 $A = \Lambda^T\eta\Lambda - \eta$ 仍为对称的，因而 $A = 0$

$\Lambda^T \eta \Lambda = \eta$ (Four vector formulation $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$)

① $\det(\Lambda^T \eta \Lambda) = \det(\eta) = -1$

$\parallel$

$\det(\Lambda^T)\det(\eta)\det(\Lambda)$ ← $\det(\Lambda^T) = \det(\Lambda)$

$\downarrow$

$\det(\Lambda^T) = \det(\Lambda) = \pm 1$

we want the 4-dim Volume $d^4x' = cdt'dx'dy'dz' = \det(\Lambda)\, c\,dt\,dx\,dy\,dz$ invariant $\Rightarrow \det(\Lambda) = 1$

• 不含一次反演

$SO_+(1,3)$

(验证其是否群，只是没验证来说明封闭性)

② $\Lambda^T \eta \Lambda = \eta$ matrix eq ⇕ component eq

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}, \quad \Lambda^T = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^1{}_0 & \Lambda^2{}_0 & \Lambda^3{}_0 \\ \Lambda^0{}_1 & \Lambda^1{}_1 & \Lambda^2{}_1 & \Lambda^3{}_1 \\ \Lambda^0{}_2 & \Lambda^1{}_2 & \Lambda^2{}_2 & \Lambda^3{}_2 \\ \Lambda^0{}_3 & \Lambda^1{}_3 & \Lambda^2{}_3 & \Lambda^3{}_3 \end{pmatrix}$$

$(\Lambda^0{}_0)^2 = \sum_{i=1}^{3}(\Lambda^i{}_0)^2 + 1$ #1

$(\Lambda^0{}_j)^2 = \sum_{i=1}^{3}(\Lambda^i{}_j)^2 - 1$ #3 $(j = 1,2,3)$

$\Lambda^0{}_k \Lambda^0{}_j = \sum_{i=1}^{3}\Lambda^i{}_k \Lambda^i{}_j$ #6/2 = #3 $(k \neq j)$ $(k, j = 1,2,3)$

$\Lambda^0{}_0 \Lambda^0{}_j = \sum_i \Lambda^i{}_0 \Lambda^i{}_j$ #3

total number of Constraints # = 1+3+6 = 10, LT has 16 Components/parameters.

we have 6 free parameters

3 free ⇒ Lorentz boost v_x, v_y, v_z

3 free ⇒ Lorentz rotation x, y, z-axis

• (实际的洛伦兹变换应该还包含平移)

• rotation 和 boost，再加上反射就是所有的 Lorentz Transf.

$(\Lambda^0{}_0)^2 \geq 1 \Rightarrow \Lambda^0{}_0 \geq +1$, $\Lambda^0{}_0 \leq -1$

In physical treatment, we choose $\Lambda^0{}_0 \geq 1$

$\begin{cases} \Lambda^0{}_0 \geq +1 \\ \det(\Lambda) = 1 \end{cases}$: $SO_+^{\uparrow}(1,3)$ proper ortho-chronus LT (LG)

e.g. 1. Lorentz boost along x-direction $\begin{cases} ct' = \gamma ct - \beta\gamma x \\ x' = -\beta\gamma \cdot ct + \gamma x \\ y' = y \\ z' = z \end{cases}$ $\Lambda^0{}_0 = \gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1$

To derive Angular-momentum tensor, we must use LG (angular-momentum conservation)

eg 2. Lorentz rotation around z-axis $\begin{cases} ct' = ct \\ x' = \cos\theta\, x + \sin\theta\, y \\ y' = -\sin\theta\, x + \cos\theta\, y \\ z' = z \end{cases}$ ⇒ $\Lambda^0{}_0 = 1$ (rotation 的特点)

Thm 2. The representation theory of $SO_+^{\uparrow}(1,3)$

$SO_+^{\uparrow}(1,3) \approx SU(2) \otimes SU(2)$

locally isomorphic to

The representation of $SU(2)$

$|j, m\rangle$ j_z: third component of $\vec{J}$; j^2: total angular momentum

note: $SU(2) \approx SO(3)$

note: 表示空间一致，并不代表表示矩阵一致，如 $SO_+^{\uparrow}(1,3)$ 和 $SU(2) \otimes SU(2)$

$SO_+^{\uparrow}(1,3) \approx SU(2) \otimes SU(2)$

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ (LG is just angular momentum)

$j_1\, j_2$ integers or half-integers

The representation of $SO_+^{\uparrow}(1,3)$ is denoted by (j_1, j_2)

e.g.

representation $(j_1\, j_2)$	name	examples	
(1,0) or (0,1)	vector	$x_\mu, \partial_\mu, P_\mu$	(vector is well-defined name $x' = \Lambda x$ 只有这种表示叫 vector)
$(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$	Weyl spinor (Majorana spinor)		(neutrino is Weyl spinor)
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	Dirac-bi-spinor	$\psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$, $\psi'_D = S(\Lambda)\psi_D$, $S(\Lambda) \neq \Lambda$; $S(\Lambda)$ trasf. electron, Λ transf. Vector	(electron)
(0,0)	Scalar	x^2, p^2, m_0^2 → scalar; $\Box = \partial_\mu \partial^\mu$, d^4x; $\psi'_D = S(\Lambda)\psi_D$, $\bar{\psi}'_D = \bar{\psi}_D S^\dagger(\Lambda)$ (Dirac conjugation); $\bar{\psi}_D \psi_D$ → scalar	(Higgs, meson)

(1,0), (0,1) > massive spin-1 $W^\pm$ ← Three polarization

$(\frac{1}{2}, \frac{1}{2})$ massless spin-1 γ → 2 polarization

$(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ Weyl/majorana spinor ν

$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ Dirac bispinor   e, nucleons

(0,0) Spinless particle ⊕ (0,0)...   φ meson, Higgs, pseudo-nucleon

⑩

VII. Classical Field Theory

· P10 - P11
1. Definition { Classical Field ①② / Classical local Field ③④ } (and example)

Classification of Classical Field { scalar / vector / Dirac-bispinor } (→ 不同的变换律 / 对应不同的表示 of LG / 代表不同的粒子) ~ translation 不能分类粒子

2. Lagrangian Formulation { Fields / Independent DOF / c. L. S (Lorentz invariant) }

· Classical Field = function of $t, \vec{x}$

$\delta S=0$ Action principle ~ Euler Lagrangian eq. { CFT - pick up path / QFT - every path has contribution; EOM is assumption }

3. Hamiltonian Formulation { π_a / H / EOM { Canonical EOM / Poisson Bracket EOM } }

VII.1. Definitions

Def 1 A classical field is a physical object (quantity) described by a function of space-time.

e.g.1. $\vec{E}(t,\vec{x})$, $\vec{B}(t,\vec{x})$

Def 2 A classical field describes a physical object with infinity number of DoF (Degrees of freedom).

where space are just labels instead of observables in QM.

e.g.2. $\vec{E}_{\vec{x}}(t) = \vec{E}(t,\vec{x})$, $\vec{B}_{\vec{x}}(t)$, $\vec{x}\in\mathbb{R}^3$; but in QM, $\vec{x}\to\hat{\vec{x}}$

· right-formulation

here $\vec{x}$ just labels, not observables

· QM.

· QFT and CFT

e.g.3. $N_0\,\vec{x}$, $q(t)=\phi(t)$ → particle mechanics (finite number of freedom degrees)

Def 3 A classical Local Field is a physical Quantity whose dynamics at a space-time point is determined by field around such space-time point, namely, is described by PDE.

e.g. Maxwell eqs. are PDEs, that means ED is local field theory.

Def 4 A classical local field represents a non-instantaneous (非瞬时) interaction between particles at different space-time points.

e.g. Q source charge, $\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$; q test charge; $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}$

Definition of Classical (local) Field theory ↑

VII.2. Classification of classical fields

Classical fields	representation of LG	transformation law	example
real scalar fields: $\phi(x)$ complex scalar fields: $\psi(x)$	$(0,0)$ ↑ spin-less	$x\to x'$ $\phi(x)\to\phi'(x')=\phi(x)$ $\psi(x)\to\psi'(x')=\psi(x)$	meson(s) pseudo-nucleon(s) # Higgs
vector maxwell fields $A_\mu(x)$ (gauge potential)	$(1,0)$ or $(0,1)$ ↑ spin 1 粒子	$x\to x'=\Lambda x$ $A_\mu(x)\to A'_\mu(x')=\Lambda_\mu{}^\nu A_\nu(x)$ $\{g(\Lambda^{-1}\cdot T)\}_\mu{}^\nu$	photon(s)
Dirac bi-spinor fields $\psi_D(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$ (just column matrix)	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ ↑ spin $\frac{1}{2}$ 粒子的描述	$x\to x'=\Lambda x$ $\psi_D(x)\to\psi'_D(x')=S(\Lambda)\psi_D(x)$ $S(\Lambda)\neq\Lambda$	electron(s) and nucleon(s)

→ 粒子的变换律不同

Remark: What about space-time translation group $x\to x'=x+\varepsilon$ (infinitesimal translation)

$$\begin{cases}\phi(x)\to\phi'(x')=\phi(x)\\ \psi(x)\to\psi'(x')=\psi(x)\\ A_\mu(x)\to A'_\mu(x')=A_\mu(x)\\ \psi_D(x)\to\psi'_D(x')=\psi_D(x)\end{cases}$$

· 平移变换时，各种粒子(场)有相同的变换律，无法用来分类

{ Space-time translation ⟷ Energy-momentum tensor / Lorentz boost ⟷ angular-momentum tensor }

· How to derive FDs is not as easy as draw them.

VIII.3 Lagrangian and Hamiltonian formulations

* Fields: (In general case) $\phi^a(t,\vec{x}) = \phi^a_{\vec{x}}(t)$: $\mathbb{R}^1_{\text{time}}\times\mathbb{R}^3\to\mathbb{R}^n$ or $\mathbb{C}^n$

$a=1,2,\dots,N$ are components

N-component field: { ψ_D: 4-component / $A_\mu(x)$: 4-component / ϕ: 1-component (real) / ψ: 1-component (complex) }

· 经典力学：连续介质的力学
电动力学：连续介质(电磁场) → Lagrangian and Hamiltonian 表述

· Remark: particle mechanics = No spatial label $q^a(t)=\phi^a(t)$

{ field → general object / particle → special object }

* States (Independent DoF) ← why not $\phi^a(t,\vec{x})$

$\phi^a(t,\vec{x})$, $\partial_\mu\phi^a(t,\vec{x})$ (like q^a $\dot{q}^a$)

* Lagrangian density ($\neq$ Lagrangian): $\mathcal{L}=\mathcal{L}(\phi^a,\partial_\mu\phi^a)$

Lagrangian: $L=\int d^3x\,\mathcal{L}(\phi^a,\partial_\mu\phi^a)$

Note: $L=\int d^3x\,d^3y\,\mathcal{L}(\phi^a_x\phi^a_y)$ → non-local theories

· 个人觉得，L 能写成 $\mathcal{L}$ 积分，已经表明场的局域性

* Action: $S=\int_{t_1}^{t_2}dt\,L=\int d^4x\,\mathcal{L}(\phi^a,\partial_\mu\phi^a)$

Remark: QFT is QM+SR

Action is LT invariant, because $d^4x'=d^4x$, $\mathcal{L}'(\phi',\partial'_\mu\phi')=\mathcal{L}(\phi,\partial_\mu\phi)$

Action is translation-invariant $d^4x'=d^4x$

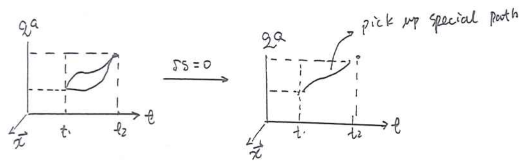
Action is everything.

* Action Principle :

particle mechanics $\xrightarrow{\text{generalization}}$	Field theory
$q^a(t)\to q^a+\delta q^a$ Arbitrary infinitesimal transf.	$\phi^a_{\vec{x}}(t)\to\phi^a_{\vec{x}}(t)+\delta\phi^a_{\vec{x}}(t)$ ← Arbitrary infinitesimal fluctuation
Time boundary is fixed Stationary Action $\delta S=0$	Time boundary is fixed $\delta S=0$

⑪

Geometric picture (of Action principle)



* Euler-Lagrangian eq.

$S = \int L(q, \dot{q})\, dt$

$\delta S = 0 = \int dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right)$

$S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$

$\delta S = 0 = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta(\partial_\mu\phi^a) \right\}$

Suppose: $\delta(\partial_\mu\phi) = \partial_\mu\delta\phi$ is usually right

$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta(\partial_\mu\phi^a) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\right)\delta\phi^a$

$\delta S = 0 = \int d^4x \left\langle \left\{ \frac{\partial \mathcal{L}}{\partial\phi^a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\right) \right\}\delta\phi^a + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\right] \right\rangle$

with four-dim Gaussian theorem, total derivative is replaced by the corresponding boundary term, which is supposed to vanish (assumption)

EOM (Euler-Lagrangian eq.)

$\frac{\partial \mathcal{L}}{\partial\phi^a} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\right)$

• attention:

Remarks: EOM in particle mechanics $\frac{\partial L}{\partial q^a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)$

(1st homework (a) use the same logical)

* Hamiltonian Formulation $\mathcal{H}$ (very important, first component of $T_{\mu\nu}$)

Conjugate momentum (Canonical momentum) $p^a = \frac{\partial L}{\partial \dot{q}^a} \rightarrow \pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a}$

$H = p_a\dot{q}^a - L \rightarrow \mathcal{H} = \pi^a\dot{\phi}_a - \mathcal{L} = \mathcal{H}(\phi_a, \pi_a)$

EOM (Hamilton's equation)

$\dot{p}^a = -\frac{\partial H}{\partial q^a} \qquad \rightarrow \qquad \dot{\pi}^a = -\frac{\partial \mathcal{H}}{\partial \phi^a}$

$\dot{q}^a = \frac{\partial H}{\partial p^a} \qquad \rightarrow \qquad \dot{\phi}^a = \frac{\partial \mathcal{H}}{\partial \pi^a}$

(1st homework (b))

Note: $\mathcal{H} = \mathcal{H}(\phi_a, \pi_a, \partial_k\phi_a)$

$\begin{cases} \dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi_a} \\ \dot{\pi}_a = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \partial_k\frac{\partial \mathcal{H}}{\partial(\partial_k\phi_a)} \end{cases}$

Today: Homework 1st, Due Nov. 8 a) b) c) d) optional

Lorentz transf., LG. (by matrix $\Lambda^T\eta\Lambda = \eta$; $\det(\Lambda) = \pm 1$, $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$ → $\det(\Lambda) = 1$, $\Lambda^0{}_0 \geq 1$ in physics)

The definition of Classical Fields (4 dof = 3 math + 1 physics)

Classical Fields' classification

$\mathcal{L}$. $\mathcal{H}$. EOM.

Next Tuesday: Oct. 25 * Examples for the CFT: real scalar, complex scalar, vector (Electrodynamics), Dirac bispinor dynamics

* Space-time translation invariant with conservation of energy-momentum tensor.

Lec. VIII, Classical Field Theory

Lec IX, Symmetry & Conservation Law

2016 10 25

Remark (about Homework 1st): due Nov. 8.

Doing Homework is doing research. Doing research is doing person.

- Focus on a small problem. ----> perspective
- Solve the problem Completely ----> persistence
- write down the solution step by step ----> patience
- Quote your reference carefully ----> power (honesty)

Recall $SO^\uparrow_+(1,3)$, $SO(4)$, $SO(3)$, $SU(2)$

$SO(4)$ = Real Special Orthogonal 4×4 matrix, $\det(R) = 1$, $RR^T = I_4$

$SO^\uparrow_+(1,3)$: $\Lambda^T\eta\Lambda = \eta$, $\det(\Lambda) = 1$, $\Lambda^0{}_0 \geq 1$ ($\det\eta = -1$)

$\Lambda^0{}_0 = 1$. $ct = ct'$ ⇒ Lorentz rotation $SO^\uparrow_+(1,3)$ → $SO(3)$ Spatial rotation } three Euler angles, three real param.; $SO(3) \simeq SU(2)$ isomorphic (locally)

$SO^\uparrow_+(1,3) \simeq SU(2) \otimes SU(2)$, 6 real param. = 3 + 3

Example of Classical Field

① Real Scalar Field / Def: $\mathbb{R}$ — Quant → meson

$\begin{cases} \mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \\ \text{EOM}: (\Box + m^2)\phi = -\frac{\partial V}{\partial\phi} \xrightarrow{V=0} \text{K-G eq.} \\ \pi_\phi = \dot{\phi},\ \mathcal{H},\ P_i \end{cases}$

② Complex Scalar Field / Def: $\mathbb{C}$ — Quant → Higgs, pseudo-nuc

$\begin{cases} \mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi - V(\psi^\dagger\psi) \\ \text{EOM}: (\Box + m^2)\psi = -\frac{\partial V}{\partial\psi^\dagger}\ \&\ \text{c.c.} \\ \pi_{\psi^\dagger} = \dot{\psi},\ \pi_\psi = \dot{\psi}^\dagger,\ \mathcal{H},\ P_i \end{cases}$

⟺ 2 real Scalar Field ϕ_1, ϕ_2

③ Dirac Bispinor Field / Def: $\mathbb{C}^4$, $D(\Lambda)$ — Quant → e, p, n, spin-½

+ Algebra: $\{\alpha_i\} = \beta_i$, $\gamma^0 = \beta$

$\mathcal{L} = \bar{\psi}_D(i\gamma^\mu\partial_\mu - m_e)\psi_D$

EOM: $(i\gamma^\mu\partial_\mu - m_e)\psi_D = 0$ → $i\partial_t\psi_D = (\vec{\alpha}\cdot\vec{p} + m\beta)\psi_D$ Dirac eq.

$\pi_{\bar{\psi}_D} = 0$, $\pi_{\psi_D} = i\psi_D^\dagger$, $\mathcal{H}$

• $SO^\uparrow_+(1,3)$

④ Maxwell Vector Field / Def: $\mathbb{C}^4$

$\vec{E}, \vec{B}$ ⟺ $\phi, \vec{A}$ ⟺ $A^\mu, F^{\mu\nu}$ (gauge transf., gauge fixing)

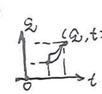
$\mathcal{L} = -\frac{1}{4}F^2$

EOM ⇒ $\begin{cases} \partial_{[\mu}F_{\alpha\beta]} = 0 \\ \partial_\mu F^{\mu\nu} = 0 \end{cases}$ ⟺ maxwell eq.

Recall Action principle

Classical Action principle (field or particle) picks up a specified path by EOM.

particle mechanics



In QM (QFT), every path has a contribution (path Integral) ← no EOM.

In QM, Schrödinger Equation (or Heisenberg Eq.) is a assumption instead of an Equation derived from the fundamental principle.

In QFT, EOM by Canonical quantization is assumption.

Question: What about Quantum Action Principle. (Research Project).

VIII. Classical Field Theory (Example for Classical field)

VIII.1 Real Scalar Field Theory

Note: Quantum Real Scalar Field Theory (suitable for describing meson)

(12)

Eg 1.

- $\phi(t,\vec{x}) \in \mathbb{R}$ (real)

  $\phi'(x') = \phi(x)$, $x' = \Lambda x$ (scalar)

  $\phi'(x+\epsilon) = \phi(x)$

- $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

  $\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\, \partial^\mu \phi - m^2\phi^2\right)$ → mass of meson, real physics

  $\mathcal{L}_{int} = -V(\phi)$ $\qquad V(\phi) \geq 0$ (potential energy)

  e.g. $V(\phi) = \frac{\lambda}{4!}\phi^4$ (λ: Coupling constant, ϕ to 4 model)

\# Note: Hamiltonian of real Scalar field $\mathcal{H} = \frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi)$

$\pi_\phi = \dot{\phi}$

use Hamilton Formalism to derive the same EOM as Lagrange Formalism

- $S = \int d^4x\, \mathcal{L}$, $\quad \phi \to \phi + \delta\phi$ (infinitesimal transformation, and Action Principle)

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{\partial V(\phi)}{\partial \phi} = -m^2\phi - V'(\phi)$$ (是函数的导数，不是普通的)

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \ , \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \Box\phi = \partial_\mu \partial^\mu \phi$$

EOM: $(\Box + m^2)\phi = -V'(\phi)$

when $V(\phi) = 0 \Rightarrow (\Box + m^2)\phi = 0$

real Klein-Gordon equation

(with no interaction)

Exact Solution : known

- In QFT, We have the same eq. but not derived from action principle, it is assumption

Eg. 2.

$V(\phi) = \frac{\lambda}{4!}\phi^4$

$(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3$ $\qquad$ Nonlinear equation

Exact Solution: unknown up to now. (That's why we use perturbative QFT, Feynman diagram)

We are weak in solving the EOM.

Feynman diagram $\begin{cases} \text{we know the exact solution of Real Klein-Gordon Eq.} \\ \text{we don't know... of } \phi^4 \text{ situation} \end{cases}$

∇ Ⅲ. 2 Complex Scalar Field Theory

Note: Quantum Complex Scalar Field is used for $\begin{cases} \text{Higgs particle} \\ \text{pseudo-nucleon "N" "}\bar{N}\text{"} \end{cases}$

- $\psi(x) \in \mathbb{C}$ (complex Field)

  $\psi'(x') = \psi(x)$, $x' = \Lambda x$ (Scalar Field)

  $\psi'(x+\epsilon) = \psi(x)$

- $\mathcal{L}(\psi) = \mathcal{L}_0 + \mathcal{L}_{int}$

  $\mathcal{L}_0 = \partial_\mu^\dagger \psi^\dagger \partial_\mu \psi - m_H^2 \psi^\dagger \psi$ → Higgs particle (mass)

  $\mathcal{L}_{int} = -V(\psi^\dagger\psi)$, $V(\psi^\dagger\psi) \geq 0$

$\psi^\dagger = \psi^*$ in Classical field, ψ is number

- EOM should derive from $\delta S = 0$

Action: $S = \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{int}\right)$, ψ and $\psi^\dagger$ are Independent DOF (why?)

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) \qquad \frac{\partial \mathcal{L}}{\partial \psi^\dagger} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\right)$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi} &= -m_H^2\psi^\dagger - \frac{\partial V(\psi^\dagger\psi)}{\partial \psi} \\ \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) &= \partial_\mu(\partial^\mu \psi^\dagger) = \Box\psi^\dagger \end{aligned}\right\} (\Box + m_H^2)\psi^\dagger = -\frac{\partial V}{\partial \psi}(\psi^\dagger\psi)$$

$$\downarrow \dagger$$

$$(\Box + m_H^2)\psi = -\frac{\partial V(\psi^\dagger\psi)}{\partial \psi^\dagger}$$

- Hamiltonian and Canonical Conjugate momentum

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^\dagger \qquad \Pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} = \dot{\psi}$$ (two conjugate momentum)

And Hamiltonian density $\mathcal{H} = \Pi_\psi \dot{\psi} + \Pi_{\psi^\dagger}\dot{\psi}^\dagger - \mathcal{L}$

$$= \underbrace{\dot{\psi}^\dagger\dot{\psi} + \nabla\psi^\dagger\cdot\nabla\psi + m_H^2\psi^\dagger\psi}_{\geq 0} + \underbrace{V(\psi^\dagger\psi)}_{\geq 0} \geq 0 \text{ and real}$$

$H = E = \int d^3x\, \mathcal{H}$ must be positive/non-negative number

Remark: $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ $\qquad \phi_1, \phi_2 \in \mathbb{R}$

$\psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$

$\psi^\dagger\psi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$, non-negative

- 复数可分成两个实数，因而复标量场可写成两个实标量场 (不同的场之间有相互联系)

$\mathcal{L}(\psi, \psi^\dagger) = \mathcal{L}_0(\phi_1) + \mathcal{L}_0(\phi_2) + -V(\phi_1^2 + \phi_2^2)$

and equation of motion: $(\Box + m_1^2)\phi_1 = -\frac{\partial V}{\partial \phi_1}$

$(\Box + m_2^2)\phi_2 = -\frac{\partial V}{\partial \phi_2}$

and Hamiltonian (density): $\pi_{\phi_1} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \dot{\phi}_1$

$\pi_{\phi_2} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = \dot{\phi}_2$

$\mathcal{H} = \mathcal{H}(\phi_1, \phi_2) = \pi_{\phi_1}\dot{\phi}_1 + \pi_{\phi_2}\dot{\phi}_2 - \mathcal{L}$

$= \mathcal{H}_0(\phi_1) + \mathcal{H}_0(\phi_2) + V(\phi_1^2 + \phi_2^2) \geq 0$

Complex scalar Field theory $\psi(x) \approx (\phi_1, \phi_2)$ two real scalar Field

maybe someday photon $\approx$ two electron

∇ Ⅲ. 3. Dirac Bispinor Field Theory

Note: Quantization of Dirac Bispinor Field is For $\begin{cases} \text{Electron} \\ \text{nucleon} \\ \text{nutrino} \\ \vdots \\ \text{other spin-}\frac{1}{2}\text{ particle} \end{cases}$

and spin is nature even in Classical Field

- Dirac Bispinor Field $\psi_D(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$ $\qquad \psi^a(x) \in \mathbb{C}$, $a = 1,2,3,4$

Four Complex field.

$\psi_D'(x') = D(\Lambda)\psi_D(x)$

$D(\Lambda) \neq \Lambda$

$x' = \Lambda x$

- Gamma Algebra

Gamma matrix γ^μ ← 是四维时空矢标, $\mu = 0,1,2,3$, 4×4 matrices, satisfying

① $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

② $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$, $i = 1,2,3$ (时空部分不一样)

Dirac Conjugation $\bar{\psi}_D(x) \equiv \psi_D^\dagger(x)\gamma^0$
↑ Hermitian Conjugation

$\bar{\gamma}^\mu \equiv \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$ (在 Dirac Conjugation 下, γ^μ 不变 这就是引入 DiracC 的原因)

$\underset{1\times4\ \text{matrix}}{\bar{\psi}}$ $\underset{1\times4}{\psi^\dagger}$ $\underset{4\times4}{\gamma^0}$

$\bar{\psi}^D_a(x) = \psi^{\dagger D}_a(x)(\gamma^0)_{ab}$ summation with a

- Lagrangian density for Dirac Field

$\mathcal{L}_0(\psi_D) = \bar{\psi}_D(x)\, i\gamma^\mu\partial_\mu\psi_D(x) - m_e\bar{\psi}_D(x)\psi_D(x)$
↳ electron mass $m_e \neq m_e^2$

$= \bar{\psi}_D(x)(i\gamma^\mu\partial_\mu - m_e)\psi_D(x)$ (everything is matrix, when doing calculation)

$= \bar{\psi}^D_a\left(i(\gamma^\mu)_{ab}\partial_\mu - m_e\delta_{ab}\right)\psi^D_b(x)$ ①

计算时要带指标
写公式时不用 (clarity)

$= \psi^\dagger_{D,a}\, i\dot{\psi}^D_a + i\psi^{D\dagger}_a(\gamma^0\gamma^i)_{ab}\partial_i\psi^D_b(x) - m_e\psi^\dagger_a\gamma^0_{ab}\psi_b(x)$ ②
注意指标

若不用 Dirac Conjugation, 是4维

EOM: $\psi_D(x)$ and $\bar{\psi}_D(x)$ are independent DoF
(matrix ψ_D, $\bar{\psi}_D(x)$, we usually write down component $\psi_D^a(x)$, $\bar{\psi}_D^a(x)$)

- EOM:

$\dfrac{\partial\mathcal{L}}{\partial\psi^D} = 0 \iff \dfrac{\partial\mathcal{L}}{\partial\psi^D_a(x)} = 0$

· 按FD一样, 经计算, 都是同样结果

怎么知道计算? 应该用分量计算

$\dfrac{\partial\mathcal{L}}{\partial\bar{\psi}^D} = 0 \iff \boxed{\dfrac{\partial\mathcal{L}}{\partial\bar{\psi}^D_a(x)} = 0}$ 只有一个项, 比较简单

EOM should be (called Dirac Equation)

$\dfrac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m_e)\psi = 0$ (matrix)

$(i\gamma^\mu\partial_\mu - m)\psi = 0$ ⇝ 两边取 Dirac Conjugate, 结果是一样的
$-\partial_\mu\bar{\psi}_D\, i\gamma^\mu - m_e\bar{\psi}_D = 0$

· Try to derive Hamiltonian Density

Two diff Canonical momentum

$\Pi_{\psi_D} = \dfrac{\partial\mathcal{L}}{\partial\dot{\psi}_D(x)} \overset{注②}{=} i\psi^\dagger_{Da}(x)$, $\Pi_{\bar{\psi}_D} = \dfrac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}_{Da}} \overset{注①}{=} 0$

注 Component formulation 的好处 < 计算简单 / conceptual 清楚, 不过把它看成4个 complex field

$\mathcal{H} = \dot{\psi}_{D,a}\Pi_\psi + \underbrace{\Pi_{\bar{\psi}}\dot{\bar{\psi}}}_{0} - \mathcal{L} = i\psi^\dagger_D\dot{\psi}_D - \underbrace{\bar{\psi}_D(i\gamma^\mu\partial_\mu - m)\psi_D}_{\text{use Dirac eq} = 0}$

$= i\psi_D^\dagger\dot{\psi}_D$ (满足量子力学的 Hamiltonian)

$H = \int\mathcal{H}\, d^3x = \int d^3x \cdot i\psi_D^\dagger\frac{\partial}{\partial t}\psi_D$

Remark: In Quantum Mechanics, $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$, $\hbar = c = 1$

In Field (classical) theory $i\gamma^\mu\partial_\mu\psi_D = m_e\psi_D$

$\Rightarrow i\gamma^0\frac{\partial}{\partial t}\psi_D + i\gamma^i\partial_i\psi_D = m_e\psi_D$ $(\gamma^0)^2 = 1$

$\Rightarrow i\frac{\partial}{\partial t}\psi_D = -i\gamma^0\gamma^i\partial_i\psi_D + m_e\gamma^0\psi_D$ $\begin{cases}\beta = \gamma^0 \\ \alpha^i = \gamma^0\gamma^i\end{cases}$

$\Rightarrow i\frac{\partial}{\partial t}\psi_D = (-i\vec{\alpha}\cdot\vec{\nabla} + m_e\beta)\psi_D$

$= (\vec{\alpha}\cdot\vec{p} + m_e\beta)\psi_D$ $\vec{p} = -i\vec{\nabla}$

$H = -i\vec{\alpha}\cdot\vec{\nabla} + m_e\beta$
$= \vec{\alpha}\cdot\vec{p} + m_e\beta$

· original form of Dirac eq
But use γ^μ, Lorentz transf is more simple

- Detail of Electro-Dynamics, see the solution of the first homework

VIII.4 Maxwell Vector Field Theory
(Electric-magnetic field theory)

Note: Quantum version of ... is for photon

- use gauge potential Field $A(x)$. $A(x) \equiv A_\mu(x) = \begin{pmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ A_3(x)\end{pmatrix}$ $A_\mu(x) \in \mathbb{R}$

(four real field theory) nothing strange because we have already know EM Field.

Remark 1. $\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \nabla\times\vec{A}$ E, B strength

$A^\mu(x) = (\varphi(x), \vec{A}(x))$

Remark 2: Two maxwell equations $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\nabla\cdot\vec{B} = 0$ can be derived use Remark 1's def of $\vec{E}, \vec{B}$

Remark 3: use $A_\mu(x)$, more convenient for Lorentz transf.

Field strength (Tensor) $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, $E^i = F^{i0} = F_{0i}$, $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$

Gauge Transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\xi(x)$, $\xi(x) \to \lambda(x)$ ↳ Arbitrary Smooth function, $F_{\mu\nu}(x)$ is unchanged! Totally Anti-Symmetric Tensor

Gauge invarice is not good thing, it means redundant DoF

How to deal with Gauge freedom → Gauge fixing (it makes the QED complicated)

First Gauge fixing → Lorentz gauge $\partial_\mu A^\mu(x) = 0$

second Columb gauge $\nabla\cdot\vec{A} = 0$

- Lagrange density for EM field

/ 利用反对称性

$$\mathcal{L}(A_\mu) = \mathcal{L}(F_{\mu\nu}) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu}$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\nu A^\mu)(\partial^\mu A^\nu)$$

对于 equation of motion

π_{A_μ} 没有帮助

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}$$

$LHS = 0 \quad RHS = -\partial_\nu(F^{\nu\mu}) = \left(-\frac{1}{2}-\frac{1}{2}\right)(F^{\nu\mu})$

$\partial_\mu F^{\mu\nu} = 0$

Lec IX & Lec X Symmetries and Conservation Laws 20161101

Lec XI Conomical quantization

Lie Group P14-15

1. Def { Lie Group ① $g = e^{i\alpha^a T^a} = 1 + \alpha^a T^a$
Lie Algebra ①② T^a (generator)

2. Example ① SU(2) SO(3) $g = e^{i\alpha^a S^a}$ $\vec{S}^2$ $[S_i, S_j]$ 表示 $|jm\rangle$

Lie Group 表示 → ○
↑ 指数 ↑ 决定
Lie Algebra 表示 → ○
↓ 对数 ↓ 决定
O(Lie Group) 表示 → ○

Homework 1 due Nov. 8th, 2016

Recall

- Maxwell Vector field theories (to be talked again in the solution of homework)
- Lorentz group $SO^\uparrow_+(1,3)$ which is locally isomorphic to $SU(2) \otimes SU(2) \approx SU(4)$

to understand this and symmetries, we've got to learn Lie group.

② $SO^\uparrow_+(1,3)$ $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$, $[M^{\mu\nu}, M^{\rho\sigma}] = \cdots$ 表示演化 $\{\vec{L}, \vec{K}\}$, $\Lambda = e^{-i\vec{L}\cdot\vec{\theta}}e^{-i\vec{K}\cdot\vec{\beta}}$

{ K_1, K_2 } $SO^\uparrow_+(1,3) \approx SU(2) \otimes SU(2)$

表示

$M^{\mu\nu} \to J^{\mu\nu}$ $S^{\mu\nu}$ $L^{\mu\nu}$

四矢表示 旋量表示 标量表示

Def of Lie group G :

$\forall g \in G$, $g(\alpha) = 1 + i\sum_{a=1}^{N}\alpha^a T^a + O(\alpha^2)$ (maybe a)

T^a : Lie group generator α^a : infinitesimal group parameter (Continous parameter)

(0,0) Spinless
$(\frac{1}{2},0), (0,\frac{1}{2})$. ⊗ Spin-½
(1,0), (0,1) $(\frac{1}{2},\frac{1}{2})$ Spin 1

Def of Lie algebra :

Lie group generators T^a, $a = 1, 2, \cdots, N$.

1) $[T^a, T^b] = i f^{abc} T^c$ (structure constant)

③ Poincare Group $M_{\mu\nu}, P_\mu \to P_{17}$

2) Jacobi identity $[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$

Note: Casimir operator, commute with T^a $C = \sum_{a=1}^{N} T^a T^a = T^a T^a$ (maybe :=)

Note: group element $g = \exp(i\alpha^a T^a)$ (can be compared with translation $e^{-\frac{i}{\hbar}\hat{P}\cdot d\vec{r}_0}$)

Representation of Lie group is given by representation of Lie algebra.

Eg 1. $SU(2) \approx SO(3)$

- SU(2) Lie group $[T^a, T^b] = i\varepsilon^{abc}T^c$ (totally antisymmetric)
- Two dimensional Spinor representation of SU(2)

$[S^a, S^b] = i\varepsilon^{abc}S^c$, $S^a = \frac{1}{2}\hbar\sigma^a \overset{\hbar\to 1}{=\!=} \frac{1}{2}\sigma^a$ (Pauli matrix)

$g = \exp(i\alpha^a S^a) = \exp(\frac{i}{2}\alpha^a\sigma^a)$

- Casimir operator $C = (S^1)^2 + (S^2)^2 + (S^3)^2 = \vec{S}^2$, $[\vec{S}^2, S^a] = 0$
- $|sm\rangle$, $m = -s, -s+1, \cdots, s$

$\vec{S}^2|sm\rangle = s(s+1)|sm\rangle$, $S_z|sm\rangle = m|sm\rangle$

- spinor spin-½ representation $|\frac{1}{2}, \frac{1}{2}\rangle$ $|\frac{1}{2}, -\frac{1}{2}\rangle$

Eg 2 $SO^\uparrow_+(1,3)$

- $SO^\uparrow_+(1,3)$ Algebra $g = \exp\left[-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right]$ (generator of LG)

$\omega^{\mu\nu} = -\omega^{\nu\mu}$, $M_{\mu\nu} = -M_{\nu\mu}$

$M_{\mu\nu}$ 共有16个自由度

1) Structure Constant ⇒ 6个约束 C_4^2

2) Jacobi ⇒ C_4^3 4个约束

$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$
orbital spin

Total Angular momentum Tensor.

- Commutative relation $[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho}$
- Define new generator

- $L_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$ $i, j, k = 1, 2, 3$

$[L_i, L_j] = i\varepsilon_{ijk}L_k$ (spatial rotation)

- $K_i = M_{0i}$, $i = 1, 2, 3$

$[K_i, K_j] = -i\varepsilon_{ijk}L_k$ $[L_i, L_j] = i\varepsilon_{ijk}K_k$ (maybe wrong), $[L_i, K_m] = \frac{i}{2}\varepsilon_{imk}K_k$

generated by $\{M_{\mu\nu}\}$ equivalent $\{L_i, K_i\}$

$g = \exp(-i\vec{L}\cdot\vec{\theta})\exp(-i\vec{\beta}\cdot\vec{K})$
Lorentz rotation Lorentz boost

Note: $\theta_1 = \omega_{23}$ …
$\beta_1 = \omega_{01}$ …

- $\vec{J}_1 = \vec{J}(+) = \frac{1}{2}(\vec{L} + i\vec{K})$ $\vec{J}_2 = \vec{J}(-) = \frac{1}{2}(\vec{L} - i\vec{K})$

$[J_i^{(\pm)}, J_j^{(\pm)}] = i\varepsilon_{ijk}J_k^{(\pm)}$. Similar to $[S^a, S^b] = i\varepsilon^{abc}S^c$ in SU(2)

$[J_i^{(+)}, J_j^{(-)}] = 0$ → they commute with each other.

$\Rightarrow \{M_{\mu\nu}\} \approx \{\vec{L}, \vec{K}\} \approx \{\vec{J}^{(+)}, \vec{J}^{(-)}\}$ $SU(2) \otimes SU(2)$ ⊗: Commute

Transformation of Fields
$$\begin{cases} \delta x^\mu \text{ 坐标的变化 } x^\mu \to x'^\mu,\ \delta x^\mu = x'^\mu - x^\mu \\ \delta\Phi \text{ 场的内禀变化 } \Phi'_a(x') - \Phi_a(x) \\ \delta_0\Phi \text{ 场的总变化 } \delta_0\Phi = \delta\Phi - \delta x^\rho \cdot \partial_\rho \Phi = \Phi'_a(x) - \Phi_a(x) \end{cases}$$

1. Space time Translation

Translation group $g = e^{-iP_\mu \cdot \varepsilon^\mu}$, $[P_\mu, P_\nu] = 0$
$$\begin{cases} \text{坐标的变化 } \delta x^\mu = -iP_\mu \cdot \varepsilon^\mu x^\mu,\ P_\mu = i\partial_\mu ;\ x'^\mu = x^\mu + \varepsilon^\mu \\ \text{场的变化 } \delta\Phi = 0,\ \delta_0\Phi = -iP_\mu \cdot \varepsilon^\mu,\ P_\mu = -i\partial_\mu ;\ \delta_0\Phi = \phi(x-\varepsilon) - \phi(x) \end{cases}$$

2. Lorentz transformation

Transformation Group $g = e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}$

$[M_{\mu\nu}, M_{\rho\sigma}] = \ldots$

$(\mathcal{J}_{\mu\nu})^\rho{}_\sigma = (\delta_\mu{}^\rho \eta_{\nu\sigma} - \delta_\nu{}^\rho \eta_{\mu\sigma}) \cdot i$

$(L_{\mu\nu}) = (x_\mu \partial_\nu - x_\nu \partial_\mu) \cdot i$

$(S_{\mu\nu}) = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$

坐标的变化 $\delta x^\mu = -\frac{i}{2}\underset{\uparrow\ \text{四矢表示}}{(\mathcal{J}_{\rho\sigma})^\mu{}_\nu}\omega^{\rho\sigma}x^\nu = -\frac{i}{2}(\underset{\uparrow\ \text{轨道表示}}{-L_{\rho\sigma}}\omega^{\rho\sigma})x^\mu = \Lambda^\mu{}_\nu x^\nu - x^\mu = \omega^\mu{}_\nu x^\nu$

$$\frac{\delta x^\mu}{\delta \omega^{\rho\sigma}} = \frac{1}{2}(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho)$$

标量场的变化 $\Phi'_a(x) = \Phi_a(\Lambda^{-1}x)$

$\delta_0\Phi = -\frac{i}{2}\underset{\uparrow\ \text{标量场表示}}{L_{\mu\nu}}\omega^{\mu\nu}\Phi \qquad \delta\Phi = 0$

$= -\delta x^\rho \cdot \partial_\rho \Phi$

Maxwell Vector field的变化 $A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$

$\delta_0\Phi^\mu = -\frac{i}{2}(L_{\rho\sigma} + \underset{\text{矢量表示}}{\mathcal{J}_{\rho\sigma}})\omega^{\rho\sigma}A^\mu$

$\delta\Phi^\mu = -\frac{i}{2}\mathcal{J}_{\rho\sigma}\omega^{\rho\sigma}A^\mu = \omega^\mu{}_\nu A^\nu$

Bispinor field的变化 $\psi'_D(x) = D(\Lambda)\psi_D(\Lambda^{-1}x)$

$\delta_0\psi_D = -\frac{i}{2}(L_{\rho\sigma} + \underset{\text{双旋量表示}}{S_{\rho\sigma}})\omega^{\rho\sigma}\psi_D$

$\delta\psi_D = -\frac{i}{2}S_{\rho\sigma}\omega^{\rho\sigma}\psi_D$

Symmetries & Conservation's laws

场 $\Phi_a(x)$ $\partial_\mu\Phi_a(x)$

动力学量 $\mathcal{L}$, L, S

transformation δx^μ, $\delta_0\Phi_a$, $\delta\Phi_a$, $\delta\mathcal{L}$

1. Symmetry

Def: $\delta_0\mathcal{L} = \partial_\mu F^\mu \longrightarrow$ S. EOM 不改变

Noether's Thm: $j^\mu{}_a = \underbrace{\left[-\mathcal{L}\delta^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_q)}\partial_\rho\Phi_q\right]\frac{\delta x^\rho}{\delta\omega^a}}_{\text{轨道项 位移项}} - \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_q)}\frac{\delta\Phi_q}{\delta\omega^a}}_{\text{自旋项 内禀项}}$

连续的对称性 => 守恒流 j^μ_a
守恒荷 $\int j^0{}_a d^3\vec{x}$ } not unique

$$\begin{cases} \text{Defining Symmetry: Poincaré Symmetry } (\Lambda^\mu{}_\nu,\ \text{Translation}) \\ \text{Anomaly : Chiral Symmetry} \end{cases}$$

2. Classification of Symmetry

$\begin{cases} \text{Continues} \\ \text{discrete} \end{cases}$ $\begin{cases} \text{Space-time} \\ \text{Internal} \end{cases}$

3. Examples $\begin{cases} \text{Real Scalar Field} \\ \text{vector Field} \\ \text{Bispinor Field} \end{cases}$

① All Field → Space time Translation invariant → Energy momentum Tensor $T^{\mu\nu}$

$\delta x^\mu = \varepsilon^\mu$
$$\begin{cases} \frac{\delta x^\rho}{\delta\varepsilon^\nu} = \delta^\rho_\nu \\ \frac{\delta\Phi_q}{\delta\varepsilon^\nu} = 0 \end{cases}$$

$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_q)}\partial^\nu\Phi_q$
$$\begin{cases} \int T^{00} d^3\vec{x} = H \text{ 能量} \\ \int T^0{}_k d^3\vec{x} = \int \pi_q \partial_k \Phi_q d^3\vec{x} = P_k \text{ 动量} \end{cases}$$

② All Field → Lorentz invariant ⟶ angular momentum tensor

$\delta x^\mu = \omega^\mu{}_\nu x^\nu$
$$\begin{cases} \frac{\delta x^\rho}{\delta\omega^{\sigma\nu}} = \frac{1}{2}(\delta^\rho_\sigma x_\nu - \delta^\rho_\nu x_\sigma) \\ \frac{\delta\Phi_q}{\delta\omega^{\sigma\nu}} = -\frac{i}{2}(S^{\sigma\nu})^a{}_b\Phi_b \end{cases}$$

$j^{\mu\sigma\nu} = \frac{1}{2}(x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu}) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_q)}\frac{\delta\Phi_q}{\delta\omega^{\sigma\nu}}$

$M^{\mu\sigma\nu} = -2 \cdot j^{\mu\sigma\nu} = L^{\mu\sigma\nu} + S^{\mu\sigma\nu}$

③ Complex Scalar Field → $U(1) = \{e^{-i\lambda}\}$ invariant

C.S 守恒流

$j^\mu = i(\partial^\mu\psi^* \cdot \psi - \partial^\mu\psi \cdot \psi^*)$

2-Real Scalar Field → $SO(2) = \{e^{i\sigma_y \cdot \lambda}\} = \left\{\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}\right\}$

C.S 守恒流

$j^\mu = \partial^\mu\phi_1 \cdot \phi_2 - \partial^\mu\phi_2 \cdot \phi_1$

$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$
$\psi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$ $\sim \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

for internal transform

$\frac{\delta x^\rho}{\delta\omega^a} = 0$

$\frac{\delta\phi_q}{\delta\omega^a} \neq 0$

$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_q)}\frac{\delta\Phi_q}{\delta\omega^a}$

④ Discrete Symmetry (C, P, T) ⟿ (Strong) CPT theorem

⟿ Parity / Def: $x^\mu \to x_\mu$

matrix $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\begin{cases} \in SO(1,3),\ \text{保 } ds^2 \\ \notin SO_+(1,3)\ \det = -1 \end{cases}$

Scalar field: $(P\Phi)(x^\mu) = \eta\Phi(x_\mu)$

vector: $(PA_\mu)(x^\mu) = A^\mu(x_\mu)$

Dirac: $(P\psi_D)(x^\sigma) = \eta\gamma^0\psi_D(x_\sigma)$

$(P\bar{\psi}_D)(x^\sigma) = \bar{\psi}_D(x_\sigma)\eta\gamma^0$

$$g = \exp\left(-i(\vec{J}^{(+)} + \vec{J}^{(-)})\cdot\vec{\theta}\right)\exp\left(-(\vec{J}^{(+)} - \vec{J}^{(-)})\cdot\vec{\beta}\right)$$

$$\begin{array}{ccc} SO^{\uparrow}_{+}(1,3) & \approx SU(2) & \otimes\, SU(2) \\ \downarrow & \downarrow & \downarrow \\ \{M_{\mu\nu}\} & \{\vec{J}^{(+)}\} & \{\vec{J}^{(-)}\} \\ (j_1 j_2) & |j_1 m_1\rangle & \otimes\, |j_2 m_2\rangle \end{array}$$

- spinless particle (scalar) $(0,0) \approx |00\rangle \otimes |00\rangle$
- spin-$\frac{1}{2}$ $\left.\begin{array}{l}(\frac{1}{2},0)\\(0,\frac{1}{2})\end{array}\right\} \to$ Weyl / Majorana Spinor (neutrino) $\quad |\frac{1}{2},\pm\frac{1}{2}\rangle \otimes |00\rangle$, $\; |00\rangle \otimes |\frac{1}{2},\pm\frac{1}{2}\rangle$

  $(\frac{1}{2},0)\oplus(0,\frac{1}{2}) \to$ Dirac bispinor (electron) $\quad |\frac{1}{2},\pm\frac{1}{2}\rangle \otimes |00\rangle \oplus |00\rangle \otimes |\frac{1}{2},\pm\frac{1}{2}\rangle$

  $\downarrow$ perhaps e^- is not fundamental particle.

- spin 1

  $\left.\begin{array}{l}(1,0)\\(0,1)\end{array}\right\}$ massive spin-1 ($W^{\pm}$ Three polarization : $\pm 1, 0$)

  $(\frac{1}{2},\frac{1}{2})$ massless spin-1 (photon Two polarization: ± 1)

Symmetries & Conservation Laws

IX.1. Transformation of fields

IX.1.a space time translation

- Translation group $g \in G$, $g = \exp\left(-iP_\mu \varepsilon^\mu\right)$
  - Lie Algebra $[P_\mu, P_\nu] = 0 \qquad P^2 = m^2 \leftarrow$ Casimir operator?
  - Space time translation $x^\mu \xrightarrow{T} x'^\mu = x^\mu + \varepsilon^\mu$

    $\delta x^\mu = \varepsilon^\mu = -i\varepsilon^\rho P_\rho x^\mu \qquad P_\rho = i\frac{\partial}{\partial x^\rho}$ coordinate change

    Nothing different
    Just notation

- $\left.\begin{array}{l}\phi(x)\\ \psi(x)\\ \psi_D(x)\\ A_\mu(x)\end{array}\right\}$ all fields $\Phi(x) \xrightarrow{U_T} \Phi'(x') = U_T \Phi(x) U_T^{+}$

  field change

  $\Phi \to \Phi'(x) = \Phi(T^{+}x) = \Phi(x-\varepsilon)$

  $\delta\Phi(x) \equiv \Phi'(x') - \Phi(x) = 0$ 流形上的场的变化，或场的形式的变化

  $\delta_0\Phi(x) = \Phi'(x) - \Phi(x) = -\varepsilon^\mu\partial_\mu\Phi(x) = i\varepsilon^\mu P_\mu\Phi(x)$ 观察者看到的场的变化，或场因位置改变导致的变化 + 形式变化

- $\delta\Phi(x) = \Phi'(x') - \Phi(x)$

  Note: $\delta = \delta_0 - (-\delta x^\mu \cdot \partial_\mu)$
  形式的变化 = 总变化 - 位变化

  suppose $\Phi' = \Phi + \delta\Phi = \Phi + O(\delta x^\mu)$

  $\delta\Phi(x) = \Phi'(x') - \Phi'(x) + \Phi'(x) - \Phi(x) = \delta x^\mu \partial_\mu \Phi(x) + \delta_0\Phi(x) + O((\delta x^\mu)^2)$

  $\delta = \delta_0 + \delta x^\mu \partial_\mu + O((\delta x^\mu)^2)$

  Note: $\delta x^\mu = \varepsilon^\mu \Leftrightarrow \frac{\delta x^\mu}{\delta\varepsilon^\rho} = \eta^\mu_\rho \qquad \delta\Phi(x) = 0 \Leftrightarrow \frac{\delta\psi}{\delta\varepsilon^\rho} = 0$

IX.2 Lorentz transformation of fields

- (Real or Complex) Scalar fields

  $x \xrightarrow{\Lambda} x' = \Lambda x$

  $\phi(x) \xrightarrow{U_\Lambda} \phi'(x') = \phi(x) \qquad \psi(x) \xrightarrow{U_\Lambda} \psi'(x') = \psi(x)$

- $\delta\phi = \phi'(x') - \phi(x) = 0 \quad (\delta\psi = 0)$

  $\delta_0\phi = \phi'(x) - \phi(x) = \phi(\Lambda^{-1}x) - \phi(x)$

  $x'^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu_\nu + w^\mu{}_\nu)x^\nu = x^\mu + w^{\mu\nu}x_\nu$

  $\delta x^\mu = \underbrace{w^{\mu\nu}x_\nu}_{①} = \underbrace{\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}x^\mu}_{②} \qquad M_{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)$

  ① use group language, ② generator of LG

  $\Leftrightarrow \frac{\delta x^\mu}{\delta w^{\rho\sigma}} = \frac{i}{2}M_{\rho\sigma}x^\mu = \frac{i}{2}\left(\eta^\mu_\rho x_\sigma - \eta^\mu_\sigma x_\rho\right)$

  Def of Scalar field $\frac{\delta\phi(x)}{\delta w^{\rho\sigma}} = 0 \Leftrightarrow \delta\phi(x) = 0$

  $\delta_0\phi(x) = \phi(\Lambda^{-1}x) - \phi(x) = \phi(x_\mu - w_{\mu\nu}x^\nu) - \phi(x) = -w^{\mu\nu}x_\nu\partial_\mu\phi(x)$
  $= -\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}\phi(x)$) 利用 $w^{\mu\nu}$ 的反对称性

- (Dirac) bispinor field

  $x \xrightarrow{\Lambda} x' = \Lambda x \qquad \psi_D(x) \xrightarrow{U_\Lambda} \psi_D'(x') = D(\Lambda)\psi_D(x)$

  $D(\Lambda) = \exp\left(-\frac{i}{2}w^{\mu\nu}S_{\mu\nu}\right)$ — $w^{\mu\nu}$: group parameter, $S_{\mu\nu}$: generator spin-$\frac{1}{2}$

  $S_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$

  $\delta\psi_D(x) = \psi_D'(x') - \psi_D(x) = -\frac{i}{4}w^{\mu\nu}\sigma_{\mu\nu}\psi_D(x)$

  $\Updownarrow$ equivalent

  $\frac{\delta\psi_D(x)}{\delta w^{\mu\nu}} = -\frac{i}{4}\sigma_{\mu\nu}\psi_D(x)$

  $\to$ transformation

  $\delta_0\psi_D(x) = \psi_D'(x) - \psi_D(x) = D(\Lambda)\psi_D(\Lambda^{-1}x) - \psi_D(x) = -\frac{i}{2}w^{\mu\nu}M_{\mu\nu}\psi_D(x)$

  $M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$

- Maxwell Vector field

  $x \xrightarrow{\Lambda} x' = \Lambda x \qquad A^\mu(x) \xrightarrow{U_\Lambda} A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)$

  $\Lambda = \exp\left(-\frac{i}{2}w^{\rho\sigma}S_{\rho\sigma}\right)$, $S_{\rho\sigma}$: spin-1

  $(S_{\rho\sigma})_\mu{}^\nu = i(\eta_{\rho\mu}\eta_\sigma{}^\nu - \eta_{\sigma\mu}\eta_\rho{}^\nu)$ (矢量表示的生成元)

  $\delta A_\mu(x) = A'_\mu(x') - A_\mu(x) = w_{\mu\nu}A^\nu(x) = -\frac{i}{2}w^{\rho\sigma}(S_{\rho\sigma})_\mu{}^\nu A_\nu(x)$ (方向的改变由Λ决定)

  $\frac{\delta A_\mu(x)}{\delta w^{\rho\sigma}} = -\frac{i}{2}(S_{\rho\sigma})_\mu{}^\nu A_\nu(x)$

  You can derive the solution of 1st home work in this way.

  $\delta_0 A_\mu(x) = A'_\mu(x) - A_\mu(x) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x) - A_\mu(x) = -\frac{i}{2}w^{\rho\sigma}(M_{\rho\sigma})_\mu{}^\nu A_\nu(x)$ (总的变化，由Λ($J_{\mu\nu}$, $L_{\mu\nu}$)共同决定)

  $(M_{\rho\sigma})_\mu{}^\nu = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\eta_\mu{}^\nu + (S_{\rho\sigma})_\mu{}^\nu$

IX.3. Symmetries & Conservation Laws

· Symmetries play fundamental roles in modern theoretical physics, because they are capable of determining the formulation of the action, such as the action of non-Abelian gauge field theories in Standard Model in particle physics.

· Notation

General fields: $\Phi_a(x)$

States: $\Phi_a(x)$, $\partial_\mu\Phi_a$

Lagrangian: $\mathcal{L}(\Phi_a, \partial_\mu\Phi_a)$

Action: $S = \int d^4x\, \mathcal{L}$

$\Phi'(x) = \Phi(x) + \delta_0\Phi(x)$ → 形的变化 (指函数形式之下)

transformation: $x \to x' = x + \delta x$ $\quad \Phi(x) \to \Phi'(x') = \Phi(x) + \delta\Phi(x)$ $\quad \mathcal{L} \to \mathcal{L}'(\Phi', \partial_\mu\Phi')$

总的变化

这里的变换 $\delta_0\Phi$ 并不是
坐标系变换导致的变换 $\delta\Phi$
因而 $\mathcal{L}$ 作为标量也可能改变

· Def of Symmetry:

A transformation is called a symmetry if and only if

· $S = \int d^4x'\, \mathcal{L}'(\Phi'(x'), \partial_\mu\Phi') = \int d^4x\, \mathcal{L}(x)$ ← 相差一个散度也可 (与 g 无关的)

· $\mathcal{L}'(\Phi, \partial_\mu\Phi) = \mathcal{L}(\Phi, \partial_\mu\Phi) + \partial_\mu F^\mu$

$F^\mu(\Phi, \partial_\mu\Phi)\big|_{\text{space time boundary}} = 0$ ← 是一个散度也可

Remark: it means $\int d^4x\, \partial_\mu F^\mu = \oint d s^\mu F_\mu = 0$

亦即，不同的坐标系下观察不同位置的 $\mathcal{L}$
但给出相同的运动规律

Remark: why we require S be unchanged? we require formalism of EOM unchanged.

· Noether's theorem: Every continuous symmetry is associated with a conserved current.

$\delta x^\rho = \frac{\delta x^\rho}{\delta\omega^a}\delta\omega^a$ $\quad \delta\Phi = \frac{\delta\Phi}{\delta\omega^a}\delta\omega^a$ ← 场的方向变化 $\quad \delta_0\Phi - \delta\Phi = -\delta x^\rho \cdot \partial_\rho\Phi$ ← 位的增值变化

$\delta_0\mathcal{L} = -\delta x^\rho \cdot \partial_\rho\mathcal{L}$

The conserved current density j^μ_a, $\partial_\mu j^\mu_a = 0$, j^μ_a is not unique.

Scalar field ← j^μ_a, $j'^\mu_a = j^\mu_a + t^\mu_a$ $(\partial_\mu t^\mu_a = 0)$ are also conserved currents.

$$j^\mu_a = -\left[\mathcal{L}\eta^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\rho\Phi\right]\frac{\delta x^\rho}{\delta\omega^a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\Phi}{\delta\omega^a}$$

conserved charge: $Q_a = \int d^3x\, j^0_a$

这一部分不含自旋的因素 是坐标变换导致的

the second term means spin angular momentum contribution

(这一次 $\frac{\delta\Phi}{\delta\omega^a}$ 是 $\delta_0\Phi$ 中除去坐标变换带来的另一部分)
是场的内自由度变化部分

$\frac{d}{dt}Q_a = 0$

e.g. Spacetime translation invariance

=> Energy-momentum tensor $T_{\mu\nu}$

$\delta x_\mu = \varepsilon_\mu$, $\frac{\delta x_\mu}{\delta\varepsilon^\nu} = \eta_{\mu\nu}$, $\frac{\delta\Phi}{\delta\varepsilon_\mu} = 0$

Replace Φ with $A_\mu(x)$ in homework.

· $T_{\mu\nu} = \left[-\mathcal{L}\eta_{\mu\rho} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\rho\Phi\right]\eta^\rho_\nu = -\mathcal{L}\eta_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi$

$\partial^\mu T_{\mu\nu} = 0$, $\partial_\mu T^{\mu\nu} = 0$

e.g. 2: Lorentz invariance and angular momentum

$$\frac{\delta x^\mu}{\delta\omega^{\rho\sigma}} = \frac{1}{2}\left(\eta^\mu_\rho x^\sigma - \eta^\mu_\sigma x_\rho\right)$$

$\frac{\delta\Phi}{\delta\omega^{\rho\sigma}} = \to 0$ Scalar field

$\leadsto$ Spin-½ Spin-1

$$M_{\mu\rho\sigma} = \left[-\mathcal{L}\eta_{\mu a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_a\Phi\right]\cdot\frac{1}{2}\left(\eta^a_\rho x_\sigma - \eta^a_\sigma x_\rho\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\Phi}{\delta\omega^{\rho\sigma}}$$

$$M_{\mu\rho\sigma} = \frac{1}{2}T_{\mu a}\left(\eta^a_\rho x_\sigma - \eta^a_\sigma x_\rho\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\Phi}{\delta\omega^{\rho\sigma}}$$

$$= -\frac{1}{2}\left[\left(x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}\right) + 2\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\Phi}{\delta\omega^{\rho\sigma}}\right]$$

$L_{\mu\rho\sigma} = x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}$ $\quad$ homework, just replace

Nov. 8th, 2016 $\quad$ plan: Internal Symmetries
Discrete Symmetries
Canonical Quantization

Lec X. Symmetries and Conservation Laws

Lec XI. Canonical Quantization

Homework 1st, Due today

Homework 2nd, Due Nov 29, 2016

Homework 2nd. * Peskin & Shroeder's book
PDF file in instructor's homepage

Problem 1. P & S's book, P.P 39-40

1.1 verify (3.18) to satisfy the Lorentz Algebra (3.17)

1.2 verify (3.20) & (3.21) as examples of (3.19) $\quad$ (about Dirac equation)

Problem 2. P & S's book, P.P 40-42

2.1 Verify (3.23) to satisfy the Lorentz Algebra (3.17)

2.2 Verify (3.26) (3.27) & (3.29) $\quad$ (about Dirac Lagrangian)

2.3 verify Dirac equation (3.31) and Lagrangian (3.34) Lorentz invariant.

Problem 3. P & S's book P.P 48-49 $\quad$ (about solution of Dirac eq.)

3.1 Verify the (3.59) as a positive energy solution of Dirac eq. (3.31) in momentum space. and verify (3.60)

3.2 verify (3.62) as negative energy solution of (3.31) in momentum space

3.3 Verify (3.63) (3.64) (3.66) and (3.67)

(get other book for solution)

(17)

Recall Last

* space time translation group

the coordinate translation $x \xrightarrow{T} x' = x + \varepsilon = T x$

the field transformation $\Phi(x) \xrightarrow{U_T} \Phi'_a(x) = U_T \Phi_a U_T^{+}(x) = \Phi_a(T^{-1} x)$

unitary transformation

Infinitesimal transformation

$U_T = \exp\left(-i P_\mu \varepsilon_\mu\right)$

momentum, Can be expressed as $P_\mu = i\partial_\mu$ for coordinate Transform
$P_\mu = -i\partial_\mu$ for field Transform

* Lorentz transformation

the coordinate $x \xrightarrow{\Lambda} x' = \Lambda x$

the field $\Phi(x) \xrightarrow{U_\Lambda} \Phi'_a(x) = U_\Lambda \Phi_a(x) U_\Lambda^{+}(x) = D(\Lambda)_{ab} \Phi_b(\Lambda^{-1} x)$

general field
- Scalar
- vector
- ...

spin representation including gravity
- electron → $\frac{1}{2}$ representation
- grivition → 2 representation
- ...

Infinitesimal Transformation

$D(\Lambda) = \exp\left(-\frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$

spin momentum $\begin{cases} \frac{1}{2} \cdots \\ 1 \end{cases}$

$\Phi'_a(x') = D(\Lambda)_{ab} \Phi_b(x)$

$U(\Lambda) = \exp\left(-\frac{i}{2} w_{\mu\nu} M^{\mu\nu}\right)$

general representation of Lorentz group

$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$

[illegible] [illegible]

* The Space-time Poincare's group Algebra is generated by $\{P_\mu\}$ and $\{M_{\mu\nu}\}$

a larger group include translation and Lorentz transf.

$[P_\mu, P_\nu] = 0$

$[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho$

$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\rho\nu} + \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}$ ← Home 2nd prove the representation of this group

* Poincare Algebra 's Representation

← Poincare group 's representation is the entire particle physics

Two Casimir Operator (Conserved Operator) $[C, P_\mu] = [C, M_{\mu\nu}] = 0$

$P^2 = P_\mu P^\mu = m_0^2$ ← just number,

$W^2 = W_\mu W^\mu$, $W_\mu = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$

$W^2 = -\frac{1}{2} S^{\rho\sigma} S_{\rho\sigma} P^2 + P_\rho S^{\rho\sigma} S_{\nu\sigma} P^\nu$

very good quantity, has representation of mass, spin angular momentum (intrinsic quantity, not include $\vec{L}$)

1) if. $m_0 \neq 0$ choose rest frame

it's possible but not available for photon, neutrino

$P^\mu = (m, 0, 0, 0)$

$W^2 = -m^2 S^k S^k$, $k = 1, 2, 3$

$S^k = \frac{1}{2} \varepsilon^{ijk} S^{ij}$

only spin-angular momentum survived

$W^2 = -m^2 \left((S^1)^2 + (S^2)^2 + (S^3)^2\right)$
$= -m^2 \vec{S}^2$, $\vec{S} = (S^1, S^2, S^3)$

particle in rest, the representation is simpler

$|m, S, S_z\rangle$ $S^2 |S_z\rangle = S(S+1) |S_z\rangle$

2) if $m_0 = 0$. $P^2 = 0$, $W^2 = 0$, $W_\mu P^\mu = 0$

casimir operator is zero
Something must be changed

define $\lambda = \frac{W^0}{P^0} = \frac{\vec{S} \cdot \vec{P}}{|\vec{P}|}$, it is Helicity operator, its representation is

$\frac{\vec{S} \cdot \vec{P}}{|\vec{P}|} |x\rangle = \lambda |x\rangle$, $\lambda = \pm\frac{1}{2}, \pm 1, \pm 2, 0$

mass
- $|0, \pm\frac{1}{2}\rangle$ massless neutrino
- $|0, \pm 1\rangle$ photon
- $|0, \pm 2\rangle$ graviton

Helicity

- When talk about massive spin-1 particle, $\begin{cases} +1 \\ 0 \\ -1 \end{cases}$
  But massless particle $\begin{cases} 1 \\ -1 \end{cases}$
- All the particle can be classified with the Poincare group
  It define Standard model
  (Conformal group. generalization of it)

X. Symmetries & Conservation laws

X1 Transformation of Fields

		space-time coordinate	dynamical fields	examples
Classification of transf. 1	Space time transf.	changed	changed	translation, Lorentz transf.
	Internal transf. (Very Very Important)	fixed	changed	$U(1)$, $SU(2)$. $SU_F(2) \times SU_C(3) \times U_Y$

		Transformation parameter	Spacetime transf	Internal transf.
Classification of transf. 2	Continuous transf.	Continuous	translation, Lorentz transf.	$U(1)$, $SU(2)$ $SU_F(2) \otimes SU(3) \otimes U_Y$ [illegible]
	discrete transf.	No parameter	Parity, Time Reversal	Charge conjugation (particle ↔ anti-particle) big achievement of QFT it can predict anti-particle

X.2 Def of Symmetries

· Symmetries almost determine(s) the formalism of the action in the Standard model.

↑ not completely

X.3 Noether's theorem

Every continuous symmetry is associated with the conserved current j^μ_a,

$$\partial_\mu j^\mu_a = 0$$

μ is spacetime index, a is for transformation, and

$$j^\mu_a = -\left[\mathcal{L}\eta^\mu_\rho - \sum_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \partial_\rho \phi^\alpha\right] \frac{\delta x^\rho}{\delta \omega^a} - \sum_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \frac{\delta \phi^\alpha}{\delta \omega^a}$$

· powerful thm, because the formulation of j^μ_a is determined.

Note: Conserved current is not unique,

$$\tilde{j}^\mu_a = \begin{cases} \lambda j^\mu_a \\ j^\mu_a + t^\mu_a \quad \text{with } \partial_\mu t^\mu_a = 0 \end{cases}$$

is also conserved current.

The Formalism is not unique, but the physical consequence is the same.

(Choose the formalism is an 'ART')

X.3.1. Space-time translation invariant and Energy-momentum tensor.

the translation (by variation) $\quad \dfrac{\delta x^\mu}{\delta \varepsilon^\nu} = \eta^\mu_\nu ; \quad \dfrac{\delta \Phi}{\delta \varepsilon^\nu} = 0 ;$

Immediately we get the cosequence of Noether's thm:

$$T^{\mu\nu} = -\mathcal{L}\eta^{\mu\nu} + \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial^\nu \phi^a$$

· the second part is vanished !

Note 1: The (0,0) component of $T^{\mu\nu}$

$$T^{00} = \sum_a \frac{\partial \mathcal{L}}{\partial(\dot\phi^a)} \dot\phi^a - \mathcal{L} = \mathcal{H}$$

· It seems we get the right answer because $T^{00} = \mathcal{H} = \text{Energy}$

$$E := H = \int d^3x\, T^{00}$$

Energy is the conserved quantity.

Note 2: The (0,i) component of $T^{\mu\nu}$

$$T^{0i} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)} \partial^i \phi^a = \sum_a \pi^a \partial^i \phi^a$$

maybe $\vec{p} = (T^0{}_i) = \sum_a \pi^a \nabla \phi^a$

$$\vec{p} = (T^{0i}) = \sum_a \pi^a \nabla\phi^a \quad \text{is momentum density}$$

$$\vec{P} = \int d^3x\, \vec{p} = \int d^3x \sum_a \pi^a \nabla \phi^a \quad \text{is conserved, Total momentum}$$

↓ Canonical / Conjugate momentum

it is Not the conserved quantity.

Example: Real Scalar Field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi^2)$$

↑ free Lagrangin $\quad$ ↑ $V(\phi^2) \geq 0$

Canonical momentum $\quad \pi_\phi = \dot\phi$

Hamiltonian density $\quad \mathcal{H} = \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi) \geq 0$

$= T^{00}$

Now In quantum field theory, we first talk T^{00}, without $\mathcal{H}$

momentum $\quad (T^{0i}) = \vec{p} = \dot\phi \nabla \phi$

maybe $T^0{}_i$

X.3.2. Lorentz-transf invariant and Angular momentum Tensor

the Lorentz transform

$$\frac{\delta x^\mu}{\delta \omega_{\rho\sigma}} = \frac{1}{2}(\eta^\mu{}_\rho x_\sigma - \eta^\mu{}_\sigma x_\rho)$$

$$\frac{\delta \Phi}{\delta \omega_{\rho\sigma}} = \sum_b \left(-\frac{i}{2}\right)(S_{\rho\sigma})_a{}^b \Phi_b$$

↑ spin-1 for photon
spin-½ for proton, electron

The angular momentum Tensor

$$\partial_\mu M^{\mu\rho\sigma} = 0 \quad \rightarrow a \rightarrow \rho\sigma$$

and up the index

$$M^{\mu\rho\sigma} = L^{\mu\rho\sigma} + 2\sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \frac{\delta \phi^a}{\delta \omega_{\rho\sigma}}$$

↓ total Angular momentum $\quad$ ↓ Orbital Angular momentum $\quad$ ↘ Spin angular momentum

$$L^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

Spin angular momentum $= -i\sum_a \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} (S^{\rho\sigma})_a{}^b \Phi_b$

↑ we must introduce $S^{\rho\sigma}$ operator
so it must be spin angular momentum

Lorentz 變換之下. 決定 $S^{\rho\sigma}$ 並找出它
決定自旋

Application $\begin{cases} \text{spinless} \rightarrow \text{scalar} \\ \text{Spin-one} \rightarrow \text{maxwell vector field} \\ \text{Spin-½} \rightarrow \text{Bispinor field} \end{cases}$

Example. Real scalar field (spinless, $S^{\rho\sigma} = 0$)

$M^{\mu\rho\sigma} = L^{\mu\rho\sigma}$ will be known if $T^{\mu\nu}$ is determined.

Example. Maxwell vector field (Homework)

Example, Dirac Bispinor field (Done by ·· self)

X.3.3 Internal Symmetries and Conserved Current.

↳ fields → changed

coordinate of spacetime → unchanged

$\delta x^\rho = 0$

$\delta x^\rho / \delta \omega^a = 0$

So the Conserved Current

$$j^\mu_a = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \frac{\delta \phi^a}{\delta \omega^a}$$

• the first term is cancelled
it is simplified
But the transform of $\delta\phi^a$ is complicated

Example 1. $U(1)$ invariant Complex Scalar field

just phase factor

$U(1) = \{ e^{i\lambda} \mid \lambda \in \mathbb{R} \}$

parameter $e^{-i\lambda} \to \lambda$

it is a Lie group

The Lagrangian of Complex Scalar field

$$\mathcal{L} = \partial_\mu \psi^+ \partial^\mu \psi - m^2 \psi^+ \psi - V(\psi^+ \psi)$$

$V(\psi^+\psi) \geq 0$

The $U(1)$ transformation

$$\psi(x) \xrightarrow{U(1)} \psi'(x) = e^{-i\lambda} \psi(x)$$

$$x \longrightarrow x' = x$$

$$\psi^+(x) \longrightarrow \psi^{+\prime}(x) = e^{i\lambda} \psi^+(x)$$

→ we can check $\psi^+\psi \to \psi^+\psi$ unchanged

$\partial_\mu \psi^+ \partial^\mu \psi \to \partial_\mu \psi^+ \partial^\mu \psi_+$

$\mathcal{L} \to \mathcal{L}$ unchanged (4-d divergence)

The Conserved Current $\quad S \to S$ unchanged, we can conclusion the existence of j^μ_a given by Noether's thm

$$\frac{\delta\psi(x)}{\delta\omega^a} \Rightarrow \frac{\delta\psi}{\delta\lambda} = -i\psi(x)$$

$$\frac{\delta\psi^+}{\delta\lambda} = i\psi^+$$

$\delta\psi = \psi'(x) - \psi(x)$

$= -i\lambda\psi(x)$

$\delta\psi^+ = i\lambda\psi^+$

variation equations ⟺ Lie group coef.

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\lambda} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^+)}\frac{\delta\psi^+}{\delta\lambda} = \partial^\uparrow_\mu \psi^+(-i\psi) + \partial^\uparrow_\mu \psi(i\psi^+)$$

$$= i\left((\partial_\mu\psi)\psi^+ - (\partial_\mu\psi^+)\psi\right)$$

$$j^0 = i(\dot\psi\psi^+ - \dot\psi^+\psi)$$

$$= (\dot\phi_1\phi_2 - \dot\phi_2\phi_1)$$

expressed by corresponding real scalar field

Example 2. $SO(2)$ invariant Real Scalar Field ⟷ $U(1)$ invariant Complex Scalar field

↳ 2×2 matrix, satisfied the Orthogonal Condition

$$SO(2) = \{ R_{2\times2} \mid R^T R = R R^T = \mathbb{1}_2,\ \det R = 1 \}$$

$$R = \exp(i\sigma_y \cdot \lambda), \quad i\sigma_y = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$\in SO(2)$, it is a Lie group

$$= \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}$$

Define the field Vector

$$\vec\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \vec\phi^T = (\phi_1, \phi_2)$$

The corresponding Lagrangian Density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec\phi^T\partial^\mu\vec\phi - \frac{1}{2}m^2\vec\phi^T\cdot\vec\phi - V(\vec\phi^T\vec\phi)$$

check the $SO(2)$ invariant

$\vec\phi \xrightarrow{R} R\vec\phi$

$\vec\phi^T \xrightarrow{R} \vec\phi^T R^T$

$\vec\phi^T\vec\phi \xrightarrow{R} \vec\phi^T\vec\phi$

$\partial_\mu\vec\phi^T\partial^\mu\vec\phi \xrightarrow{R} \partial_\mu\vec\phi^T\partial^\mu\vec\phi$

$\mathcal{L} \to \mathcal{L}$
$S \to S$
so we have the Conserved current.

use the $\vec\phi' = R\vec\phi = \exp(i\sigma_y\lambda)\vec\phi$ to give the variation equations

$$\delta\vec\phi = \vec\phi' - \vec\phi = \lambda(i\sigma_y)\vec\phi, \quad \frac{\delta\vec\phi}{\delta\lambda} = (i\sigma_y)\vec\phi$$

$$\frac{\delta\phi_a}{\delta\lambda} = (i\sigma_y)_{ab}\phi_b$$ ← component equation

We take the Noether's Current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\vec\phi)}\cdot\frac{\delta\vec\phi}{\delta\lambda} = \partial_\mu\vec\phi^T\cdot(i\sigma_y)\vec\phi$$ ← do everything by component

$$= \partial^\mu\phi_1\cdot\phi_2 - \partial^\mu\phi_2\cdot\phi_1$$

Note: $j_\mu = i((\partial_\mu\psi)\psi^+ - (\partial_\mu\psi^+)\psi)$ for Complex Scalar field,

we make the relationship $\psi = \frac{1}{2}(\phi_1 + i\phi_2)$
$\psi^+ = \frac{1}{2}(\phi_1 - i\phi_2)$

we can prof $j_\mu = (\partial^\mu\phi_1)\phi_2 - (\partial^\mu\phi_2)\cdot\phi_1$ ← we have proved it share the same Lagrangian density now, the j^μ is also the same.

X.3.4 The Discrete Symmetries and CPT Theorem

No Noether's Thm
No Conserved Current

Def (of Discrete Symmetries) Parity, time Reversal, Charge Conjugation
(P) (T) (C)

CPT theorem: Any Local Quantum Field theories are invariant under the Joint C.P.T

Quantum field theory

very very Strong Constrain on QFT

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CPT thm $\Rightarrow$ parity violation

3.4.1. Parity (Mirror Symmetry)

Def : $P: t \to t,\ \vec{x} \to -\vec{x}$ → it can be defined this way $P: x^\mu \to x_\mu$

matrix of P: $\begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Delta s^2 = c^2t^2 - \vec{x}^2 \xrightarrow{P} \Delta s^2$ unchanged

which means $P \in SO(1,3)$

But $\det(P) = -1$

which means $P \notin SO_+^\uparrow(1,3)$ physical Lorentz group

Next we talk about the field's change.

Scalar field :

$\phi(x^\mu) \xrightarrow{P} \eta\, \phi(x_\mu)$, $\eta^2 = 1$ $\begin{cases} \eta = 1 & \text{Scalar} \\ \eta = -1 & \text{pseudo-scalar} \end{cases}$

Because $P^2 = \mathrm{Id}$ which means η^2 must be 1.

example 1. $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) - \frac{1}{2}m^2\phi - V(\phi^2)$

To show real Scalar field is invariant under P, we have

$\phi^2(x^\mu) \xrightarrow{P} \eta^2\phi^2(x_\mu) = \phi^2(x_\mu)$

$\partial_\mu\phi \xrightarrow{P} \eta\,\partial^\mu\phi(x_\mu)$

$\partial^\mu\phi \xrightarrow{P} \eta\,\partial_\mu\phi(x_\mu)$

$\mathcal{L}_{(x^\mu)} \xrightarrow{P} \mathcal{L}(x_\mu)$ ← only coordinates change, the function is not changed.

$d^4x \xrightarrow{P} d^4x$

$S \xrightarrow{P} S$

Vector field :

example 2. Maxwell Vector field $A_\mu(x)$

$P: x^\mu \xrightarrow{P} x_\mu$ $\qquad A_\mu(x^\mu) \xrightarrow{P} A^\mu(x_\mu)$

$\partial^\mu \xrightarrow{P} \partial_\mu$

the massive photon $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_\gamma^2 A_\mu A^\mu$

mass term: $A_\mu(t,\vec{x})A^\mu(t,\vec{x}) \to A^\mu A_\mu(t,-\vec{x})$

the ∂_μ term: $\partial_\mu A_\nu(t,\vec{x}) \to \partial^\mu A^\nu(t,-\vec{x})$

$F_{\mu\nu}(t,\vec{x}) \to F^{\mu\nu}(t,-\vec{x})$

注意，计算中 $A \to A'$

$\partial^\mu A'^\nu(x^\sigma)$

$= \partial^\mu A_\nu(x_\sigma)$ → 对自身坐标

$= \partial_\mu A_\nu \big|_{x_\sigma}$ 而不是对对偶坐标求导

the Lagrangian Density

$\mathcal{L}(t,\vec{x}) \to \mathcal{L}(t,-\vec{x})$

$d^4x \to d^4x$

$S \to S$ unchanged , but it does change in some interaction situation

Dirac bi-spinor field

$P: x^\mu \to x_\mu$ $\qquad \underset{4\times1\ \text{matrix}}{\psi(x^\mu)} \to \eta\, \underset{4\times4\ \text{matrix}}{\gamma^0}\, \psi(x_\mu)$

$\eta^2 = 1$

$(\gamma^0)^2 = 1$

example 3.

the Dirac Density $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$

$\bar\psi = \psi^\dagger\gamma^0$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times4}$ ✓ attention.

$(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$

$(i\partial^\mu\partial_\mu - m)\psi = 0$

How it changes is the second homework (under Lorentz transf.)

the change of field: $\psi^\dagger(x^\mu) \to \eta\,\psi^\dagger(x_\mu)\gamma^0$

$\bar\psi(x^\mu) \to \eta\,\bar\psi(x_\mu)\gamma^0$

the mass term: $\bar\psi\psi(x^\mu) \to \eta^2\bar\psi(x_\mu)\underbrace{\gamma^0\gamma^0}_{I_{4\times4}}\psi(x_\mu) = \bar\psi(x_\mu)\psi(x_\mu)$

unchanged except the index of x^μ

other term: $\bar\psi(x^\mu)\gamma^\mu\psi(x^\mu)$

$\to \bar\psi(x_\mu)\gamma^0\gamma^\mu\gamma^0\psi(x_\mu)$ $\qquad \gamma^0\gamma^\mu\gamma^0 = \bar\gamma^\mu = \gamma_\mu$

$= \bar\psi(x_\mu)\gamma_\mu\psi(x_\mu)$ $\qquad \gamma^\mu$ acts like the vector

$\bar\psi(x^\mu)\gamma^\mu\partial_\mu\psi(x^\mu)$

$\to \bar\psi(x_\mu)\gamma_\mu\partial^\mu\psi(x_\mu)$

The Dirac Lagrangian density is unchanged

$\mathcal{L}(x^\mu) \to \mathcal{L}(x_\mu)$

$S \to S$

Time Reversal / Charge Conjugation will be talked in Canonical Quantization $\begin{cases} \text{Scalar} \\ \text{vector} \\ \text{Dirac Bispinor field} \end{cases}$

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Lec XI - XII Canonical Quantization

NOV 15 - 23. 2016

* Homework 2nd, Due NOV, 29, 2016

PDF file for P & S's book, has the name of ... QFT - Non-Commercial ... pdf

look for other books for solution

* Homework 3rd. Due Dec, 20, 2016

Two problems, either problem 1 or problem 2 is encouraged to be choosen. Not necessary to choose both problems.

Problem 1: Research report or project on specific topic in QFT.

1) The topic has to be approved by instructor;
2) The file has to be written in PDF file (.tex, .doc), and E-mailed to Instructor.
3) Excellent research reports are encouraged to be talked by students themselves in the last class (Jan, 3, 2017)

Recommended topics: ① Lorentz group and its representation.

② Poincare group and its representation.

(why photon has two degrees of freedom?)

③ Spin - Statistics theorem. ✓

④ Noether's theorem (proof)

⑤ CPT theorem

⑥ Majorana Fermion (Weyl fermion) (important is Condensed physics and neutrino)

⑦ Vaccum energy and Cosmological constant problem. ✓

⑧ Vaccum energy and Casimir force ✓

(refer to the text book in Quantum field)

⑨ Lorentz invariant Cross section and decay width.

⑩ Columb Scattering in QFT.

⑪ Rutherford scattering in QFT. (Both in Classical mechanics, QM, QFT)

⑫ Dyson's formulao.

(derives it, refer to Dyson's original paper, and find simpler way)

⑬ Wick's theorem.

⑭ One-loop Calculation in QED.

⑮ Regularization and Renormalization in QED.

⑯ Ward Identity in QED.

(and get gauge invariance of QED)

⑰ LSZ reduction and S-matrix

P21-28

1. Def | Canonical Quantization: Quantization, Canonical Quantization

Classical → Quantum (① / ②) not good, not unique

Hamiltonian Formalism (Canonical) V.S. ⟺ Path Integral

2. Steps of Canonical Quantization

Classical ①~⑥ → Quantum ①~⑤

3. Quantization of Free Real Scalar Field

Classical Theory ϕ, π, H, P, M ⇒ K-G eq., Canonical eq. ⇒ plane wave Solution $\alpha(\vec{k})$, $a(\vec{k})$, $\beta(\vec{k})$

Quantization $\hat{\phi}, \hat{\pi}, \hat{H}, \hat{P}, \hat{M}$ ⇒ Heisenberg eq., K-G eq. (Lorentz invariant) ⇒ Plane wave Solution $\alpha(\vec{k})$, $a(\vec{k})$, $\beta(\vec{k})$

$[a, a^\dagger] = \delta \Leftrightarrow [\phi, \hat{\pi}] = i\delta$

$\hat{H}, \hat{\vec{P}} = \hat{H}, \hat{\vec{P}}(a, a^\dagger) \Rightarrow \infty$ SHO

(2개 classical SHO)

⑱ Green function and Generating functions

(G_c, Σ, Γ

Generating function for Green function and special Green function)

⑲ Multi-loop Feynman diagrams.

More topics, which are planed to be talked in the semester, but time is not enough.

Problem 2: The Calculation of differential cross section in the Bhabha Scattering $e^+e^- \to e^+e^-$

(see text book of QFT) ← it is a big subject

(Luke's problem #6, problem 3) ← it is simpler

(OR peskin and Shroeder's book, problem 5.2, p.p. 120)

* Question: Why not talk about the entire Contents of online lec. notes, like what has been done in the past four years?

Answer: Knowledge & Calculation is NOT that important for beginning/beginners in QFT.

Basic Concepts, self-study, and research experience are important for most student.

Note: In fact, for interested students, they are encouraged to check calculations (step by step) in online lec. notes. (do it by yourself)

XI. 1 Def of Canonical Quantization

Def. Quantization: A systematic procedure of constructing quantum theory from classical theory.

→ Quantum theory (big) is somehow depend on classical theory (small), it is impossible tasks).

Note: Quantum Mechanics → Quantum Field theory

Quantization ↑ ↑

Classical particle mechanics → (Continuum limit) Classical Field theory

Note: Quantization is a procedure from the simplest to the Complicated, or from the specific to the General, or from the boundary to the domain; hence, it is essentially a type of Guesswork (impossible) instead of rigrous mathematical derivation.

→ QM, QFT is somehow very ill/bad defined.

Quantization is very ugly thing!

Def. Canonical Quantization : Construction of the Hamiltonian formulation of quantum theory, from the Hamiltonian formulation of classical theory.

→ Because Hamiltonian is energy, but Lagrangian is not obvious.

Note: Why the Canonical Quantization (Name of)

Heisenberg's Matrix QM is constructed by Canonical quantization → Canonical: essential important

Note: In principle, we are able to Construct ∞ number of quantization procedures, for example, path integral quantization, which selects the Lagrangian formulation and plays the essential roles in modern theoretical physics (particles p. standard model condensed p.)

Note: In the Instructor's opinion, no quantization → perfect. best choices

or New quantization → better

or Canonical / Path integral → bad.

Questions: Why choose Canonical Quantization, rather than Path Integral Quantization, for beginners in QFT?

Canonical Quantization: ugly, but OK. best choice for beginner's

Path Integral Quantization: essential in modern theoretical physics

Answer: Comparison of Canonical Quantization and/with Path Integral Quantization. ← Instructor's Opinion

	Canonical Quantization	Path Integral Quantization
Physics	particle - Field duality (obvious) (wave)	The understanding of The Action Principle (QM, all the Path have Contributions the Action principle is wrong)
particle Interpretation	Explicit (defined exactly) (we talk about particle physics. we must see particles)	Not obvious
Probability Amplitude	Not first (particle be derived first)	First (Amplitude is not easy to understand)
Classical - quantum Correspondence	$\hbar \to 0$ Heisenberg Equations → Canonical equations go back to	$\hbar \to 0$ stationary phase → The action principle (part of distribution be selected)
Formulation	Hamiltonian (Energy → Explicit physical meaning)	Lagrangian (Action → very Important advantage.)
Mathematics	Operator-valued function (Everything is matrix)	ordinary function
Symmetry	Lorentz invariance is NOT explicit (Some mathematical calculation needed)	Lorentz invariance is EXPLICIT
Causality	Not obvious	Not obvious
Vaccum energy	∞	∞
physical Quantities (experiments measurements)	∞	∞ ← · Need Extra Specification regularization of Canonical Quantization renormalization of Path Integral · Computatation is useless in INFINITY situation

XI. 2 Recall The Hamiltonian Formulation of CFT (Review) (classical Field Theory)

1°) General Coordinates and Canonical Momentum

$\phi^a(t,\vec{x}) \to \phi^a$ general field coordinate

$\pi^b(t,\vec{y}) \to \pi^b$ Canonical field momentum

Satisfied the Poisson Brackets

$\{\phi^a(t,\vec{x}),\ \phi^b(t,\vec{y})\} = 0, \quad \{\pi^a(t,\vec{x}),\ \pi^b(t,\vec{y})\} = 0$

$\{\phi^a(t,\vec{x}),\ \pi^b(t,\vec{y})\} = \delta^{ab}\delta^3(\vec{x}-\vec{y})$

2°) Hamiltonian $\mathcal{H}$. $H = \int d^3x\, \mathcal{H}(\phi^a, \pi^a)$ ← Conserved energy associated with Space time trasform.

$\mathcal{H} \geq 0$. $H \geq 0$

3°) The Canonical equations

The def of Poisson Brackets $\{f,g\} = \sum_a \left(\frac{\partial f}{\partial \phi^a}\frac{\partial g}{\partial \pi^a} - \frac{\partial f}{\partial \pi^a}\frac{\partial g}{\partial \phi^a}\right)$ ← may be a '−' in other courses

$$\begin{cases} \dot{\phi}^a = \{\phi^a, H\} = \frac{\partial H}{\partial \pi^a} \\ \dot{\pi}^a = \{\pi^a, H\} = -\frac{\partial H}{\partial \phi^a} \end{cases}$$ ← H is independent of time

For Observables $O(\phi^a, \pi^a, t)$

$$\frac{dO}{dt} = \frac{\partial O}{\partial t} + \{O, H\}$$

4°) Symmetry → translation invariance, → Lorentz invariance

Lorentz invariants is not obvious $(\partial^\mu, \partial_\mu)$

Not obvious → Because Hamiltonian formulation select the special time!

5°) Causality (Special Relativity) obvious! in Maxwell equations!

6°) Finite Physical quantities : Finite energy ! OK!

XI 3. Canonical Quantization of Real Scalar Field Theory

→ one by one
not talk general fields's canonical quantization, it is complicated.

1° From ordinary (classical) field coordinates and field momentums to

operator-valued field coordinates and momentums,

satisfying $\hat{\phi}^a(t,\vec{x})$, $\hat{\pi}^b(t,\vec{x})$

$[\hat{\phi}^a(t,\vec{x}), \hat{\phi}^b(t,\vec{y})] = 0$, $[\hat{\pi}^a(t,\vec{x}), \hat{\pi}^b(t,\vec{y})] = 0$

$[\hat{\phi}^a(t,\vec{x}), \hat{\pi}^b(t,\vec{y})] = i\hbar\, \delta^{ab} \delta^3(\vec{x}-\vec{y})$

→ replace $\{\ \} \to [\ ]$
$, \to i\hbar$

2° The Hamiltonian operator

$\mathcal{H} \to \hat{\mathcal{H}}$, $H \to \hat{H}$

$\forall\, |\phi\rangle$, $\langle\phi|\mathcal{H}|\phi\rangle \geq 0$, $\langle\phi|H|\phi\rangle \geq 0$

Note: The operator product ordering ambiguity ← suffer from subtle problem, we have different choice

- $\phi^a\phi^a \to \hat{\phi}^a\hat{\phi}^a$
- $\phi^a\pi^b \to \hat{\phi}^a\hat{\pi}^b$

$\qquad \| $

$\pi^b\phi^a \to \hat{\pi}^b\hat{\phi}^a$

commutative — not commutative

- $\phi^a\pi^b\phi^a \to \hat{\phi}^a\hat{\pi}^b\hat{\phi}^a$

$\qquad \| $

$(\phi^a)^2\pi^b \to \hat{\phi}^a\hat{\phi}^a\hat{\pi}^b$

$\qquad \| $

$\pi^b(\phi^a)^2 \to \hat{\pi}^b\hat{\phi}^a\hat{\phi}^a$

This is unsolved and unavoidable problem in quantization.
(unsolvable)

because quantization is impossible in principle.

3° The Equation of Motion (Heisenberg's Equation of Motion)

$i\hbar\dot{\hat{\phi}}^a = [\hat{\phi}^a, \hat{H}] = \frac{\partial \hat{H}}{\partial \pi^a}$

$i\hbar\dot{\hat{\pi}}^a = [\hat{\pi}^a, \hat{H}] = -\frac{\partial \hat{H}}{\partial \phi^a}$

no logical
just replacement

for general observables $\hat{O}(\hat{\phi}^a, \hat{\pi}^a, t)$

$i\hbar\dot{\hat{O}} = i\hbar\frac{\partial \hat{O}}{\partial t} + [\hat{O}, \hat{H}]$

4° Symmetry

Lorentz invariance NOT EXPLICIT

→ maybe, we construct time-dependent one

- we talk about equal-time Commutators, and Heisenberg's EoM prefers the Special choice of Time.
- But we can prove the Lorentz invariance.

Note: Classical Symmetries maybe violated (in QFT), which is called Anomalies (Check text book)

Quantization is very subtle

Oct, 22, 2016. teaching plan

Canonical quantization (II)

Hamiltonian, Vacuum-Energy, Fock space
micro causality ...

can be derived from Canonical quantization

$\langle q',t'|q,t\rangle = \int Dq(t)\, \exp\frac{i}{\hbar}S[\ ]$ (action)

$= \sum \exp i\frac{S(p)}{\hbar}$

all the paths from $(q,t) \to (q',t')$

In QFT: all paths

In CFT: several Paths

Lec XI - XII Canonical Quantization

2016 11 02

* Answer questions on previous lec.

1) space time translation

space time coordinate translation $x \xrightarrow{T} x' = x + \varepsilon$

field translation $\phi(x) \xrightarrow{U_T} \phi'(x') \equiv U_T\, \phi\, U_T^{+}(x) = \phi(T^{-1}x)$

general fields
(spinor, vector, scalar...)

$U_T \equiv \exp(-i\hat{P}_\mu \varepsilon^\mu)$, $\hat{P}_\mu$: Generator of space time translation

expand U_T in terms of ε^μ

$U_T\,\phi\,U_T^{+}(x) = (1 - i\hat{P}_\mu\varepsilon^\mu)\,\phi\,(1 + i\varepsilon_\mu \hat{P}^\mu)(x)$

$= \phi(x) - i[\hat{P}_\mu, \phi](x)\,\varepsilon^\mu + o(\varepsilon^2)$

$= \phi(x) - \varepsilon^\mu\partial_\mu\phi(x) + o(\varepsilon^2)$

and we have

$i[\hat{P}_\mu, \phi] = \partial_\mu\phi$

$\hat{P}_\mu \to -i\partial_\mu$

generator — representation

2) Lorentz transformation

coordinate $x \xrightarrow{\hat{\Lambda}} x' = \Lambda x$

field $\Phi_\alpha(x) = \Phi'_\alpha(x) = (U_\Lambda \Phi U_\Lambda^{+})_\alpha(x) = D(\Lambda)_{\alpha\beta}\,\Phi_\beta(\Lambda^{-1}x)$

spinor representation

$D(\Lambda) = \exp(-i\omega_{\mu\nu}\frac{S^{\mu\nu}}{2})$ $\qquad S^{\mu\nu}$ → spin angular momentum

$U(\Lambda) = \exp(-i\omega_{\mu\nu}\hat{M}^{\mu\nu})$ $\qquad \omega_{\mu\nu}$ → antisymmetric infinitesimal tensor

→ angular momentum generator

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$$\Phi'_\alpha(x) = \Phi_\alpha(x) - i\omega^{\mu\nu}[\hat{M}_{\mu\nu}, \Phi_\alpha](x) + O(\omega^2)$$

$$= \Phi_\alpha(x) - \frac{i}{2}\omega^{\mu\nu}\left[i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}\right]\Phi_\alpha + O(\omega^2)$$

easy to derive total angular momentum

$$[\hat{M}_{\mu\nu}, \Phi] = \left(i\,(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}\right)\Phi$$

$\hat{M}_{\mu\nu}$: generator of Lorentz group; $i(x_\mu\partial_\nu - x_\nu\partial_\mu)$: orbital angular momentum representation; $S_{\mu\nu}$: spin angular momentum

$\hat{M}_{\mu\nu}$ is operator, generator of Lorentz group.

$[\hat{M}_{\mu\nu}, \Phi] = (L_{\mu\nu} + S_{\mu\nu})\Phi(x) = M_{\mu\nu}\Phi(x)$

$\hat{M}_{\mu\nu}$: group; $M_{\mu\nu}$: representation on this field

3) Representation of group theory

group G, element $g \in G$, $g: V \to V$

$V = \mathrm{span}\{|i\rangle\}$

$g|i\rangle = \sum_j |j\rangle g_{ji}$

representation → representation space V (e.g. Hilbert space)

→ state basis: $|i\rangle$

→ representation matrix g_{ji}

don't get confused

when talk about representation we talk about $\left\{ \begin{array}{l} V \\ |i\rangle\text{s} \\ g_{ji} \end{array} \right.$ together

4) Representation of Lorentz group $SO(1,3)$

$SO(1,3) \approx SU(2) \otimes \bar{SU}(2)$

locally isomorphic

$SO(1,3) \approx SU(2) \oplus SU(2)$?

A good research topic for Homework 3rd (to determine which one is right)

5) Definition of $\left\{ \begin{array}{l} \text{Scalar} \\ \text{Vector} \\ \text{Spinor} \end{array} \right.$, they are just representation of a specific group (under transformation, its transformation Rule determine which field it is)

e.g. $SO(N)$ invariant real scalar field theory

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}, \quad \phi_a,\ a = 1, 2, \cdots, N \text{ are field components}$$

transformation $\phi'_a(x) = R_{ab}\phi_b(x)$, R: $N \times N$ matrix

$\vec{\Phi}$: vector representation under $SO(3)$. $\quad R^T R = R R^T = \mathrm{Id}$, $R \in SO(N)$

e.g. $\Phi'_a(x) = \Phi_a(\Lambda^{-1}x)$, scalar under LT $SO^\uparrow_+(1,3)$

(fixed)

6) Parity (Mirror transf.)

t time component → fixed t

$\vec{x}$ space component → $-\vec{x}$ reverse

$\} \; x^\mu \to x_\mu$

$\vec{v} = \frac{d\vec{x}}{dt} \to -\vec{v}$

$\vec{p} = m\vec{v} \to -\vec{p}$

$\vec{A} \to -\vec{A}$

$A^0 \to A^0$

$) \; A^\mu \to A_\nu$ (upperscripts → lower scripts)

parity in EM

parity in QEM $\quad A_\mu(x_\mu) \to A^\mu(x_\mu)$) experience in Newtonian theory determine the rule in QEM

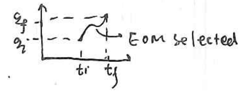
7) Poisson Bracket $\{f, g\} = \sum_a \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p^a} - \frac{\partial f}{\partial p^a}\frac{\partial g}{\partial q^a}$

In QM, $[f, g]_\pm = fg \pm gf$

$\left\{ \begin{array}{l} + \to \text{anticommutator (we don't use } \{\ \}\text{)} \\ - \to \text{commutator} \end{array} \right.$

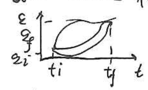
8) Path Integral and action principle

particle picture $\delta S = 0$



→

QM wave picture



$\langle q_2 t_2 | q_1 t_1 \rangle = \int \mathcal{D}q(t) \exp \frac{i}{\hbar} S$

every path has contribution

Path Integral Quantization : particle → field / wave

Canonical Quantization : wave (field) → particle

easily describe particle picture

(this is particle physics)

9) Question 9. History of Canonical Quantization

$\left\{ \begin{array}{l} \text{EOM} \xrightarrow{\text{Replace}} \text{Heisenberg EOM (from Heisenberg) 1P26} \\ \text{Poisson Bracket} \to \text{Commutator (from Dirac)} \end{array} \right.$

• History of Path Integral Quantization

(Realized) $\left\{ \begin{array}{l} \text{Observed by Dirac in 1P30S} \\ \text{defined and Applied by Feynman in 1P40S} \end{array} \right.$

Home word 2nd, Due Nov 29, 2016
Home word 3rd, Due Dec, 20. 2016

* Recall lec XI:

XI 1. Definition of Canonical Quantization

XI 2. Review of Hamiltonian formulation of CFT (Classical field theory)

XI 3. Canonical Quantization of Real Scalar field.
and of General field
(vector, scalar, spinor)

1° From ordinary field Functions $\Phi^a(t,\vec{x})$, $\pi^b(t,\vec{y})$ to operator-valued field functions $\hat{\Phi}^a$ and $\hat{\pi}^b$ (general field).
They satisfying Equal-time Commutator / Anticommutator.

$$[\hat{\Phi}^a(t,\vec{x}), \hat{\Phi}^b(t,\vec{y})]_\pm = 0$$
$$[\hat{\pi}^a(t,\vec{x}), \hat{\pi}^b(t,\vec{y})]_\pm = 0$$
$$[\hat{\Phi}^a(t,\vec{x}), \hat{\pi}^b(t,\vec{y})]_\pm = i\hbar\,\delta^3(\vec{x}-\vec{y})\,\delta^{ab}$$

Thm (spin-statistical theorem)

For Spin-Integer relativistic(al) quantum field theory, Canonical Quantization chooses Commutator $[A,B]_- = AB-BA$;

For Spin-half Integer relativistic quantum field theory, Canonical Quantization chooses anti-commutator $[A,B]_+ = AB+BA$;

Remark:
Commutator $[A,B]_- = AB-BA := [A,B]$ ← prefered
anti Commutator $[A,B]_+ = AB+BA$

← It is very important theorem, answering when we choose $[\ ]_\pm$

Note: Quantization of Scalar, vector → Commutator
Quantization of Dirac-Spinors → Anti Commutator

2° Hamiltonian Operator of General field

$$\mathcal{H}(\Phi^a, \pi^b) \longrightarrow \hat{\mathcal{H}}(\hat{\Phi}^a, \hat{\pi}^b)$$
$$H = \int d^3x\, \mathcal{H} \longrightarrow \hat{H} = \int d^3x\, \hat{\mathcal{H}}$$

Note: Operator-product ordering ambiguity.

Reason: Quantization is a Guesswork from the special limit to General theory.
(boundary theory)

It is impossible in principle, but it can be tested by experiment.

Remark 2: The Hamiltonian operator must be non-negative and finite.

$0 \leq \hat{H} \leq +\infty$ (requirement by physics, by physical meaning of Energy)
which means, for any $|\phi\rangle$, the mean value
$+\infty \geq \langle\phi|H|\phi\rangle \geq 0$

But in fact, in canonical quantization, $\hat{H} \to +\infty$, or minus. (Bad!)
we must do something additional to cancel the divergence.

3° Heisenberg EOM

$$\dot{\Phi}^a = \{\Phi^a, H\} = \frac{\partial H}{\partial \pi^a} \longrightarrow \dot{\hat{\Phi}}^a = -\frac{i}{\hbar}[\hat{\Phi}^a, H] = \frac{\partial \hat{H}}{\partial \hat{\pi}^a}$$

$$\dot{\pi}^a = \{\pi^a, H\} = -\frac{\partial H}{\partial \Phi^a} \qquad \dot{\hat{\pi}}^a = -\frac{i}{\hbar}[\hat{\pi}^a, H] = -\frac{\partial \hat{H}}{\partial \hat{\Phi}^a}$$

↑ valid for {Bosons, Fermions} both.

Note: for any operator,

$$\dot{O} = \frac{\partial}{\partial t}O + \{O, H\} \longrightarrow i\hbar\,\dot{\hat{O}} = \frac{\partial}{\partial t}\hat{O} + [\hat{O}, \hat{H}]$$

4° Micro-Causality and Quantum Measurement

Def: The micro-causality relation between two observables $\hat{O}_1^a(t,\vec{x})$ and $\hat{O}_2^b(t,\vec{y})$ for microscopic world are specified by the commutator $[\hat{O}_1(t,\vec{x}), \hat{O}_2(t,\vec{y})] = 0$ for $(x-y)^2 < 0$

↑ Commutator for any field {vector, bi-spinor}

Note: $(x-y)^2 < 0$ Two points are space-like separated.
(no instantaneous interactor)

$[\hat{O}_1(x), \hat{O}_2(y)] = 0$ for $(x-y)^2 < 0$, The share the same eigenstates. Two quantum measurements by $\hat{O}_1$ and $\hat{O}_2$ do not interfer with each other.

Note: In canonical Quantization, Equal-time Commutator/anti Commutator are specified.
Micro-causality is NOT obvious, and must be verified.

5° Defining Symmetry and Anomaly

Def. Defining Symmetry is obeyed in classical physics, and survives the quantization procedure.
such as Poincaré Symmetry (space time translation and Lorentz Invariance)

Def. Anomaly is a symmetry obeyed in classical physics but violated in quantum physics.
Such as Chiral Anomaly in Chiral Gauge Field Theory.

Note: Defining Symmetry is Crucial

能量.动量.角动量是必须有的
↗ 因而必须存在对应的 Defining Symmetries.

Reason: They define Hamiltonian $\hat{H} = \int d^3x\, \hat{T}^{00}$
Momentum $\hat{P}^i = \int d^3x\, \hat{T}^{0i}$ ⇒ conserved charge of space-time translation
Angular Momentum $\hat{J}^{\rho\sigma} = \int d^3x\, \hat{M}^{0\rho\sigma}$

Reason 2: NOT obvious but must ~~be~~ verified — obviously given in path integral

physical quantities { Expectation value $\langle\cdot|\hat{O}|\cdot\rangle$; probability Amplitude $\langle\cdot|\cdot\rangle$

must be invariant under defining symmetries. (transformation) ← general principle for even quantum gravity

6° Finite physical Quantities ← big problems in QFT

$\langle\cdot|\hat{O}|\cdot\rangle$ expectation value, $\langle\cdot|\cdot\rangle$ amplitude > must be finite in QFT

Note: After Canonical and Path Integral Quantization, we (unavoidably) get ∞ quantities.

what's the reason why we always get ∞ quantities?

QFT is a theory with ∞ number of DOF ← How to overcome it?

Note: Artificial procedures Additional to Quantization are introduced/devised

{ 1) Normal ordering of operator products
2) Regularization (Canonical)
3) renormalization (Path Integral) }
to remove divergences in QFT, to obtain finite physical quantities.

Quantization is also artificial, but always favored by experiements. (up to now).

Note: QFT is Artificially Constructed but always supported by High Energy Experiment up to today,

this means QFT { partly right. partly wrong.

Open question: What's well-defined QFT?

Does it exist? Does it become fundamental?

What about quantum gravity?

XI- XII: 4 Canonical Quantization of free real scalar field theory.

(free: no interaction; scalar: Scalar on Lorentz transformation)

4.1. Classical Theory:

Lagrangian: $\mathcal{L}=\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}m^2\phi^2$

Conjugate momentum: $\pi_\phi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\dot{\phi}$

Hamiltonian density: $\mathcal{H}=\frac{1}{2}(\dot{\phi}^2+(\nabla\phi)^2+m^2\phi^2)\geq 0$ ($m^2\geq 0$, $\mathcal{H}\geq 0$)

$(\dot{\phi}\pi_\phi-\mathcal{L})$

momentum density: $\mathbf{P}=\pi_\phi\nabla\phi$

K-G equation $(\Box+m^2)\phi=0$

4.2 Quantum theory (QFT and Canonical Quantization)

equal-time commutator (spinless, [ ]) $[\hat{\phi}(t,\vec{x}),\hat{\pi}_\phi(t,\vec{y})]=i\hbar\delta^3(\vec{x}-\vec{y})$

Hamiltonian $\hat{\mathcal{H}}(\hat{\phi},\hat{\pi})=\frac{1}{2}(\hat{\dot{\phi}}^2+(\nabla\hat{\phi})^2+m^2\hat{\phi}^2)$ ← no operator product ambiguity. ($\hat{\dot{\phi}}^2=\hat{\pi}_\phi^2$)

Hamiltonian operator $\hat{H}=\int d^3x\,\hat{\mathcal{H}}=\int d^3x\,\frac{1}{2}(\hat{\dot{\phi}}^2+\nabla\hat{\phi}\cdot\nabla\hat{\phi}+m^2\hat{\phi}^2)\geq 0$

EOM: $\begin{cases}\hat{\dot{\phi}}(t,\vec{x})=\hat{\pi}_\phi(t,\vec{x})\\ \hat{\dot{\pi}}_\phi(t,\vec{x})=i[\hat{H},\hat{\pi}_\phi]\Rightarrow(\nabla^2-m^2)\hat{\phi}=\hat{\ddot{\phi}}\end{cases}$ solved like ODE.

$\Rightarrow(\partial_\mu\partial^\mu+m^2)\hat{\phi}(t,\vec{x})=0$ (* K-G eq)

Canonical equation are not explicitly Lorentz invariant.

But (* K-G eq) is Explicitly Lorentz invariant.

Next lec. Tue, 29, Nov, 2016

{ plain-wave solution → on shell condition $k^2=m^2$, to explain the meaning of m^2 (particle mass)
Fock space, Vaccum
Energy, normal ordering, micro-causality. Feynman propagator.
Canonical Quantization of Dirac field theory. }

Lec XIII & XIV Canonical quantization

Nov 29, 2016

*Homework 2nd, Due today

*Homework 3rd, Due Dec. 3rd, 2016

• Teaching plan

Lec XIII. Canonical quantization of Free scalar Field Theory

4.1. Free real scalar field

4.2. Canonical quantization

4.3. Vacuum energy and normal ordering

4.4. Fock space

Lec XIV.

4.5 Micro causality

4.6 Feynman propagator

4.7 Poincare's Algebra

Lec XV Dec 13, 2016

Solution to Homework 1st (Maxwell vector field theory)

Lec XVI Dec 20, 2016

Solution to Homework 2nd (Dirac bispinor field theory)

Lec XVII Dec, 27, 2016

Feynman Rules, S-matrix and Cross-section

Lec XVIII, Jan. 3, 2016

Solution to Homework 3rd

- Research reports
- differential cross-section of Bhabha scattering (get answer in text book of QFT)

• teaching idea/strategy

2012~2016: calculation ⇓ online lec. notes

2016: basic Concepts, Self-study, research experience

(27)

- What we have done on Canonical Quantization ?
  1) Def. Canonical quantization
  2) Hamiltonian Formulation of General Classical field theory.
  3) Canonical quantization of General Classical Field.

4.1. Classical theory of real scalar field

- $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, $m^2 \geq 0$
  (it is particle mass)

$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$

$\mathcal{H} = \dot\phi\,\pi_\phi - \mathcal{L} = \frac{1}{2}(\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2) \geq 0$

(energy must be non-negative number)

EOM (Klein-Gordan equation):

$$(\partial_\mu\partial^\mu + m^2)\phi = 0$$

※ Note: Solution of K-G eq.

$$\phi(t,\vec{x}) = \int \frac{d^4k}{(2\pi)^{3/2}} \phi(k)^{(4)} e^{-ik\cdot x}\,\delta(k^2-m^2)$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\, 2w_{\vec{k}}} \left(\phi(\vec{k})e^{-ik\cdot x} + \phi^*(\vec{k})e^{ik\cdot x}\right)$$

- Plain wave solution (of Klein-Gordan equation):

$$\phi(t,\vec{x}) = \int d^3\vec{k}\, \left(\alpha(\vec{k})\, e^{-ik\cdot x} + \alpha^*(\vec{k})\, e^{ik\cdot x}\right)$$

- just fourier transform of Klein Gordan equation/field
- we need other Constrains

$$(\Box^2 + m^2)\phi = 0 \iff k^2 = m^2$$

- this is mass-shell relation
- only after quantization, we will know meaning of m^2
- $\vec{k}$ is wave vector, $k = (w_{\vec{k}}, \vec{k})$, $w_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

Def. $a(\vec{k}) = (2\pi)^{3/2}\sqrt{2w_{\vec{k}}}\;\alpha(\vec{k})$, Define new coefficient, (no particle here)

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \cdot \frac{1}{\sqrt{2w_{\vec{k}}}} \cdot \left(a(\vec{k})\, e^{-ik\cdot x} + a^*(\vec{k})\, e^{ik\cdot x}\right)$$

- after quantization, this is Hamonic oscillator, and will simplify the solution
- Different solution formulations exist.

一般用四矢记号 $\beta(k)$

Note: Def $\beta(\vec{k}) = (2\pi)^{3/2}\sqrt{2w_{\vec{k}}}\; a(\vec{k})$

$$\phi(t,\vec{x}) = \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3\, 2w_{\vec{k}}}}_{} \frac{1}{}\left(\beta(\vec{k})\, e^{-ik\cdot x} + \beta^*(\vec{k})\, e^{ik\cdot x}\right)$$

this is Lorentz invariant integral measure. easy for { calculation of Cross section ...

find the proof of Lorentz invariant on lec. notes.

- Space-time translation of Energy-momentum tensor and

$x' = Tx = x + \varepsilon$

$\phi'_T(x) = U_T\phi\, U_T^\dagger(x) = \phi(T^{-1}x)$

$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \dashrightarrow \partial_\mu T^{\mu\nu} = 0$

we now can get Harmiltonian density and momentum density (Conserved charge)

$T^{00} = \dot\phi^2 - \mathcal{L} = \mathcal{H} \geq 0$ (energy)

$T^{0i} = \dot\phi\,\partial^i\phi = p^i$ (momentum) , $\vec{p} = \pi\nabla\phi$

- Space-time Lorentz invariance and Angular-momentum tensor

$x' = \Lambda x$

$\phi'_\Lambda(x) = U_\Lambda\phi\, U_\Lambda^{-1}(x) = \phi(\Lambda^{-1}x)$

the angular momentum tensor

$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \dashrightarrow \partial_\mu M^{\mu\rho\sigma} = 0$ (Conserved current)

$J^{\rho\sigma} = \int d^3x\, M^{0\rho\sigma}$

(Conserved charge)

4.2 Canonical Quantization of real Scalar field

- Equal time commutator : $[\hat\phi(t,\vec{x}), \hat\pi_\phi(t,\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y})$

Equal: not obvious Lorentz invariance
commutator: for Dirac field, we use anti-commutator

Hamiltonian $\hat{\mathcal{H}}(\hat\phi, \hat\pi_\phi) = \frac{1}{2}(\dot{\hat\phi}^2 + (\nabla\hat\phi)^2 + m^2\hat\phi^2) \geq 0$

EOM :
$$\begin{cases} \dot{\hat\phi}(t,\vec{x}) = \hat\pi_\phi(t,\vec{x}) & \text{(def of momentum)} \\ \dot{\hat\pi}_\phi(t,\vec{x}) = -i[\hat\pi, \hat H] = \nabla^2\hat\phi - m^2\hat\phi \end{cases} \Rightarrow (\partial_\mu\partial^\mu + m^2)\hat\phi(t,\vec{x}) = 0$$

not explicit Lorentz invariant

the EOM is explicit Lorentz invariant. (physical result)

this formulation like Klein Gordan eq. but, it is for operator

- The plane wave solution (Satisfying the on-shell condition $k^2 = m^2$) → no particle here until talking about Fock space

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}(\vec{k}) e^{-ik\cdot x} + \hat{a}^{+}(\vec{k}) e^{ik\cdot x} \right)$$

→ harmonic osillator

$$= \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_{\vec{k}}} \left(\hat{\beta}(\vec{k}) e^{-ik\cdot x} + \hat{\beta}^{+}(\vec{k}) e^{ik\cdot x} \right)$$

→ easy for Lorentz invariant { decay, Feynman rules, physical quantity computation }

$$= \int d^3\vec{k} \left(\hat{\alpha}(\vec{k}) e^{-ik\cdot x} + \hat{\alpha}^{+}(\vec{k}) e^{ik\cdot x} \right)$$

→ pure fourier transform
when use transform, computation could be easy.

Lemma 1: In canonical quantization. we talk about field operator

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}_\phi(t,\vec{y})] = i\hbar\, \delta^3(\vec{x}-\vec{y})$$

$$[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0$$

The commutator can be replaced by commutator between $\hat{a}(\vec{k})$, $\hat{a}^{+}(\vec{k})$

$$\Leftrightarrow \quad [a(\vec{k}), a^{+}(\vec{k}')] = \delta^3(\vec{k}-\vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = [a^{+}(\vec{k}), a^{+}(\vec{k}')] = 0$$

Note: $\hat{a}(\vec{k}), \hat{a}^{+}(\vec{k})$ are associated with Harmonic Oscillator (this is why we choose $\hat{\beta}(\vec{k}), \hat{\alpha}(\vec{k})$)

Check online lec. notes to get proof. (crucial)

Lemma 2:

$$\hat{H} = \frac{1}{2}\int d^3x \left(\hat{\pi}_\phi^{\,2} + (\nabla\phi)^2 + m^2\hat{\phi} \right) = \frac{1}{2}\int d^3\vec{k}\, \hbar\omega_{\vec{k}} \left(a(\vec{k})a^{+}(\vec{k}) + a^{+}(\vec{k})a(\vec{k}) \right)$$

↑ use field operator ↑ in momentum space

Note: we get Hamiltonian for simple Harmonic oscillator.

Theorem: free real scalar field theory (quantum) is equivalent to infinity number of independent simple harmonic osillators (by $a(\vec{k})$ and $a^{+}(\vec{k})$)
↓
naive but crucial

Note:	Harmonic oscillator	QFT (free real scalar field theory)
$a(\vec{k})$	lowering operator	Annihilation operator
$a^{+}(\vec{k})$	raising operator	Creation operator
$\lvert 0\rangle$ $\lvert \text{Vacuum}\rangle$	ground state $a(\vec{k})\lvert 0\rangle = 0$	$a(\vec{k})\lvert vac\rangle = 0$

↓
maybe Black hole
entire theoretical physics (Harmonic oscillator)

4.3 Vacuum energy and Normal ordering

P28-29 Vacuum energy problem { $E_{vac} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}\hbar\omega_{\vec{k}}\cdot V \to \infty$, $\rho_{vac} \to \infty$ }

Box periodic condition $E_{vac} \to \infty$ } ∞ DOF ↓ ∞ $\vec{k}$, ∞ SHO ↓ ∞ E_{vac} ↕ cosmological constant problem

Casimir force ↔

- The vacuum $\lvert vac\rangle$ defined by

$$a(\vec{k})\lvert vac\rangle = 0, \quad \forall\, \vec{k}$$

way out

$\lvert vac\rangle$ is destroyed by entire domian of $a(\vec{k})$

very strong constrain

normal ordering

def: $N[\;] = :\;:$

∵ 对 boson 的 [illegible] { artificial / when $\hbar \to 0$ both go to classical }

The Hamiltonian (infinity num of SHOs)

$$\hat{H} = \int d^3\vec{k}\, \frac{1}{2}\hbar\omega_{\vec{k}} \left(a^{+}(\vec{k})a(\vec{k}) + a(\vec{k})a^{+}(\vec{k}) \right)$$

$:H: = \int d^3k\, a^{+}_{\vec{k}} a_{\vec{k}} \omega_{\vec{k}}$, $E_{vac} = 0$

$$[a(\vec{k}), a^{+}(\vec{k}')] = \delta^3(\vec{k}-\vec{k}')$$

$$= \int d^3\vec{k}\, \hbar\omega_{\vec{k}} \left[a^{+}(\vec{k})a(\vec{k}) \right] + \frac{1}{2}\hbar\omega_{\vec{k}}\, \delta^3(0)$$

↓
zero point energy, infinity number of ∞
(we choose β or α, then no SHO. SHO is artificial)

- The vacuum energy

$$E_{vac} = \langle vac \rvert \hat{H} \lvert vac\rangle$$

$$= \int d^3\vec{k}\, \frac{1}{2}\hbar\omega_{\vec{k}}\, \delta^3(0)$$

Note: Dirac-delta function is defined by $\delta^3(\vec{k}) = \int \frac{d^3\vec{x}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}$

$$\delta^3(0) = \int \frac{d^3\vec{x}}{(2\pi)^3}\cdot 1 = \frac{V}{(2\pi)^3}$$

So that vacuum energy $E_{vac} = V\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}\hbar\omega_{\vec{k}}$

$V \to +\infty \Rightarrow E_{vac} \to +\infty$

This is called Infrared divergence (associated with Volumn)

The energy density $\rho_{vac} = \frac{E_{vac}}{V} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}\hbar\sqrt{\vec{k}^2+m^2} \to +\infty$

$\vec{k} \to +\infty \quad \rho_{vac} \to +\infty$

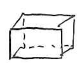
This is still a divergence.

It is called Ultra-violet divergence.

QFT is not well defined from the starting point.

Remark 1: The oringin of divergence in E_{vac}, ρ_{vac}

The box normalization Condition (QFT in a box with periodic boundary condition)

 $\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$

The Hamiltonian $H = \sum_{\vec{k}} \frac{\hbar\omega_{\vec{k}}}{2}\left(a_{\vec{k}} a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger} a_{\vec{k}}\right)$, $[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta^3_{\vec{k}\vec{k}'}$

$= \frac{1}{2}\sum_{\vec{k}} \hbar\omega_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} + \frac{1}{2}\sum_{\vec{k}} \hbar\omega_{\vec{k}}$

finite, just zero point energy of a SHO

The oringin of ∞ Evac. ρvac is

∞ number of SHO

The origin of ∞ num of SHO is

Field has ∞ number of DOF

Maybe there is no way for solving the problem, for creating correct QFT. Or QFT is not well defined. String theory, but also has infinity number.

Remark 2 : The Cosmological Constant problem

The observed vacuum energy $\hat{}$ density of the Universe is almost zero

$\rho_{vac}^{experiment} \approx 0$

But the theoretical calculation/prediction

$\rho_{vac}^{theory} \to \infty$

$\gg \rho_{vac}^{experiment}$

What's the way out? (search google) no hope!

Remark 3 : Casmir force (verified by experiment)

QFT in a box with periodic boundary conditional. When the vacuum is disturbed, there is observed phenomenon.

Remark 4. With gravity, the vacuum energy in the current space time can be observed. Quantum gravity must get rid of the divergence of vacuum energy.

4.3. Normal ordering & Normal ordered product.

In flat space-time, the vacuum energy is the absolute energy, and only the relative energy difference is the observed quantity. This means the vacuum energy in the flat-space time can be removed by artificial procedures such as normal ordering.

Def. Normal ordered product (in free real scalar field theory) is defined in the following way:

$: a^{\dagger}(\vec{k})\, a(\vec{k}) : = : a(\vec{k})\, a^{\dagger}(\vec{k}) : = a^{\dagger}(\vec{k})\, a(\vec{k})$

In side normal ordering $: \dots :$, the bosonic operators for free spin-integer fields are always commutative.

As the result, the annihilation operators are always on LHS, and the Creation operators are always on RHS.

just artificial, in flat space-time to get rid of infinity num of DOFs and infinity energy.

Example : The normal ordering of $\hat{H}$

$: \hat{H} : = \int d^3\vec{k}\, \frac{1}{2}\hbar\omega_{\vec{k}} : a_{\vec{k}}^{\dagger} a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^{\dagger} : = \int d^3\vec{k}\, \frac{1}{2}\hbar\omega_{\vec{k}} \left(a_{\vec{k}}^{\dagger} a_{\vec{k}} + a_{\vec{k}}^{\dagger} a_{\vec{k}}\right)$

Killed $[a(\vec{k}), a^{\dagger}(\vec{k})]$

$= \int d^3\vec{k}\, \hbar\omega_{\vec{k}}\, a_{\vec{k}}^{\dagger} a_{\vec{k}}$

$E_{vac} = \langle vac | :\hat{H}: | vac \rangle = 0$

now, everything becomes well-defined...

but it can't be used to talk about gravity/cosmological problem.

Note: Normal ordering is artificial, Physics is not artificial } With normal-ordering and without normal-ordering, the same classical theory must be obtained when $\hbar \to 0$

(Physics can't be changed)

(So normal-ordering is just looking like artificial)

Check: $H = \frac{1}{2}(p^2 + q^2)$ ← SHO in classical physics

$:H: = \hbar\omega \hat{a}^{\dagger}\hat{a}$ ← with normal ordering

$H = \hbar\omega(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})$ ← without normal ordering

$:H: = \hbar\omega \hat{a}^{\dagger}\hat{a} = \hbar\omega \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \frac{\hat{q} + i\hat{p}}{\sqrt{2}}$

$H = \hbar\omega(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}) = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$

Physical Interpretion of Normal ordering

Teaching Plan

① Fock Space (particle is derived)
② micro-causality
③ Feynman propagator
④ Poincare Algebra.

about renormalization

QFT → divergence (types) { Vacuum Energy ⇒ removed by normal-ordering (normalization); Probability Amplitude (Feynman Rules, S-matrix) ⇒ removed by regularization and renormalization }

about group

QFT (Mathematics) — Ill-defined ← supports QFT — particle physics (experiments) Result of HEP

QFT, needed to be reformulated (beautiful theory)

ugly theory → beautiful result

(30)

Lec XIV Canonical quantization   2016 1206

Homework 3rd, due Dec 20, 2016

Last Tuesday Canonical quantization of real free scalar field theory

- 4.1 Classical Theory
- 4.2 Canonical quantization
- 4.3 Vacuum Energy and Normal ordered procedure

Today's topic

- 4.4 Fock ~~space~~ particle space in momentum representation
- 4.5 Fock space in coordinate representation
- 4.6 Micro Causality and quantum measurement
- * 4.7 Time-ordered product and Feynman propagator (special example) (Motivation, kill divergence of vacuum energy)
- * 4.8 Realization of Poincare Algebra.

Quantum measurement ⇒ QM
causality ⇒ SR
(micro ⇒ small particle)
} QFT = QM + SR

Subject for research report (Topic)

what about definitions of vector and spinors in $SO(1,3)$
$(\Lambda^{\mu}{}_{\nu})$ $S(\Lambda)_a{}^b$

4.4 Fock space in momentum representation

Def. Fock space is the state space for quantum field theory (and/or particle physics), and is the state space for any number of particles, denoted by

$$\mathcal{F} = \bigoplus_{n=0}^{+\infty} \mathcal{H}^{(n)} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(n)} \oplus \cdots$$

· particle can be destroyed or created in high energy.

1. $\mathcal{H}^{(0)}$: no-particle space (vacuum) $|vac\rangle \in \mathcal{H}^{(0)}$

   $\mathcal{H}^{(n)}$: n-particle space

   $\mathcal{H}^{(n)}$, usually NOT the Hilbert space in QM (particle number can be changed) and the Completeness relation is strange, defined for the Entire Fock Space

2. Fock space defines the concept of particles for particle physics.

Def. Fock space in momentum representation

1. particles in Free Real Scalar Field Theory: Freely moving spinless particles obeying the Bose-Einstein statistics

2. Observables: The Hamiltonian $\hat{H} = \int d^3x \,\hat{\mathcal{H}}$
   The momentum $\hat{P} = \int d^3x \,\pi_\phi \nabla \hat{\phi}$ } from translation
   The Angular momentum $\hat{J} = \begin{cases} \text{orbital} \\ \text{spin} \end{cases}$, from LT. (exist but always zero) see P & S's book

   · Feynman diagrams: particles + propagators

Normal ordering:
$$:\hat{H}: = \int d^3k \, \hbar\omega_{\vec{k}} \, a^{+}_{\vec{k}} a_{\vec{k}}$$
$$:\hat{P}: = \int d^3k \, \vec{k} \, a^{+}(\vec{k}) a(\vec{k})$$

Note: $:\hat{H}: \neq \hat{H}$, but $:\hat{P}: = \hat{P}$

$$\hat{P} = \int d^3k \, \vec{k} \, \tfrac{1}{2}\left(a^{+}_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^{+}_{\vec{k}}\right)$$

$\mathcal{H}^{(0)}$: No particle space $|vac\rangle$ $|vac\rangle \neq |0\rangle$ ground state

$a(\vec{k})|vac\rangle = 0$, for $\forall \vec{k}$ ⇒ QFT's statement

properties: $\langle vac|vac\rangle = 1$

$:\hat{P}:|vac\rangle = 0$, $:\hat{H}:|vac\rangle = 0$, $:\hat{J}:|vac\rangle = 0$

$\mathcal{H}^{(1)}$: one-particle space

$|\vec{k}\rangle = a^{+}(\vec{k})|vac\rangle$

properties: $\langle \vec{k}'|\vec{k}\rangle = \delta^3(\vec{k}-\vec{k}')$ ⇒ same as QM;
$\Leftrightarrow [a_{\vec{k}}, a^{+}_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}')$ equivalent to commutator below

$\vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$, $:\hat{H}:|\vec{k}\rangle = \hbar\omega_{\vec{k}}|\vec{k}\rangle$

Note: $k^{\mu} = (k^0, \vec{k})$, $k^0 = \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

$\mathcal{H}^{(2)}$: Two-particle space

$|\vec{k}_1, \vec{k}_2\rangle = a^{+}(\vec{k}_1) a^{+}(\vec{k}_2)|vac\rangle$ ⇒ advantage of Canonical Quantization; Impossible in Path Integral

$[a^{+}_{\vec{k}}, a^{+}_{\vec{k}'}] = 0$

$\Leftrightarrow |\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$

Bose-Einstein statistics
(derived in QFT, but assumed in QM)

properties:
$$\langle \vec{k}_1' \vec{k}_2' | \vec{k}_1 \vec{k}_2\rangle = \delta^3(\vec{k}_1 - \vec{k}_1')\delta^3(\vec{k}_2 - \vec{k}_2') + \delta^3(\vec{k}_1 - \vec{k}_2')\delta^3(\vec{k}_2 - \vec{k}_1')$$
↑ Einstein Statistics

$:\hat{H}:|\vec{k}_1, \vec{k}_2\rangle = (\hbar\omega_{\vec{k}_1} + \hbar\omega_{\vec{k}_2})|\vec{k}_1, \vec{k}_2\rangle$

$:\hat{P}:|\vec{k}_1, \vec{k}_2\rangle = (\vec{k}_1 + \vec{k}_2)|\vec{k}_1, \vec{k}_2\rangle$

Note: These (assumption) are NOT assumption; can be derived by commutator of $a_{\vec{k}}, a^{+}_{\vec{k}'}$

Properties of the Entire Fock space:

1. The completeness relation
$$Id = |vac\rangle\langle vac| + \sum_{n=1}^{\infty} \frac{1}{n!}\int d^3k_1 \cdots d^3k_n \, |\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n\rangle\langle \vec{k}_1 \cdots \vec{k}_n|$$

It is defined for the entire Fock space
$\frac{1}{n!}$ means Statistics
$\mathcal{H}^{(n)}$ is not the usual Hilbert space.

P30-34 Fock Space

1. Def
   - Fock space $F = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$
   - $a_{\vec{k}}$, $a^{+}_{\vec{k}}$. $|vac\rangle$, $|\vec{k}\rangle$, $|\vec{k}_1 \vec{k}_2\rangle$. [illegible]
   - observable $:\hat{H}:$ $:\hat{P}:$ $:\hat{J}:$

2. Momentum Representation
   $|\vec{k}\rangle = a^{+}_{\vec{k}}|vac\rangle$ 是具有动量[illegible]/能量[illegible]

   - $\mathcal{H}^{(0)}$ – $|vac\rangle$ – $P/H\,|vac\rangle = 0$
   - $\mathcal{H}^{(1)}$ – $|\vec{k}\rangle$ – $P/H\,|\vec{k}\rangle = \vec{p}/\omega_{\vec{k}}\,|\vec{k}\rangle$
   - $\mathcal{H}^{(2)}$ – $|\vec{k}_1, \vec{k}_2\rangle$ – $P/H\,|\vec{k}\rangle = \vec{p}_1+\vec{p}_2 / \omega_{\vec{k}_1}+\omega_{\vec{k}_2}\,|\vec{k}_1, \vec{k}_2\rangle$

   sp{ $a^{+}_{\vec{k}}|vac\rangle = |\vec{k}\rangle, \ldots$ } rotation invariant
   sp{ $\beta_k|vac\rangle = |k\rangle, \ldots$ } Lorentz invariant
   $\delta(\vec{k}-\vec{p})(2\pi)^3 2\omega_{\vec{k}}$ Lorentz invariant

3. Coordinate Representation
   $\hat{\phi}(0,\vec{x})|vac\rangle$ { 在x位置产生一个粒子 / [illegible]
   $m \gg k$ → $|\vec{x}\rangle$; $\langle p|\hat{\phi}|vac\rangle$ → $e^{-i\vec{p}\cdot\vec{x}}$

   Propagator
   $\hat{\phi}(t,\vec{x})$ → $\hat{\phi}^{(+)}(t,\vec{x})$, $a(\vec{k})$; → $\hat{\phi}^{(-)}(t,\vec{x})$, $a^{+}(\vec{k})$

   norm

   $D(x-y) = \langle vac|\hat{\phi}(x)\hat{\phi}(y)|vac\rangle = \langle vac|[\hat{\phi}^{(+)}(x), \hat{\phi}^{(-)}(y)]|vac\rangle = [\phi^{(+)}(x), \phi^{(-)}(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik(x-y)}$

   代表粒子传播 [illegible] 是 Green func

   $D(x-y) - D(y-x) = \langle vac|[\hat{\phi}(x), \hat{\phi}(y)]|vac\rangle = [\hat{\phi}(x), \hat{\phi}(y)]$

   $D(x-y)$ Lorentz invariant, $\neq 0$
   Causality Broken

   way out 1 → $D_F(x-y)$ Time ordered product

   way out 2 → micro-causality (measurement)
   $[\phi(x), \phi(y)]_{(x-y)^2<0} = D(x-y) - D(y-x)\big|_{(x-y)^2<0} = 0$
   ⇕
   equal time commutation
   ⇕ existence of anti-particle pair; ⇕ CPT theorem $D(x-y) \xleftrightarrow{CPT} D(y-x)$

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2. Thm 1

The Fock space created by $a(\vec{k})$ and $a^\dagger(\vec{k})$ is rotation-invariant under $SO(3)$.

namely, $\langle \vec{p} | \vec{k} \rangle = \delta^3(\vec{p}-\vec{k})$ is invariant under $SO(3)$

↳ the probability amplitude is invariant under $SO(3)$ automatically

Note: $\int d^3\vec{k}\, \delta^3(\vec{p}-\vec{k}) = 1 \to SO(3)$ invariant (↑ $SO(3)$ invariant)

verifying can be got on online lec notes.

$d^3\vec{k}^R = \det(R)\, d^3\vec{k}$, $\det(R) = 1$ $SO(3)$ invariant

$\delta^3(\vec{p}-\vec{k}) = \langle \vec{p} | \vec{k} \rangle$ also Lorentz $SO(3)$ invariant

基矢的内积一致 Lorentz invariance $\Rightarrow$ 基矢 Lorentz invariance $\Rightarrow$ Fock space Lorentz invariance

3. Thm 2. (Construct Lorentz invariant Fock space)

The Fock space created by $\beta(\vec{k}) = (2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}\, a(\vec{k})$

and $\beta^\dagger(\vec{k}) = (2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}\, a^\dagger(\vec{k})$

is invariant under Lorentz transformation.

namely $\langle p | k \rangle = (2\pi)^3 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{p})$ is invariant under LT.

$|k\rangle = \beta^\dagger(k)|vac\rangle$, $|p\rangle = \beta^\dagger(p)|vac\rangle$

Note: see online Lec. Notes.

$$\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \cdot 2\omega_{\vec{k}}\, \delta(\vec{k}-\vec{p}) = 1 \to \text{Lorentz invariant}$$

also Lorentz invariant

also LT invariant; it is what we want (probability amplitude)

Feynman diagrams is created/constructed using this Lorentz invariant Fock space. It is automatically LT Invairant.

In particle physics calculations of probability amplitude, we choose Fock space by $\beta(k)$ and $\beta^\dagger(k)$, for example, Feymann Rules.

4.5 Fock space in Coordinate representation

Statement 1: $|k\rangle = \beta^\dagger(k)|vac\rangle$ means creating one-particle in momentum representation

Statement 2: $\hat{\phi}(0,\vec{x})|vac\rangle$ means creating ① a particle at the ② position $\vec{x}$ in Schrödinger picture.

↑ field operator

Note: $\hat{\phi}(0,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \left(\beta(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + \beta^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right)$

$\hat{\phi}(0,\vec{x})|vac\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{-i\vec{k}\cdot\vec{x}} |k\rangle$

Guess: $\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}$, $|\vec{k}| \ll m \Rightarrow 2\omega_{\vec{k}} \approx 2m$ (low energy limit)

$\hat{\phi}(0,\vec{x})|vac\rangle \propto \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} |\vec{k}\rangle \propto |\vec{x}\rangle$ in QM

proof: $\langle p | \hat{\phi}(0,\vec{x}) | vac \rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{-i\vec{k}\cdot\vec{x}} \langle p | k \rangle = e^{-i\vec{p}\cdot\vec{x}}$

$\langle p|k\rangle = (2\pi)^3 2\omega_p\, \delta(\vec{p}-\vec{k})$

it means $\hat{\phi}(0,\vec{x})|vac\rangle$ is a particle at position $\vec{x}$

Note: In Non-relativistic QM, we define $|\vec{x}\rangle$ by

$\hat{x}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$

$\langle \vec{p} | \vec{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{p}\cdot\vec{x}}$

Statement 3: Normalization in coordinate rep.

$D(x-y) = \langle vac | \hat{\phi}(x)\hat{\phi}(y) | vac \rangle$

called D-function

1) The probability amplitude

→ propagation of a free particle from $\vec{y}$ to $\vec{x}$ (probability amplitude) $y \to x$

→ creating a particle at y and then destroyed at x

Calculation: $\phi(t,\vec{x}) = \phi^{(+)}(t,\vec{x}) + \phi^{(-)}(t,\vec{x})$

positive frequency ← $a(\vec{k})$; associated with $a^\dagger(\vec{k})$ → negative frequency

although we use field operator, the commutator $[a(\vec{k}), a^\dagger(\vec{k}')]$ is importantant

$$\phi^{(+)}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\, a(\vec{k})\, e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}}$$

$$\phi^{(-)}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\, a^\dagger(\vec{k})\, e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}}$$

In QM, $i\partial_t \leftrightarrow \hat{H}$.

$i\partial_t e^{-i\omega_{\vec{k}}t} = \omega_{\vec{k}}$ positive energy

$i\partial_t e^{i\omega_{\vec{k}}t} = -\omega_{\vec{k}}$ negative energy

In relativistic QM, it suffers from negative-energy problem

In QFT, no such problem

negative/positive energy $\Rightarrow$ just what we call it / just teminal

2) Compute $D(x-y) = \langle vac | \phi(x)\phi(y) | vac \rangle = \langle vac | \phi^{(+)}(x)\phi^{(-)}(y) | vac \rangle$

$= \langle vac | [\phi^{(+)}(x), \phi^{(-)}(y)] | vac \rangle$

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$$= \langle vac| \int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^3 \sqrt{2\omega_k}\sqrt{2\omega_{k'}}} \underbrace{[a(\vec{k}), a^\dagger(\vec{k}')]}_{\delta^3(\vec{k}-\vec{k}')} e^{-ik\cdot x + ik'\cdot y} |vac\rangle$$

$$= \langle vac| \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} |vac\rangle$$

$$= [\phi^{(+)}(x), \phi^{(-)}(y)] = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} \neq 0$$

- get exact result from text book
  $D(x-y) \neq 0$ is the most important result
- 另外. 证明 [illegible] 可以看出 $\langle vac|\phi(x)\phi(y)|vac\rangle = D(x-y)$ [illegible] 有 [illegible] 不 [illegible]
- 另外 $D(x-y) = \langle vac|[\phi(x), \phi(y)]|vac\rangle = [\phi(x), \phi(y)]$ ← 常数 [illegible]
  $\langle vac|[\phi^{(+)}(x), \phi^{(-)}(y)]|vac\rangle = [\phi^{(+)}(x), \phi^{(-)}(y)]$

Note 1. $D(x-y) = \nearrow^{x}_{y} \neq 0$

$D(y-x) = \nearrow^{y}_{x} \neq 0$

Note 2. $D(x-y)\big|_{x^0=y^0} = D(y-x)\big|_{y^0=x^0}$ equal time, two D-func is the same.

Because $\phi(x)\phi(y)\big|_{x^0=y^0} = \phi(y)\phi(x)\big|_{x^0=y^0}$ ⇒ equal time commutator.

Note 3. $D(x-y) \neq 0$ for any x, y

It means $\nearrow^{x}_{y}$ probability $\neq 0$ for SPACE-LIKE interval $(x-y)^2 < 0$

This means causality for probability amplitude is broken, (trick but interesting).
violated

Note 4. $\langle vac|\hat\phi(x)\hat\phi(y)|vac\rangle$ is NOT well defined in QFT,
↑ Too dangerous to talk about

We define Time-Orderded product, it is welled defined in QFT. ⇐ one solution to violation of Causality
(Feynmen propagator)

$D_R(x-y) = \theta(x^0 - y^0)\langle vac|\phi(x)\phi(y)|vac\rangle$ (Retarded)
Constrain time ordering

$D_A(x-y) =$ or $\theta(y^0 - x^0)\langle vac|\phi(y)\phi(x)|vac\rangle$ (Advanced)

$D_F(x-y) = D_R(x-y) + D_A(x-y)$ (Feynman propagator)

4.6 Micro-Causality and quantum measurement

probability violating causality, is OK!

Quantum measurement obey causality, is Crucial! (violation doesn't appear in Measurement)

quantum measurement obey causality ⇐ another Solution to violation of Causality

Note: In Quantum Mechanics / physics, the fact that two quantum measurement by $\hat{O}_1$ and $\hat{O}_2$ do not interfer with each other means $[\hat{O}_1, \hat{O}_2] = 0$
→ Bell states?

Thm: Micro-Causality in quantum real scalar field theory cane be expressed as

$$[\hat\phi(x), \hat\phi(y)] = 0 \quad \text{for } (x-y)^2 < 0$$

亦即：$(x-y)^2<0$ 时两次测量不互相干扰

It can be verified by using the supposed equal-time commutator in Canonical quantization

$$[\hat\phi(t, \vec{x}), \hat\phi(t, \vec{y})] = 0$$

equal time Commutator
⇒ automatically result in micro-causality

proof: see online lec. notes

simple version:
$$[\hat\phi(x), \hat\phi(y)]_{(x-y)^2<0} = [\hat\phi^{(+)}(x), \hat\phi^{(-)}(y)] + [\hat\phi^{(-)}(x), \hat\phi^{(+)}(y)]$$
$$= D(x-y) - D(y-x)$$

D-func is Lorentz invariant

$$\overset{L.T.\ \Lambda}{=} D(\Lambda(x-y)) - D(\Lambda(y-x))$$
$(\Lambda x)^0 = (\Lambda y)^0$
$$= [D(\Lambda x - \Lambda y) - D(\Lambda y - \Lambda x)]_{(\Lambda x)^0 = (\Lambda y)^0}$$
$$= [\phi(\Lambda x), \phi(\Lambda y)]_{(\Lambda x)^0 = (\Lambda y)^0} = 0$$

Does causality violate in anti-particle

Scalar particle:
particle = anti-particle

In QFT with Charges
particle ≠ anti-particle

the micro-causality predicts the existence of anti-particle

In real Scalar field T
$$[\hat\phi(x), \hat\phi(y)]_{(x-y)^2<0} = 0$$

In Complex Scalar FT with charge
$$[\psi(x), \psi^\dagger(x)]_{(x-y)^2<0} = 0$$

$$\langle vac|\psi(x)\psi^\dagger(y)|vac\rangle = \langle vac|\psi^\dagger(y)\psi(x)|vac\rangle$$

$\uparrow^{x}_{y}$ particle x to y $\qquad$ $\uparrow^{y}_{x}$ anti-particle y to x

LecXV Canonical Quantization

* Homework 3rd, due Dec 20, 2016

* Final exam schedule

During, Jan 8-21, 2017

Any suggestion, ASAP

Final exam

open ; classnotes & Homeworks

Easy ; Basic concepts

Recall last lecture :

4.4 Fock space in Momentum representation.

4.5 Fock space in Coordinate representation.

4.6 Micro-Causality & Quantum measurement.

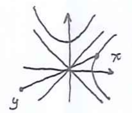
4.7 Time-ordered and Feynman Propagator.

Comments on Micro-Causality

Note 1: $D(x-y) = \langle vac | \hat{\Phi}(x)\hat{\Phi}(y) | vac \rangle$

$\neq 0$ for $(x-y)^2 < 0$ space like region ← dangerous hint

How to solve..?



Causality in SR is violated !

But not observable !

way out 1: Quantum measurement survive the micro-Causality relation:

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_{(x-y)^2<0} = 0$$

Quantum measurement in SPACE-LIKE Regions do not interfer with each other.

Namely, particle's space-like travelling can not be observed !

Remark : Gravity ! What about micro-causality?

(Standard model has nothing to do with Gravity)

Way out 2: Time ordered product

Causality (is Time-ordering problem)

we impose constrain

$$\theta(x^0-y^0) \langle vac | \hat{\Phi}(x)\hat{\Phi}(y) | vac \rangle$$

$$\theta(x^0-y^0) [\hat{\Phi}(x), \hat{\Phi}(y)]$$ ← Retarded Green Function

Time ordering is fixed by the step function. (we call it time-ordered product),

so that the violation of the causality is Automatically forbidden.

experiment supports that argument.

Remark: The micro-Causality requires the existence of Anti-particles.

free real scalar field theory, particle = antiparticle (antiparticle doesn't need to appear)

In a general QFT, the micro-Causality relation $[\hat{\Phi}(x), \hat{\Phi}^\dagger(y)]_{(x-y)^2<0} = 0$.

Plane wave :

$$\hat{\Phi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(b(\vec{k}) e^{-ik\cdot x} + c^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

$$\hat{\Phi}^\dagger(y) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(b^\dagger(\vec{k}) e^{ik\cdot y} + c(\vec{k}) e^{-ik\cdot y} \right)$$

$b(\vec{k}) \to$ annihilation of particle

$c(\vec{k}) \to$ annihilation of anti-particle

$[c, c^\dagger] = \delta^3$

$[b, b^\dagger] = \delta^3$

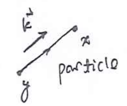
$[b, c] = 0$ ← independent harmonic oscillator

Compute propagation of y to x :

$$\langle vac | \hat{\Phi}(x) \hat{\Phi}^\dagger(y) | vac \rangle = \int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}\sqrt{2\omega_{\vec{k}'}}} e^{-ik\cdot x} e^{ik\cdot y} \langle vac | b(\vec{k}') b^\dagger(\vec{k}) | vac \rangle \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik\cdot(x-y)} \neq 0$$

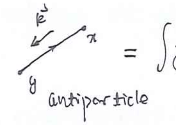
$\langle vac | \hat{\Phi}(x) \hat{\Phi}^\dagger(y) | vac \rangle =$



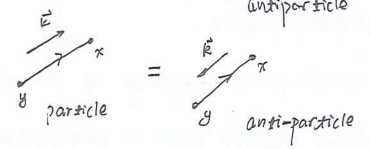
Change the ordering :

$\langle vac | \hat{\Phi}^\dagger(y) \hat{\Phi}(x) | vac \rangle \propto \langle vac | c^\dagger(\vec{k}') c(\vec{k}) | vac \rangle$ maybe wrong $c(\vec{k}')$, $c^\dagger(\vec{k})$

$=$



$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{ik\cdot(x-y)}$$



→ particle, anti-particle must creat in pair

→ otherwise, micro-causality is killed.

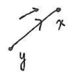
Note 3: The Micro-Causality and CPT theorem (time reversal; charge conjugation; parity)

(discrete symmetries)

(particle ↔ anti particle)

$P \cdot T[x-y] = y-x$

$C[\text{particle}] = \text{anti-particle}$ (anti-particle → particle)

$D(x-y) =$  $\xleftrightarrow{CPT}$  $= D(y-x)$

CPT theorem ⟺ micro-causality

CPT theorem is right ⟹ micro-causality is right

Note 4: Mathematical origin of micro-causality

$D(x-y) = D(y-x)$, $(x-y)^2 < 0$ — Lorentz transformation $D(\Lambda(x-y)) = D(\Lambda(y-x))$

CPT theorem (proved by representation of Lorentz group)

anti-particle — micro-causality relation

} ways to prove the $D(x-y) = D(y-x)$

P34 - P38 Feynman Propagator { Green func. / Time Ordered / Lorentz invariance

1. Def | $D_F(x-y) = \Delta_F(x-y) = \langle vac | T\hat{\phi}(x)\hat{\phi}(y) | vac\rangle$

$T[\hat{\phi}(x)\hat{\phi}(y)]$, $N[\hat{\phi}(x)\hat{\phi}(y)]$ } 内部 ~ bosonic operator 成立

2. Wick's theorem (simple form)

$D_F(x-y) = \overbrace{\hat{\phi}(x)\hat{\phi}(y)} = T\hat{\phi}(x)\hat{\phi}(y) - N[\hat{\phi}(x)\hat{\phi}(y)]$

(for micro-causality) (kill vacuum)

3. Complex Contour Integral

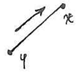
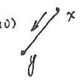
$$D_F(x-y) = \int_C \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} e^{-ik\cdot(x-y)} = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} e^{-ik\cdot(x-y)}$$

Operator of latest time on left most.

$$T\,\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) = \begin{cases} t_1 > t_2 > \cdots > t_n & \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) \\ \vdots & \\ t_n > t_{n-1} > \cdots > t_1 & \hat{\phi}(x_n)\hat{\phi}(x_{n-1})\cdots\hat{\phi}(x_1) \end{cases}$$

$$D_F(x-y) = \theta(x^0-y^0)\langle vac|\hat{\phi}(x)\hat{\phi}(y)|vac\rangle + \theta(y^0-x^0)\langle vac|\hat{\phi}(y)\hat{\phi}(x)|vac\rangle$$

$$= \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)$$

$= \theta(x^0-y^0)$  $+\ \theta(y^0-x^0)$ 

particle travelling from $y \to x$ — particle travelling from $x \to y$

← from micro-causality, we knows Time-ordering is unavoidable, and it is the essential part of modern QFT, we can naturally give S-matrix and compute the scattering amplitude.

← also, solution to divergence of E_{vac}: normal-ordering product

4.6 Time-ordered product and Feynman propagator.

Feynman propagator plays a crucial role in perturbative QFT (in the view point of Feynman diagram)

Time-ordered product is the way to solve the causality problem.

Note:
perturbative 需要 D
solve causality 需要 $\theta(x^0-y^0)$) $\xrightarrow{\text{Lorentz 不变}}$ D_F

Def. Feynman Propagator

$D_F(x-y) = \Delta_F(x-y) = \langle vac | T\hat{\phi}(x)\hat{\phi}(y) | vac\rangle$

Dyson (maybe) — Feynman

where "T" stands for time-ordering of the operator product $\hat{\phi}(x)\hat{\phi}(y)$

(check Feynman's original paper to get natural understanding of $D_F(x-y)$)

$$T\,\hat{\phi}(x)\hat{\phi}(y) \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & t_x > t_y \\ \hat{\phi}(y)\hat{\phi}(x) & t_x < t_y \end{cases}$$

Time-ordering of operator products always gives: ↙ general def. of time-ordering product.

Operator at earliest time on the right most

operator at next to earliest time on next to right most ...

Thm 1. $D_F(x-y) = \overbrace{\hat{\phi}(x)\hat{\phi}(y)}^{\text{contraction}}$

$= T\hat{\phi}(x)\hat{\phi}(y) - N[\hat{\phi}(x)\hat{\phi}(y)]$

↑ time-ordering — ↑ normal-ordering

Note: This thm is special example of the Wick's thm in perturbative QFT. ↗ essential point to derive Feynman rules.

online lec notes 2012-2015 for one-semester = for two semester course.

$\overbrace{\hat{\phi}(x)\hat{\phi}(y)}$: Contraction between $\hat{\phi}(x)$ and $\hat{\phi}(y)$

$T\hat{\phi}(x)\hat{\phi}(y)$: Time-ordering for micro-causality

$N[\hat{\phi}(x)\hat{\phi}(y)] = :\hat{\phi}(x)\hat{\phi}(y):$ Normal-ordering for Finite vacuum energy.

Note: $\langle vac| T\hat{\phi}(x)\hat{\phi}(y)|vac\rangle = D_F(x-y)$

↳ This means normal-ordering is killed in $\langle vac| \cdots |vac\rangle$

Normal-ordered product is Automatically killed by EPV of vacuum.

Proof: $\hat{\phi}(x) = \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x)$

annihilation (positive frequency) — creation (negative frequency)

$N[\hat\phi(x)\hat\phi(y)] = :\ \hat\phi^{(+)}(x)\hat\phi^{(+)}(y) + \underbrace{\hat\phi^{(+)}(x)\hat\phi^{(-)}(y)}_{\text{key point}} + \hat\phi^{(-)}(x)\hat\phi^{(+)}(y) + \hat\phi^{(-)}(x)\hat\phi^{(-)}(y)\ :$

annihilation on the right (kill vacuum energy)
creation on the left

$$= \hat\phi^{(+)}(x)\hat\phi^{(+)}(y) + \hat\phi^{(-)}(y)\hat\phi^{(+)}(x) + \hat\phi^{(-)}(x)\hat\phi^{(+)}(y) + \hat\phi^{(-)}(x)\hat\phi^{(-)}(y)$$
$$+ \hat\phi^{(+)}(x)\hat\phi^{(-)}(y) - \hat\phi^{(+)}(x)\hat\phi^{(-)}(y)$$
$$= \hat\phi(x)\hat\phi(y) + (-1)[\hat\phi^{(+)}(x), \hat\phi^{(-)}(y)]$$

replace $x \leftrightarrow y$

$$N[\hat\phi(y)\hat\phi(x)] = \hat\phi(y)\hat\phi(x) - [\hat\phi^{(+)}(y), \hat\phi^{(-)}(x)]$$

In normal-ordering, for bosonic-type operators, $N[\hat\phi(x)\hat\phi(y)] = N[\hat\phi(y)\hat\phi(x)]$

inside normal-ordering, operators commute with each other.
e.g. $N[a a^\dagger] = N[a^\dagger a] = a^\dagger a$

Inside ~~Time~~-ordering, for bosonic-type operators, $T\hat\phi(x_1)\hat\phi(x_2) = T\hat\phi(x_2)\hat\phi(x_1)$

for fermion, it is anti-commutative inside T, N

$$\begin{cases}\hat\phi(x)\hat\phi(y) = N[\hat\phi(x)\hat\phi(y)] + D(x-y)\\ \hat\phi(y)\hat\phi(x) = N[\hat\phi(x)\hat\phi(y)] + D(y-x)\end{cases}$$
$$\Updownarrow$$
$$T\hat\phi(x)\hat\phi(y) = N[\hat\phi(x)\hat\phi(y)] + D_F(x-y)$$
$$= \begin{cases} t_x > t_y, & \hat\phi(x)\hat\phi(y) = :\hat\phi(x)\hat\phi(y): + D(x-y)\\ t_x < t_y, & \hat\phi(y)\hat\phi(x) = :\hat\phi(x)\hat\phi(y): + D(y-x)\end{cases}$$

see online lec note for another type of proof. from T to N.

- Time-ordering & normal-ordering is ugly but deep, and D_F is useful.

Remark: Time-ordering $\left\langle\right.$ for micro-causality; Simplified notation for many equations; simplest form of Wick's theorem. (key point to derive Feynman Rules)

Wick's theorem : the time-ordered product is a linear combination of products of Feyman propagators and normal-ordered products.

why it is useful? because normal-ordering kills vacuum, so the Wick's theorem greatly simplifies calculation.

Dyson's formula
Scattering matrix
Feynman Rules

2 semester (Basic concept): Excellent student (Basic concepts, self study, research calculation); majority (survive)

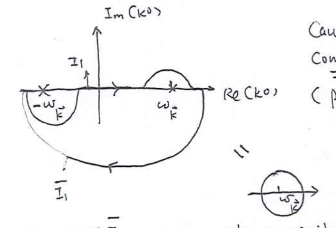
Remark: Teaching experiment — online lec notes (calculation) 2012–2015, 1 Semester: excellent students survives; Majority kicked out

Thm 2: Complex-variable Contour Integral

$$D_F(x-y) = \int_C \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik\cdot(x-y)} = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x-y)}$$
(C: Contour; $\varepsilon \to 0^+$)

$x^0 > y^0$, clockwise contour C_1 on the lower-half-plane of the complex frequence plane.

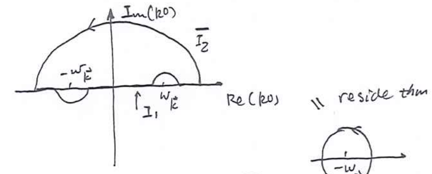
- This is complex analysis
- This is another def. of D_F



Cauchy thm.
Complex energy! (positive/minus energy)

$C_1 = I_1 \cup \bar I_1$, only pole contributes to integral (reside theorem)

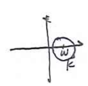
$x^0 < y^0$ Anti-clock contour C_2 on the upper-half-plane of complex frequency plane. (energy)



$C_2 = I_1 \cup \bar I_2$

proof: Def of D_F:
$$D_F(x-y) = \theta(x^0-y^0)D(x-y) + D(y^0-x^0)D(y-x)$$
$$D(x-y) = \int\frac{d^3\vec k}{(2\pi)^3 2\omega_{\vec k}}e^{-ik\cdot(x-y)}$$
$x^0 > y^0$. $$D_F(x-y) = \int\frac{d^3\vec k}{(2\pi)^3 2\omega_{\vec k}}e^{-ik\cdot(x-y)}$$

and Complex-variable Contour $$\int_{C_1}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)} = \int\frac{d^3\vec k}{(2\pi)^3}\oint_{C_1}\frac{dk_0}{2\pi}\frac{i}{k_0^2-\omega_{\vec k}^2}e^{-ik^0\cdot(x^0-y^0)}e^{i\vec k\cdot(\vec x-\vec y)}$$



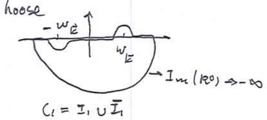
$$= \int\frac{d^3\vec k}{(2\pi)^3}\oint\frac{dk^0}{2\pi}\frac{i e^{-ik^0(x^0-y^0)}}{(k^0-\omega_{\vec k})(k^0+\omega_{\vec k})}e^{i\vec k\cdot(\vec x-\vec y)}$$
$$\oint \to -2\pi i\,\frac{i e^{-i\omega_{\vec k}(x^0-y^0)}}{2\omega_{\vec k}}$$

$e^{-ik^0(x^0-y^0)} = e^{-i\,Re(k^0)(x^0-y^0)}e^{Im(k^0)(x^0-y^0)}$, $x^0-y^0>0$

$Im(k^0) \to -\infty$, $e^{-ik^0(x^0-y^0)} \to e^{-\infty} \to 0$

$$= \int\frac{d^3\vec k}{(2\pi)^3}\frac{e^{-i\omega_{\vec k}(x^0-y^0)}}{2\omega_{\vec k}}e^{i\vec k(\vec x-\vec y)}$$
$$= \int\frac{d^3\vec k}{(2\pi)^3}\frac{e^{-ik\cdot(x-y)}}{2\omega_{\vec k}}$$

choose



$C_1 = I_1 \cup \bar I_1$

$\bar I_1 \Rightarrow$ boundary term has no contribution

when $x^0 < y^0$, $$D_F(x-y) = \int\frac{d^3\vec k}{(2\pi)^3 2\omega_{\vec k}}e^{ik\cdot(x-y)} = D(y-x)$$
the contour is changed, and anticlockwise

Solution to Homework 1st. 2016, Dec. 20

* Homework 3rd. Due Dec 20. 2016

* Final exam

Time Schedule Fixed Next Tuesday

Open test → Classnotes, Homeworks

Basic concepts and basic calculation

Notation on Electrodynamics

First type convention $\vec{E} = (E^i) = (E_x, E_y, E_z)$ field strength $\vec{x} = (x, y, z) = (x^i)$

$\vec{B} = (B^i) = (B_x, B_y, B_z)$ $\varepsilon_{123} = 1$

$\nabla = (\partial_i) = (\partial_x, \partial_y, \partial_z)$

They are physics quantities (in QED, what is physics quantities is not easy to study)

Second type convention

Gauge potential ϕ (Scalar) $\vec{A}$ (Vector)

field strength $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$. $\vec{B} = \nabla \times \vec{A}$

$B^i = \varepsilon_{ijk}\partial_j A^k$

Third type convention

Four vector Calculus $A^\mu = (A^0, \vec{A}) = (\phi, \vec{A})$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Electric/magnetic field strength $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} = -\partial_i A^0 - \partial_0 A^i = F^{i0}$

$\vec{B} = (\nabla \times \vec{A})_i = \varepsilon_{ijk}\partial_j A^k = -\frac{1}{2}\varepsilon_{ijk}F^{jk}$

$F^{ij} = -\varepsilon_{ijk}B^k$

Units on Electrodynamics { Gauss, Natural, International (not easy)

Some formula $\varepsilon_{emi}\varepsilon_{jki} = \delta_{ej}\delta_{mk} - \delta_{ek}\delta_{mj}$

General Field : Φ

Euler Lagrangian Equation for Φ^β: $\frac{\partial \mathcal{L}}{\partial \Phi^\beta} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial_\alpha \Phi^\beta}$ (Summation, component)

Lagrangian $\mathcal{L} = \mathcal{L}(\Phi^\beta, \partial_\alpha \Phi^\beta)$

In Lagrangian Formulation, Φ^β and $\partial_\alpha \Phi^\beta$ are independent variables, like q and $\dot{q}$ are independent in $L(q, \dot{q}, t)$

Note that in Hamiltonian Formulation Φ^β and π_{Φ^β} are independent variable

Maxwell Vector Field

1. Notation { $\vec{E}, \vec{B}$ → physical; $\phi, \vec{A}$ → gauge; $F^{\mu\nu}, A^\mu$ → Four vec/Tensor

2. Units { ① ② ③

3. [illegible] $\mathcal{L} = -\frac{1}{2}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) + \frac{1}{2}(\partial\cdot A)^2 + \frac{\mu_\gamma^2}{2}A^2$

$\mathcal{L}' = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu_\gamma^2}{2}A^2 = \mathcal{L} + \frac{1}{2}\partial^\mu\left((\partial\cdot A)A_\mu + (A\cdot\partial)A_\mu\right)$

gauge invariant

EOM $[\Box + \mu_\gamma^2]A_\beta = \partial_\beta(\partial\cdot A)$

$\Leftrightarrow \partial^\alpha F_{\alpha\beta} = -\mu_\gamma^2 A_\beta$ → $\mu_\gamma^2 = 0 \Rightarrow$ Maxwell eq. $\partial^\alpha F_{\alpha\beta} = 0$

$\mu_\gamma^2 \partial\cdot A = 0$, $\nu \cdot \partial X$ ⇒ { $\mu_\gamma^2 = 0$. $\partial\cdot A = 0$ (by hand), $\Box A_\beta = 0$; $\mu_\gamma^2 \neq 0$ $\partial\cdot A = 0$ (natural), $[\Box + \mu_\gamma^2]A_\beta = 0$

4. Maxwell eq. { $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\nabla\cdot\vec{B} = 0$ ⇔ def with A, Topology ⇔ Bianchi $\partial_\mu F_{\alpha\beta} = 0$

{ $\nabla\cdot\vec{E} = 0$, $\nabla\times\vec{B} = \frac{\partial \vec{E}}{\partial t}$ ⇔ $\partial^\alpha F_{\alpha\beta} = 0$

5. Symmetries { Translation δx^ρ: $\frac{\delta x^\rho}{\delta \xi_\mu} = \delta^\rho_\mu$, $\frac{\delta A_\nu}{\delta \xi_\mu} = 0$; Lorentz transf. $\frac{\delta x^\rho}{\delta \omega^{\alpha\beta}} = \frac{1}{2}(\delta^\rho_\alpha x_\beta - \delta^\rho_\beta x_\alpha)$, $\frac{\delta A_\nu}{\delta \omega^{\alpha\beta}} = \frac{\varepsilon_0}{2}\left(\delta_{\nu\alpha}A_\beta - \delta_{\nu\beta}A_\alpha\right)$, $\frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$

E-M tensor $T^{\mu\nu} = \frac{1}{4}\eta^{\mu\nu}F^2 - F^{\mu\rho}\partial^\nu A_\rho$

$\theta^{\mu\nu} = T^{\mu\nu} - F^{\mu\beta}\partial_\beta A^\nu = \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\eta^{\mu\nu} - F^{\mu\beta}F^\nu{}_\beta$

Angular Tensor

$M^{\mu\alpha\beta} = j'^{\mu\alpha\beta}(-2) = x^\alpha T'^{\mu\beta} - x^\beta T'^{\mu\alpha} + F^{\mu\beta}A^\alpha - F^{\mu\alpha}A^\beta$

Lagrangian formulation for Electrodynamics $\mathcal{L} = -\frac{1}{2}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) + \frac{1}{2}(\partial\cdot A)^2 + \frac{\mu_\gamma^2}{2}A^2$ (mass of photon)

(a) $(\Phi^\beta \leftrightarrow A^\beta)$ $\frac{\partial \mathcal{L}}{\partial A^\beta} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\beta)}$

$(A^2) = (A^0)^2 - \vec{A}^2$. $\frac{\partial (A^2)}{\partial A^0} = 2A^0$. $\frac{\partial (A^2)}{\partial A^1} = -2A^1$

Calculation step by step $\frac{\partial \mathcal{L}}{\partial A^\beta} = \frac{\mu_\gamma^2}{2}\frac{\partial (A^2)}{\partial A^\beta} \times 2A_\beta = \mu_\gamma^2 A_\beta$

or $\frac{\partial \mathcal{L}}{\partial A_\beta} = \frac{\mu_\gamma^2}{2}\frac{\partial (A^2)}{\partial A_\beta} = \frac{\mu_\gamma^2}{2}\frac{\partial}{\partial A_\beta}(A^\alpha A_\alpha)$

$\frac{\partial}{\partial A^\beta}(A^\alpha A_\alpha)$ $\underline{\underline{\text{Chain rules}}}$ $\left(\frac{\partial}{\partial A^\beta}A^\alpha\right)A_\alpha + A^\alpha\left(\frac{\partial A_\alpha}{\partial A^\beta}\right) = \eta^\alpha_\beta A_\alpha + A^\alpha \eta_{\alpha\beta} = 2A_\beta$

Right-hand side

$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -\frac{1}{2}\frac{\partial}{\partial(\partial_\alpha A^\beta)}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \frac{1}{2}\frac{\partial}{\partial(\partial_\alpha A^\beta)}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$

$\frac{\partial}{\partial(\partial_\alpha A^\beta)}\left[(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right] = \frac{\partial}{\partial(\partial_\alpha A^\beta)}\left[(\partial_\mu A^\nu)(\partial^\mu A_\nu)\right]$

$\partial_\alpha A^\beta$ and $\partial_\mu A^\nu$ are independent variables when $\alpha \neq \mu, \beta \neq \nu$

$\frac{\partial}{\partial(\partial_\alpha A^\beta)}(\partial_\mu A^\nu) = \frac{\partial}{\partial(\partial_\alpha A^\beta)}\left(\eta_\mu{}^\alpha \partial_\alpha \eta^\nu_\beta A^\beta\right) = \frac{\partial}{\partial(\partial_\alpha A^\beta)}\left(\eta^\alpha_\mu \eta^\nu_\beta \partial_\alpha A^\beta\right) = \eta^\alpha_\mu \eta^\nu_\beta$

$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -\frac{1}{2}\left(\eta^\alpha_\mu \eta_{\nu\beta}\,\partial^\mu A^\nu + \eta^\alpha_\mu \eta_{\nu\beta}\,\partial^\mu A^\nu\right) - \frac{1}{2}\left(\delta^\alpha_\beta\,\partial\cdot A + \partial\cdot A\,\delta^\alpha_\beta\right)$

$\frac{\partial}{\partial(\partial_\alpha A^\beta)}(\partial\cdot A) = \begin{cases}\alpha\neq\beta, & 0 \\ \alpha=\beta, & 1\end{cases} = \eta^\alpha_\beta = \delta^\alpha_\beta$

$\partial\cdot A = \partial_\alpha A^\alpha = \partial_\alpha(\eta^\alpha_\beta A^\beta) = \eta^\alpha_\beta \partial_\alpha A^\beta$

Now, the equation of motion

$\frac{\partial \mathcal{L}}{\partial A^\beta} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\beta)} \Rightarrow \mu_\gamma^2 A_\beta = -\partial_\alpha\left(\partial^\alpha A_\beta - \delta^\alpha_\beta(\partial\cdot A)\right)$

$\Leftrightarrow \partial_\alpha\left(\partial^\alpha A_\beta - \delta^\alpha_\beta(\partial\cdot A)\right) + \mu_\gamma^2 A_\beta = 0$

in real experimental observation

(b) get field strength equations

equation 1st. From $\partial_\alpha\partial^\alpha A_\beta - \partial_\beta(\partial_\alpha A^\alpha) + \mu_\gamma^2 A_\beta = 0$

$\Leftrightarrow \partial_\alpha(\partial^\alpha A_\beta - \partial_\beta A^\alpha) + \mu_\gamma^2 A_\beta = 0$

$\Leftrightarrow \partial_\alpha F^\alpha{}_\beta = -\mu_\gamma^2 A_\beta$ $\Leftrightarrow \partial^\alpha F_{\alpha\beta} = -\mu_\gamma^2 A_\beta$

When $\mu_\gamma^2 = 0$, we obtain $\partial^\alpha F_{\alpha\beta} = 0$

equation 2nd. $\Box A_\beta - \partial_\beta(\partial\cdot A) + \mu_\gamma^2 A_\beta = 0$

$\Leftrightarrow (\Box + \mu_\gamma^2)A_\beta = \partial_\beta(\partial\cdot A)$ just looks like Klein-Gordon equation

(How to convert it to K-G equation?)

$$\begin{cases} (\Box+\mu_r^2)A_\beta = \partial_\beta(\partial\cdot A) \\ \partial_\alpha F^{\alpha\beta} = -\mu_r^2 A_\beta \end{cases}$$

Apply $\partial\beta$ on both sides, we have $\partial_\alpha\partial^\beta F^\alpha{}_\beta = -\mu_r^2\,\partial\cdot A$

$$\underbrace{\partial_\alpha\partial_\beta}_{\text{symmetric}} \underbrace{F^{\alpha\beta}}_{\text{antisymmetric}} = 0 = -\mu_r^2\,\partial\cdot A$$

Remark 1:

when $\mu_r^2 \neq 0$, we have $\partial\cdot A = 0$ (Lorentz gauge potential)

when $\mu_r^2 \neq 0$ (photon mass), photon massive, Lorentz gauge fixing is intrinsic in Theory.

when $\mu_r^2 = 0$, photon massless, $\partial\cdot A$ is imposed by HAND

Photon is massive or massless?

Remark 2:

Research report: Lorentz gauge and photon mass

(Coulumb gauge may be also relate to photon mass)

when $\mu_r^2 \neq 0$, $(\Box+\mu_r^2)A_\alpha = 0$ Klein-Gordon eq.

when $\mu_r^2 = 0$, $\Box A_\alpha = \partial_\alpha(\partial\cdot A)$ + Lorentz gauge fixing

Same equation (K-G) $\Rightarrow$ different physical meaning

Remark 3: Here, $\mathcal{L} = -\frac{1}{2}(\partial_\alpha A^\beta)^2 + \frac{1}{2}(\partial\cdot A)^2 + \frac{\mu_r^2}{2}A^2 \Rightarrow \partial^\alpha F_{\alpha\beta} = -\mu_r^2 A_\beta$

and other forms of $\mathcal{L}' = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\mu_r^2}{2}A^2 = -\frac{1}{4}F^2 + \frac{\mu_r^2}{2}A^2$

Euler Lagrange equation: $\frac{\partial\mathcal{L}'}{\partial(A_\beta)} = \mu_r^2 A^\beta$

$$\frac{\partial\mathcal{L}'}{\partial(\partial_\alpha A^\beta)} = \frac{\partial}{\partial(\partial_\alpha A^\beta)}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$= \frac{\partial}{\partial(\partial_\alpha A^\beta)}\left[-\frac{1}{4}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right]$$

$$= -F^\alpha{}_\beta$$

$$\Rightarrow \mu_r^2 A^\beta = -\partial_\alpha F^{\alpha}{}_\beta$$

Different Langrangian density, but SAME EOM. The secret is the action

$\mathcal{L} \neq \mathcal{L}'$, $\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu$ $\quad \mu_r^2 A^\beta = -\partial_\alpha F^\alpha{}_\beta$

is the same, and EOM is derived from the action principle, the total derivative

$$S = \int d^4x\,\mathcal{L} = \int d^4x\,\mathcal{L}' = \int d^4x\,\mathcal{L} + \int d^4x\,\partial_\mu \Lambda^\mu$$

vanishs on the boundary after using the Gassian theorem.

check: $\mathcal{L} = -\frac{1}{2}(\partial_\alpha A^\beta)^2 + \frac{1}{2}(\partial\cdot A)^2 + \frac{\mu_r^2}{2}A^2$

$\mathcal{L}' = -\frac{1}{2}(\partial_\alpha A_\beta)^2 + \frac{1}{2}(\partial_\alpha A^\beta)(\partial^\beta A^\alpha) + \frac{\mu_r^2}{2}A^2$

$$(\partial_\alpha A_\beta)(\partial^\beta A^\alpha) = \partial_\alpha(A_\beta(\partial^\beta A^\alpha)) - A_\beta\partial_\alpha\partial^\beta A^\alpha$$

$$= \partial_\alpha(A_\beta\partial^\beta A^\alpha) - \partial^\beta(A_\beta\partial_\alpha A^\alpha) + (\partial\cdot A)^2$$

$$= \underbrace{\partial_\alpha(A_\beta\partial^\beta A^\alpha - A^\alpha\partial_\beta A^\beta)}_{\partial_\alpha\Lambda^\alpha} + (\partial\cdot A)^2$$

$\partial_\alpha\Lambda^\alpha$, vanishing on the boundary term in the action. not determining the EOM.

why $\mathcal{L}'$ is popular than $\mathcal{L}$?

The Reason is that, $F_{\mu\nu}$ is physical quantity (Gauge invariant)

(In QFT, gauge invariant is the most important symmetry)

Hence, $\mathcal{L} = -\frac{1}{4}F^2$ is also gauge invariant.

Get maxwell equations:

first type $\begin{cases} \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \end{cases}$ $\quad$ second type $\begin{cases} \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} \end{cases}$

$\Updownarrow$ simple $\quad$ $\Updownarrow$ Simple

Bianchi identity $\quad$ Euler Lagrangian Equations

$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$ $\quad$ $\partial_\alpha F^{\alpha\beta} = 0$

Lorentz invariance is more obviously gained.

Remark 1. Why only $\partial_\alpha F^{\alpha\beta}$ are dynamics equations? what about Bianchi identity?

It is TOPOLOGY and GEOMETRY (Identity).

prove the $\Updownarrow$:

Bianchi identity is from the definition of the gauge potential.

$\begin{cases} \vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t} \\ \vec{B} = \nabla\times\vec{A} \end{cases}$ $\quad (\vec{E},\vec{B}) \leftrightarrow (\varphi,\vec{A})$

$$\nabla\times\vec{E} = \nabla\times(-\nabla\varphi) - \frac{\partial(\nabla\times\vec{A})}{\partial t} = -\frac{\partial\vec{B}}{\partial t}$$

$\nabla\times\nabla\varphi = 0$

Because

$$\nabla\times\nabla\varphi = \varepsilon_{ijk}\partial_j\partial_k\varphi = 0$$

(ε_{ijk} totally anti-symmetric, $\partial_j\partial_k$ symmetric)

$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$

Because $\nabla\cdot(\nabla\times\vec{A}) = \nabla\cdot(\varepsilon_{ijk}\partial_j A_k) = \varepsilon_{ijk}\partial_i\partial_j A_k = 0$ (ε_{ijk} anti-symmetric, $\partial_i\partial_j$ symmetric)

Derive from Bianchi identity:

$\alpha=0,\ \beta=i,\ \gamma=j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$\Rightarrow \partial_t\left(-\varepsilon^{ijk}B^k\right) + \partial^i E^j - \partial^j E^i = 0$$

$$\Rightarrow \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$\alpha=i,\ \beta=j,\ \gamma=k$

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\Rightarrow \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0$$

$$\Rightarrow \nabla\cdot\vec{B} = 0$$

idea:

maxwell eqs (first type) $\Leftrightarrow$ Bianchi

first prove maxwell eq right $\Rightarrow$ Bianchi right

first prove Bianchi right $\Rightarrow$ maxwell right

Deriving first type has been done.

Derive the second type: $\partial_\alpha F^{\alpha\beta} = 0 \Leftrightarrow \begin{cases} \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t}\end{cases}$

write done $F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

$\alpha=i\quad \beta=0$

$$\partial_i F^{i0} = 0$$

$$\Rightarrow \nabla\cdot\vec{E} = 0 \quad \text{(Gauss Law)}$$

$\alpha=(0,j),\ \beta=i$

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0$$

$$\partial_t E^i = \partial_j(\varepsilon_{ijk}B^k) = \varepsilon_{ijk}\partial_j B^k = (\nabla\times\vec{B})^i$$

(c) The Energy-momentum tensor and angular momentum tensor

1. The Conserved current can be verified in the Lagrangian formulation (in the Action formulation) in different ways, all of which is called the proofs from Noether's theorem.

2. Without noether's theorem. (Before Noether), without Lagrangian formulation, with Newtonian dynamics (Lorentz force, work. energy), Derive conserved current.
↳ Complex, but can be got from text book.

Remark:

	Calculation	physics
Noether's thm (Lagrangian Formulation)	Simple ✓	missing ✗
without Noether's thm (Newtonian dynamics)	NOT Simple ✗	excellent ✓

3. The derivation from Lagrangian formulation:

Noether's thm (in class) Every continuous symmetry is associated with a conserved current

$$\partial_\mu j^\mu = 0,\quad j^\mu_\alpha = -\left[\mathcal{L}\eta^\mu_\rho - \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a\right]\frac{\delta x^\rho}{\delta\omega^\alpha} - \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\frac{\delta\phi_a}{\delta\omega^\alpha}$$

In Electrodynamics. $\phi_a \to A_\lambda$, $j^\mu_\alpha = -\left[\mathcal{L}\eta^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\rho A_\lambda\right]\frac{\delta x^\rho}{\delta\omega^\alpha} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\frac{\delta A_\lambda}{\delta\omega^\alpha}$

① Spacetime translation and $\partial_\mu T^{\mu\nu} = 0$

$$\frac{\delta x^\mu}{\delta\omega^\nu} = \eta^\mu_\nu,\quad \frac{\delta A^\mu}{\delta\omega^\nu} = 0$$

↓ second term vanished

$$T^{\mu\nu} = -\mathcal{L}\eta^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\partial^\nu A_\alpha$$

↓ notation

$$T^{\alpha\beta} = -\mathcal{L}\eta^{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\partial^\beta A_\gamma$$

{ first type of $\mathcal{L}$; second type of $\mathcal{L}'$ }

In first type of $\mathcal{L}$:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\gamma)} = -\partial^\alpha A^\gamma - \eta^{\alpha\gamma}(\partial\cdot A)$$

In second type of $\mathcal{L}'$:

$$\frac{\partial\mathcal{L}'}{\partial(\partial_\alpha A_\gamma)} = -F^{\alpha\gamma}$$

so that

$$T'^{\alpha\beta} = \frac{1}{4}F^2\eta^{\alpha\beta} - F^{\alpha\gamma}\partial^\beta A_\gamma$$

we what gauge invariant physical quantity, we define new energy-momentum tensor

$$\Theta^{\alpha\beta} = T'^{\alpha\beta} - \partial_\gamma(F^{\gamma\alpha}A^\beta)$$

We can check $\partial_\alpha\Theta^{\alpha\beta} = 0 = \partial_\alpha T^{\alpha\beta}$, $\Theta^{\alpha\beta}$, $T^{\alpha\beta}$ are both conserved current. (although, formulation are quit different)

$$\Uparrow$$

$$\underbrace{\partial_\alpha\partial_\gamma}_{\text{symmetric}}\overbrace{F^{\gamma\alpha}}^{\text{antisymmetric}}A^\beta = 0$$

$$\Theta^{\alpha\beta} = \frac{1}{4}F^2\eta^{\alpha\beta} + F^{\gamma\alpha}F^\beta{}_\gamma \quad \text{now is gauge invariant energy-momentum tensor}$$

② Lorentz transformation and angular momentum tensor $\partial_\mu M^{\mu\rho\sigma} = 0$

$$\frac{\delta x^\nu}{\delta\omega^{\rho\sigma}} = \frac{1}{2}(\eta^\mu_\rho x_\sigma - \eta^\mu_\sigma x_\rho),\quad \frac{\delta A_\lambda}{\delta\omega^{\rho\sigma}} = -\frac{i}{2}(S_{\rho\sigma})_\lambda{}^{\lambda'}A_{\lambda'}$$

↑ photon is spin one, we get spin-one tensor in the following

spin-1 tensor

$$(S_{\rho\sigma})_\lambda{}^{\lambda'} = i(\eta_{\rho\lambda}\eta_\sigma{}^{\lambda'} - \eta_{\sigma\lambda}\eta_\rho{}^{\lambda'})$$

$$\frac{\delta A_\lambda}{\delta\omega^{\rho\sigma}} = \frac{1}{2}(\eta_{\rho\lambda}A_\sigma - \eta_{\sigma\lambda}A_\rho)$$

This means total angular momentum tensor

$$M^{\mu\rho\sigma} = L^{\mu\rho\sigma} + 2\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\lambda)}\frac{\delta A^\lambda}{\delta\omega_{\rho\sigma}}.$$

$$\Rightarrow L^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

Take $\mathcal{L} \Rightarrow \mathcal{L}'$, $\frac{\mathcal{L}'}{\partial(\partial^\mu A^\lambda)} = -F_{\mu\lambda}$, result will be easy. ($\mathcal{L}' = -\frac{1}{4}F^2$)

$$M'^{\mu\rho\sigma} = L^{\mu\rho\sigma} + F^{\rho\mu}A^\sigma - F^{\sigma\mu}A^\rho \leftarrow \text{gauge invariant, but not obviously.}$$

To make $M'^{\mu\rho\sigma}$ gauge invariant (obviously gauge invariant)

Research report: What about obviously Gauge invariant Angular-momentum tensor.
(may occur in final exam)

Next lecture: Schedule for final exam
Solution to the 2nd homework.

Homework 1st QFT

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Consider the Lagrangian for a real vector field $A^\mu(x)$

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) + \frac{1}{2}\left(\partial_\alpha A^\alpha\right)\left(\partial_\beta A^\beta\right) + \frac{1}{2}\mu^2 A_\alpha A^\alpha$$

(a) Derive the equation of motion $\left[\eta_{\alpha\beta}(\Box+\mu^2) - \partial_\alpha\partial_\beta\right] A^\beta(x) = 0$; and verify the Lorentz Condition $\partial_\alpha A^\alpha = 0$.

(b) when $\mu^2=0$, $A^\mu = (\varphi, \vec{A})$, and $\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$, Derive Maxwell equations.

(c) With and without Noether's theorem, derive the energy-momentum tensor $T_{\mu\nu}$ for $\mu^2=0$.

(d) (Optional problem) With and without Noether's theorem, derive the angular-momentum tensor $M_{\mu\nu\rho}$ for $\mu^2=0$.

Solutions:

(a)

Method 1. From the Euler-Lagrange Equation $\frac{\partial \mathcal{L}}{\partial A^\beta} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\beta)}$,

we compute both sides of it:

- $$\frac{\partial \mathcal{L}}{\partial A^\beta} = \frac{1}{2}\mu^2 \frac{\partial}{\partial A^\beta}\left(A_\alpha A^\alpha\right) \overset{\text{chain-rule}}{=} \frac{1}{2}\mu^2\left(\eta_{\alpha\beta}A^\alpha + A_\alpha \delta_\beta{}^\alpha\right) = \mu^2 A_\beta$$

- $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\beta)} = -\frac{1}{2}\left(\delta_\rho^\mu \eta_{\sigma\beta}\right)\partial^\rho A^\sigma - \frac{1}{2}\partial_\rho A_\sigma\left(\eta^{\mu\rho}\delta_\beta{}^\sigma\right) + \frac{1}{2}\left(\delta^\mu{}_\rho \delta_\beta{}^\rho\right)\partial_\sigma A^\sigma$$
$$+ \frac{1}{2}\partial_\rho A^\rho\left(\delta^\mu{}_\sigma \delta_\beta{}^\sigma\right) = -\partial^\mu A_\beta + \partial_\alpha A^\alpha \delta^\mu_\beta$$

- $$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\beta)} = -\partial^\mu\partial_\mu A_\beta + \partial_\alpha\partial_\beta A^\alpha$$

Substituted these into Euler-Lagrange equation we obtain

$$\left(\mu^2 + \partial^\mu\partial_\mu\right)A_\beta - \partial_\alpha\partial_\beta A^\alpha = 0$$

change index $\alpha\leftrightarrow\beta$

$$\Leftrightarrow \quad \left(\Box+\mu^2\right)\eta_{\alpha\beta}A^\beta - \partial_\alpha\partial_\beta A^\beta = 0$$

The EOM has been derived above.

Method 2. From the Action Principle $\delta S = \int d^4x\, \delta\mathcal{L} = 0$.

We compute the variation of Action S:

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x \left\{ -\frac{1}{2}\partial_\alpha(\delta A_\beta)\cdot\partial^\alpha A^\beta - \frac{1}{2}\partial_\alpha A^\beta\,\partial^\alpha(\delta A_\beta) + \frac{1}{2}\partial_\alpha(\delta A^\alpha)\cdot\partial_\beta A^\beta + \frac{1}{2}\partial_\alpha A^\alpha\,\partial_\beta(\delta A^\beta) + \frac{1}{2}\mu^2\left(\delta A_\alpha A^\alpha + A_\alpha \delta A^\alpha\right)\right\}$$

$$= \int d^4x \left\{ -\partial_\alpha A_\beta\,\partial^\alpha(\delta A^\beta) + \partial_\beta A^\beta\left(\partial_\alpha \delta A^\alpha\right) + \mu^2 A_\alpha\cdot\delta A^\alpha \right\}$$

$$= \int d^4x \left\{ -\partial^\alpha\left(\partial_\alpha A_\beta\,\delta A^\beta\right) + \partial_\alpha\left(\partial_\beta A^\beta \delta A^\alpha\right) + \mu^2 A_\alpha\cdot\delta A^\alpha + \partial^\alpha\partial_\alpha A_\beta\cdot\delta A^\beta - \partial_\alpha\partial_\beta A^\beta\cdot\delta A^\alpha \right\}$$

The totally derivative vanish because the fixed boundary, so that

$$\delta S = \int d^4x \left\{ \partial^\mu\partial_\mu A_\alpha - \partial_\alpha\partial_\beta A^\beta + \mu^2 A_\alpha \right\}\delta A^\alpha$$

Since δA^α are arbitrary, we have

$$\delta S = 0$$

$$\Leftrightarrow \quad \partial^\mu\partial_\mu A_\alpha - \partial_\alpha\partial_\beta A^\beta + \mu^2 A_\alpha = 0$$

$$\Leftrightarrow \quad \left\{(\Box+\mu^2)\eta_{\alpha\beta} - \partial_\alpha\partial_\beta\right\}A^\beta = 0$$

That is EOM.

(b) Check the Lorentz Condition:

use EOM $\partial^\mu\partial_\mu A_\alpha - \partial_\alpha\partial_\beta A^\beta + \mu^2 A_\alpha = 0$, let ∂^α act both sides of it then we obtain

$$\partial^\alpha\partial^\mu\partial_\mu A_\alpha - \partial^\alpha\partial_\alpha\partial_\beta A^\beta + \mu^2\partial^\alpha A_\alpha = 0$$

Because we can change the order of derivative, which means $\partial^\alpha\partial^\mu\partial_\mu A_\alpha = \partial^\alpha\partial_\alpha\partial_\beta A^\beta$.

So when $\mu^2\neq 0$, that Lorentz condition can be proved $\partial^\alpha A_\alpha = 0$.

When $\mu^2=0$, the EOM becomes $\partial^\mu\partial_\mu A_\alpha - \partial_\alpha\partial_\mu A^\mu = 0$. Let $\partial_\mu A_\alpha - \partial_\alpha A_\mu = F_{\mu\alpha}$

and the EOM becomes $\partial^\mu F_{\mu\alpha} = 0$.

Next we shows the relation between $F_{\mu\alpha}$ and $\vec{E}, \vec{B}$.

$$E_i = -\partial_i\varphi - \frac{\partial A^i}{\partial t} = -\partial_i A_0 + \partial_0 A_i = F_{0i}$$

$$\varepsilon_{mni}B_i = +\varepsilon_{mni}(\nabla\times\vec{A})_i = -\varepsilon_{mni}\varepsilon_{ijk}\partial_j A_k = -\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)\partial_j A_k$$

$$= -\partial_m A_n + \partial_n A_m = -F_{mn}$$

So that from the EOM $\partial^\mu F_{\mu\alpha} = 0$,

when $\alpha = 0 \Rightarrow \partial^\mu F_{\mu 0} = \partial^i F_{i0} = -\partial_i F_{i0} = \partial_i E_i = \nabla\cdot\vec{E} = 0$ ......①

when $\alpha = k = 1,2,3 \Rightarrow \partial^\mu F_{\mu k} = \partial_t E_k + \partial^i F_{ik} = \partial_t E_k + \partial^i\varepsilon_{ikm}(-B_m)$

$$= \partial_t E_k - \varepsilon_{imk}\partial_i B_m = \left(\partial_t\vec{E} - \nabla\times\vec{B}\right)_k = 0$$

$$\Rightarrow \quad \nabla\times\vec{B} = \partial_t\vec{E} \quad \text{......②}$$

And the Bianchi identity is automatically contained in the definition of $F_{\mu\nu}$

$$\partial_{[\sigma} F_{\mu\nu]} = \partial_{[\sigma} F_{[\mu\nu]]} = \partial_{[\sigma}\left(\partial_\mu A_{\nu]} - \partial_\nu A_{\mu]}\right)$$

$$= \partial_{[(\sigma}\partial_{\mu)} A_{\nu]} - \partial_{[(\sigma}\partial_{\nu)} A_{\mu]} = 0$$

Let's show $\partial_{[\sigma} F_{\mu\nu]} = 0$ are equivalent to another two Maxwell equations

$$\partial_{[0} F_{ij]} = 0 \Rightarrow \partial_0 \varepsilon_{ijk} B_k + \partial_j E_i - \partial_i E_j = 0 \Rightarrow \partial_i E_j - \partial_j E_i = -\varepsilon_{ijk}\partial_t B_k$$

$$\Rightarrow \nabla\times\vec{E} = -\partial_t \vec{B} \quad \cdots ③$$

$$\partial_{[1} F_{23]} = 0 \Rightarrow -\partial_1 B_1 - \partial_2 B_2 - \partial_3 B_3 = 0 \Rightarrow \nabla\cdot\vec{B} = 0 \quad \cdots ④$$

①②③④ are Maxwell equations.

(c) Derive the Energy-Momentum tensor

Method 1. With Noether's theorem.

When we make space time translation $x'^\mu = x^\mu + \varepsilon^\mu$, the change of $A^\beta(x)$

$$\delta_0 A^\beta(x) = A^\beta(x-\varepsilon) - A^\beta(x) = \frac{\partial A^\beta}{\partial x^\rho}(-\varepsilon^\rho)$$

But $\delta A^\beta(x) = A'^\beta(x') - A^\beta(x) = 0$.

From Noether's theorem, the Conserved current is

$$J^\mu{}_a = -\left(\mathcal{L}\eta^\mu{}_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\rho A^\beta\right)\frac{-\delta x^\rho}{-\delta\omega^a} \overset{\frac{-\delta x^\rho}{-\delta\omega^a} = \delta^\rho_a}{=\!=\!=} -\mathcal{L}\eta^\mu{}_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_a A^\beta$$

So the Conserved current — energy-momentum tensor is

$$T_{\mu\nu} = -\mathcal{L}\eta_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\beta)}\partial_\nu A^\beta \quad \cdots ⑥$$

$$= -\eta_{\mu\nu}\left(-\tfrac{1}{2}\partial_\alpha A_\beta\cdot\partial^\alpha A^\beta + \tfrac{1}{2}\partial_\alpha A^\alpha\cdot\partial_\beta A^\beta\right) + \partial_\beta A^\beta\partial_\nu A_\mu - \partial_\mu A_\beta\partial_\nu A^\beta \quad \cdots ⑤$$

$$= \frac{1}{4}\eta_{\mu\nu}\left(F^{\alpha\beta}F_{\alpha\beta} + 2\partial^\alpha A^\beta\partial_\beta A_\alpha\right) + \left(-F_{\mu\beta} - \partial_\beta A_\mu\right)\left(F_\nu{}^\beta + \partial^\beta A_\nu\right) \cdots ⑦$$

The physical energy-momentum tensor should not directly contain A_μ, But should be expressed by observable quantities $F_{\alpha\beta}$, so we use Lorentz Condition to Cancel the A_ν term in ⑦ and obtain

$$T'_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F_{\mu\beta}F_\nu{}^\beta$$

Because $(F_{\alpha\beta}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$, so the $T'_{\mu\nu}$ can be written as

$$[T'_{\mu\nu}] = \frac{1}{4}(\eta_{\mu\nu})(-2\vec{E}^2 + 2\vec{B}^2) + \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & +B_3 & -B_2 \\ E_2 & -B_3 & 0 & +B_1 \\ E_3 & +B_2 & -B_1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\vec{B}^2+\vec{E}^2), & -\vec{E}\times\vec{B} \\ -(\vec{E}\times\vec{B})^T & \frac{1}{2}(\vec{B}^2+\vec{E}^2)\mathbb{1}_{3\times3} - \vec{E}\vec{E} - \vec{B}\vec{B} \end{pmatrix} \quad \cdots ⑧$$

Method 2. Without Noether's thm, but start directly from the space time translation

Coordinate transformation : $x'^\mu = x^\mu + \varepsilon^\mu$

the change of $A^\beta(x)$: $\delta_0 A^\beta = A^\beta(x'^\mu - \varepsilon^\mu) - A^\beta(x'^\mu) = \partial_\rho A^\beta\cdot(-\varepsilon^\rho)$

So the change of $\mathcal{L}$ can be written as

$$\delta_0\mathcal{L} = \mathcal{L}(x'^\mu - \varepsilon^\mu) - \mathcal{L}(x'^\mu) = \partial_\mu\mathcal{L}\cdot(-\varepsilon^\mu)$$

$$= \frac{\partial\mathcal{L}}{\partial A^\beta}\delta A^\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\delta(\partial_\mu A^\beta)$$

$$\overset{\text{use Lagrangian equation}}{=\!=\!=} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\right)\delta A^\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\mu\delta A^\beta = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\delta A^\beta\right]$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\rho A^\beta\right)(-\varepsilon^\rho)$$

From the equation above, we obtain

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\rho A^\beta - \mathcal{L}\eta^\mu{}_\rho\right)(-\varepsilon^\rho) = 0$$

Because $-\varepsilon^\rho$ is arbitrary, we has the Conserved current

$$T^\mu{}_\rho = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\rho A^\beta - \mathcal{L}\eta^\mu_\rho$$

It is the same as equation ⑥, so we have already derived the correct energy-momentum tensor.

calculate in detail

$$\begin{pmatrix} -\frac{1}{2}(\vec{E}^2-\vec{B}^2) & & & \\ & \frac{1}{2}(\vec{E}^2-\vec{B}^2) & & \\ & & \frac{1}{2}(\vec{E}^2-\vec{B}^2) & \\ & & & \frac{1}{2}(\vec{E}^2-\vec{B}^2) \end{pmatrix} + \begin{pmatrix} \vec{E}^2 & -E_2B_3+E_3B_2 & -E_3B_1+E_1B_3 & -E_1B_2+E_2B_1 \\ -E_2B_3+E_3B_2 & -E_1^2-B_1^2+B^2 & -E_1E_2-B_1B_2 & -E_1E_3-B_1B_3 \\ -E_3B_1+E_1B_3 & -E_2E_1-B_2B_1 & -E_2^2-B_2^2+B^2 & -E_2E_3-B_2B_3 \\ -E_1B_2+E_2B_1 & -E_3E_1-B_3B_1 & -E_3E_2-B_3B_2 & -E_3^2-B_3^2+B^2 \end{pmatrix}$$

$$= \frac{1}{2}(\vec{B}^2+\vec{E}^2)\mathbb{1}_{4\times4} + \begin{pmatrix} 0 & -\vec{E}\times\vec{B} \\ -(\vec{E}\times\vec{B})^T & -\vec{E}\vec{E} - \vec{B}\vec{B} \end{pmatrix}$$

Method 3. Derive Energy-momentum Tensor directly from Maxwell equations. (assume $c=\mu_0=\varepsilon_0=1$)

- We begin with Lorentz force formula $\vec{f} = \rho\vec{E} + \rho\vec{v}\times\vec{B}$.

The energy change of source particle is

$$\frac{dU}{dt} = \int d\tau\, \vec{f}\cdot\vec{v} = \int d\tau\, \rho\vec{E}\cdot\vec{v} = \int d\tau\, \vec{j}\cdot\vec{E}.$$

$$= \int d\tau \left(\nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E} = \int d\tau \left\{\nabla\cdot(\vec{B}\times\vec{E}) + \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right\}$$

$$= \int d\tau \left\{\nabla\cdot(\vec{B}\times\vec{E}) - \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right\}$$

Use Gaussian Theorem, $\int_V \nabla\cdot(\vec{B}\times\vec{E})\,d\tau = \int_{\partial V} d\vec{\sigma}\cdot(\vec{B}\times\vec{E})$ vanish, and $\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\vec{E}^2$, $\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\vec{B}^2$

So we obtain

$$\frac{dU}{dt} + \int d\tau\, \frac{\partial}{\partial t}\frac{1}{2}(\vec{E}^2+\vec{B}^2) = 0$$

If we believe the energy is conserved, then $u = \frac{1}{2}(\vec{E}^2+\vec{B}^2)$ must be the energy density of electric-magnetic field.

- We then derive the energy current $\vec{j}$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2\right) = \vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \vec{E}\cdot(\nabla\times\vec{B}) + \vec{B}\cdot(-\nabla\times\vec{E})$$

$$= \nabla\cdot(\vec{B}\times\vec{E})$$

So we can obtain $\frac{\partial u}{\partial t} + \nabla\cdot(\vec{E}\times\vec{B}) = 0$

This is a continuous equation, $\vec{E}\times\vec{B} = \vec{j}$ must be energy current.

- We then derive the momentum current $\overleftrightarrow{\mathcal{F}}$:

$$\frac{\partial\vec{j}}{\partial t} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \overset{\left\{\frac{\partial}{\partial t}\vec{E} = \nabla\times\vec{B},\ \frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E}\right.}{=} (\nabla\times\vec{B})\times\vec{B} - \vec{E}\times(\nabla\times\vec{E})$$

$$= \vec{B}\cdot\nabla\vec{B} - (\nabla\vec{B})\cdot\vec{B} - (\nabla\vec{E})\cdot\vec{E} + \vec{E}\cdot\nabla\vec{E}$$

$$\overset{\nabla\cdot\vec{E}=0,\ \nabla\cdot\vec{B}=0}{=} \nabla\cdot(\vec{B}\vec{B}) + \nabla\cdot(\vec{E}\vec{E}) - \frac{1}{2}\nabla(\vec{B}\cdot\vec{B}) - \frac{1}{2}\nabla(\vec{E}\cdot\vec{E})$$

$$= \nabla\cdot\left(\vec{B}\vec{B} + \vec{E}\vec{E} - \frac{1}{2}\vec{B}^2\mathbb{1} - \frac{1}{2}\vec{E}^2\mathbb{1}\right)$$

So we can get

$$\frac{\partial\vec{j}}{\partial t} + \nabla\cdot\left(\frac{1}{2}\vec{B}^2\mathbb{1} + \frac{1}{2}\vec{E}^2\mathbb{1} - \vec{B}\vec{B} - \vec{E}\vec{E}\right) = 0$$

This is a continuous equation, $\overleftrightarrow{\mathcal{F}} = \frac{1}{2}\vec{B}^2\mathbb{1} + \frac{1}{2}\vec{E}^2\mathbb{1} - \vec{B}\vec{B} - \vec{E}\vec{E}$ must be momentum current.

- Finally, we join $u, \vec{j}, \overleftrightarrow{\mathcal{F}}$ to form a 4x4 matrix

$$[T'_{\mu\nu}] = \begin{pmatrix} u & -\vec{j} \\ -\vec{j}^T & \overleftrightarrow{\mathcal{F}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\vec{B}^2+\vec{E}^2) & -\vec{E}\times\vec{B} \\ -(\vec{E}\times\vec{B})^T & \frac{1}{2}(\vec{B}^2+\vec{E}^2)\mathbb{1} - \vec{B}\vec{B} - \vec{E}\vec{E} \end{pmatrix}$$

It looks the same as equation (8), so it must be a real tensor, that is, it's energy-momentum tensor.

(d) Derive the angular momentum tensor.

Method 1. Using Noether's theorem

With Lorentz transformation $x' = \Lambda x$, the change of $A^\beta(x)$ is

$$\delta_0 A^\beta(x) = A^\beta(\Lambda^{-1}x) - A^\beta(x) + \delta A^\beta(x)$$

where $\frac{\delta A^\beta}{\delta\omega^{\rho\sigma}} = -\frac{i}{2}(S_{\rho\sigma})_\mu{}^\nu A_\nu \eta^{\beta\mu} = \frac{1}{2}(\eta_{\rho\mu}\eta_\sigma{}^\nu - \eta_{\sigma\mu}\eta_\rho{}^\nu)A_\nu\eta^{\beta\mu}$

$$A^\beta(\Lambda^{-1}x) - A^\beta(x) = \frac{\partial A^\beta}{\partial x^\lambda}\left\{+\frac{1}{2}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\right\}(\delta\omega^{\rho\sigma})$$

So, use Noether's thm and we obtain the conserved current

$$M^\mu{}_{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\frac{\partial A^\beta}{\partial x^\lambda}\left\{-\frac{1}{2}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\right\} - \mathcal{L}\,\eta^\mu{}_\lambda\left\{-\frac{1}{2}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\right\}$$

$$- \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\frac{1}{2}(\eta_{\rho\mu'}\eta_\sigma{}^\nu - \eta_{\sigma\mu'}\eta_\rho{}^\nu)A_\nu\eta^{\beta\mu'} \quad \cdots\cdots (10)$$

$$= T^\mu{}_\lambda\left\{-\frac{1}{2}(x_\rho\delta_\sigma{}^\lambda - x_\sigma\delta_\rho{}^\lambda)\right\} - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\rho)}A_\sigma + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\sigma)}A_\rho$$

$$= -\frac{1}{2}\left(x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho\right) - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\rho)}A_\sigma + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\sigma)}A_\rho$$

Then we obtain the angular momentum tensor

$$M_{\mu\rho\sigma} = -\frac{1}{2}(x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}) - \frac{1}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\rho)}A_\sigma\right) + \frac{1}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\sigma)}A_\rho\right) \quad \cdots\cdots (9)$$

Method 2. Using the Lorentz transformation, but without Noether's thm

With Lorentz transformation $x' = \Lambda x$, the change of A^β is

$$\delta_0 A^\beta(x) = A^\beta(\Lambda^{-1}x) - A^\beta(x) + \delta A^\beta(x)$$ see lecture notes

$$= \partial_\lambda A^\beta\,\frac{1}{2}\omega^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda + \left(-\frac{i}{2}\right)\omega^{\rho\sigma}(S_{\rho\sigma})_\mu{}^\nu A_\nu\eta^{\mu\beta}$$

$\cdots$ (11)

So the change of $\mathcal{L}$ is

$$\delta_0\mathcal{L} = \mathcal{L}(\Lambda^{-1}x) - \mathcal{L}(x) = \partial_\lambda\mathcal{L}\cdot\frac{1}{2}\omega^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\eta^\mu{}_\lambda \quad \cdots\cdots (1)$$

$$= \frac{\partial\mathcal{L}}{\partial(A^\beta)}\delta_0 A^\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\delta_0\partial_\mu A^\beta$$

$$\overset{\text{Lagrange eq}}{=} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\cdot\delta_0 A^\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\partial_\mu(\delta_0 A^\beta)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\delta_0 A^\beta\right) \quad \cdots\cdots (2)$$

So that we have

use (11) to substitute $\delta_0 A^\beta(x)$

$$(1) = (2) \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\beta)}\left\{\frac{1}{2}\omega^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\,\partial_\lambda A^\beta - \frac{i}{2}\omega^{\rho\sigma}(S_{\rho\sigma})_\mu{}^\nu A_\nu\eta^{\mu\beta}\right\} - \mathcal{L}\cdot\frac{1}{2}\omega^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\eta^\mu{}_\lambda\right) = 0$$

Cancel the arbitrary parameter $(-\omega^{\rho\sigma})$ from both sides of above equation we have the conserved current $\partial_\mu M^\mu{}_{\rho\sigma} = 0$, and

$$M^\mu{}_{\rho\sigma} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\beta)}\left(-\frac{1}{2}\cdot\partial_\lambda A^\beta\ (x_\rho\partial_\sigma - x_\sigma\partial_\rho)x^\lambda\right)$$
$$- \frac{1}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\beta)}\left(\eta_{\rho\mu'}\eta^\lambda_\sigma - \eta_{\sigma\mu'}\eta^\lambda_\rho\right)A_\lambda\,\eta^{\beta\mu'}$$
$$+ \frac{1}{2}\,\mathcal{L}\,\eta^\mu_\lambda\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right)x^\lambda$$

Which is the same as equation (10), so use the same steps in Method 1, we obtain

$$M_{\mu\rho\sigma} = -\frac{1}{2}\left(x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}\right) - \frac{1}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\rho)}A_\sigma - \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\sigma)}A_\rho\right) \quad \cdots\cdots \quad (12)$$

where

$$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\rho)} \xlongequal{\text{problem(a) has computed}} -\partial_\mu A_\rho + \partial_\alpha A^\alpha\cdot\eta_{\mu\rho}.$$

Method 3. Directly use Maxwell equations. we has already gained that $\vec{S} = \vec{E}\times\vec{B}$ is the momentum ~~density~~ density so $\vec{r}\times\vec{S}$ must be angular-momentum ~~density~~ density, and

$$\frac{\partial}{\partial t}(\vec{r}\times\vec{S}) = \vec{r}\times\frac{\partial}{\partial t}\vec{S} = -\vec{r}\times(\nabla\cdot\overleftrightarrow{\mathcal{T}}) \xlongequal{\nabla\times\vec{r}=0} -\nabla\cdot(\vec{r}\times\overleftrightarrow{\mathcal{T}})$$

where $\overleftrightarrow{\mathcal{T}} = u\overleftrightarrow{\mathrm{I}} - \vec{E}\vec{E} - \vec{B}\vec{B}$ has been derived in problem (c), so we have

$$\frac{\partial}{\partial t}(\vec{r}\times\vec{S}) + \nabla\cdot(\vec{r}\times\overleftrightarrow{\mathcal{T}}) = 0.$$

It is a continuous equation, so that we know $\overleftrightarrow{J} = \vec{r}\times\overleftrightarrow{\mathcal{T}}$ must be angular momentum current. This formula of angular momentum tensor is looked like the first term of equation (12), but a factor $-\frac{1}{2}$ and a spin-term is missing.

Reference:

1. 许定安，汪波《分析力学》，借鉴了其中连续介质的分析力学表述以理解场的拉氏密度. P153~P157.

2. 刘觉平《电动力学》第二章，借鉴了其中电磁场作用量的构造和能动张量、角动量流张量的导出方法；

3. 虞国寅，周国全《电动力学》第一章，借鉴了其中用三维 Maxwell 方程导出能动张量和角动量张量的方法；P30~P35

4. Yong zhang, 《Lecture Notes on Quantum Field Theory, V4》. Refer to the way to derive Noether's theorem, and the method for deriving the $T_{\mu\nu}$ for field ϕ_a, and $M^{\mu\rho\sigma}$ for scalar field. P57-62

Assignment 2nd , QFT

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Problem 1. Peskin and Schroeder's book, Page 39-40

1.1 Verify (3.18) to satisfy the Lorentz Algebra (3.17)

1.2 Verify (3.20) and (3.21) as examples of (3.19)

Solution:

1.1 The generator here for representation of four vector is

$$(J^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$

namely, we lift the script α and obtain

$$(J^{\mu\nu})^\alpha_{\ \beta} = i(\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta)$$

We then directly verify it to satisfy the Lorentz Algebra (3.17):

$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}]^\alpha_{\ \gamma} &= (J^{\mu\nu})^\alpha_{\ \beta}(J^{\rho\sigma})^\beta_{\ \gamma} - (J^{\rho\sigma})^\alpha_{\ \beta}(J^{\mu\nu})^\beta_{\ \gamma} \\
&= i(\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta)\cdot i(\eta^{\rho\beta}\delta^\sigma_\gamma - \eta^{\sigma\beta}\delta^\rho_\gamma) \\
&\quad - i(\eta^{\rho\alpha}\delta^\sigma_\beta - \eta^{\sigma\alpha}\delta^\rho_\beta)\cdot i(\eta^{\mu\beta}\delta^\nu_\gamma - \eta^{\nu\beta}\delta^\mu_\gamma) \\
&\overset{\text{expand}}{\underset{\text{and contract indices }\beta}{=}} i\cdot i(\eta^{\mu\alpha}\eta^{\rho\nu}\delta^\sigma_\gamma - \eta^{\nu\alpha}\eta^{\rho\mu}\delta^\sigma_\gamma - \eta^{\mu\alpha}\eta^{\sigma\nu}\delta^\rho_\gamma \\
&\quad + \eta^{\nu\alpha}\eta^{\mu\sigma}\delta^\rho_\gamma - \eta^{\rho\alpha}\eta^{\mu\sigma}\delta^\nu_\gamma \\
&\quad + \eta^{\sigma\alpha}\eta^{\mu\rho}\delta^\nu_\gamma + \eta^{\rho\alpha}\eta^{\nu\sigma}\delta^\mu_\gamma - \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^\mu_\gamma) \\
&= i\big(\eta^{\rho\nu}\cdot i(\eta^{\mu\alpha}\delta^\sigma_\gamma - \eta^{\sigma\alpha}\delta^\mu_\gamma) \\
&\quad + \eta^{\rho\mu} i(-\eta^{\nu\alpha}\delta^\sigma_\gamma + \eta^{\sigma\alpha}\delta^\nu_\gamma) \\
&\quad + \eta^{\sigma\nu} i(-\eta^{\mu\alpha}\delta^\rho_\gamma + \eta^{\rho\alpha}\delta^\mu_\gamma) \\
&\quad + \eta^{\mu\sigma} i(\eta^{\nu\alpha}\delta^\rho_\gamma - \eta^{\rho\alpha}\delta^\nu_\gamma)\big) \\
&= i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})
\end{aligned}
$$

This is just Lorentz Algebra. Here we use $\eta^{\mu\rho}$ to denote to metric rather than $g^{\mu\rho}$.

1.2. We first verify (3.20) as a example of $(V')^\alpha = (\mathbb{1} - \frac{i}{2}J^{\mu\nu}\omega_{\mu\nu})^\alpha_{\ \beta} V^\beta$

set $\omega_{12} = -\omega_{21} = \theta$ (it represents a rotation along z-axis), we have

$$
\begin{aligned}
(V')^\alpha &= \{\delta^\alpha_\beta - \tfrac{i}{2}(J^{12}\cdot\theta - J^{21}\theta)^\alpha_{\ \beta}\} V^\beta \\
&= V^\alpha - \tfrac{i}{2}\cdot 2\theta\cdot(J^{12})^\alpha_{\ \beta} V^\beta \\
&= V^\alpha + \theta(\eta^{1\alpha}\delta^2_\beta - \eta^{2\alpha}\delta^1_\beta)V^\beta \\
&= V^\alpha + \theta(\eta^{1\alpha}V^2 - \eta^{2\alpha}V^1)
\end{aligned}
$$

可以写成

$$V'^\alpha = (\mathbb{1} - \tfrac{i}{2}J^{\mu\nu}\omega_{\mu\nu})^\alpha_{\ \beta}V^\beta$$

$$\Leftrightarrow V'^\alpha = (\delta^\alpha_\beta + \omega^\alpha_{\ \beta})V^\beta$$

$$(V') = \begin{pmatrix} V^0 \\ V^1 - \theta V^2 \\ V^2 + \theta V^1 \\ V^3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & -\theta & \\ & \theta & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & -\theta & \\ & \theta & 1 & \\ & & & 1 \end{pmatrix}(V)$$

We next verify (3.21), set $\omega_{01} = -\omega_{10} = \beta$ (it represents a boost along x-axis), and we obtain

$$
\begin{aligned}
(V')^\alpha &= \{\delta^\alpha_\beta - \tfrac{i}{2}(J^{01}\beta - J^{10}\beta)^\alpha_{\ \beta}\}V^\beta = V^\alpha - \tfrac{i}{2}\cdot 2\beta(J^{01})^\alpha_{\ \beta}V^\beta \\
&= V^\alpha + \beta(\eta^{0\alpha}\delta^1_\beta - \eta^{1\alpha}\delta^0_\beta)V^\beta = V^\alpha + \beta(\eta^{0\alpha}V^1 - \eta^{1\alpha}V^0)
\end{aligned}
$$

$$(V') = \begin{pmatrix} V^0 + \beta V^1 \\ V^1 + \beta V^0 \\ V^2 \\ V^3 \end{pmatrix} = \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}(V)$$

Problem 2. Peskin and Schroeder's book Page 40-42

2.1 verify (3.23) to satisfy the Lorentz Algebra (3.17).

2.2 Verify (3.26) and (3.27), (3.29).

2.3 Verify Dirac equation (3.31) and Lagrangian (3.34) Lorentz invariant.

Solution:

2.1. We have assumed that $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}_{n\times n}$. Let's prove $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ must satisfy the Lorentz Algebra.

(Step 1) Let's firstly prove the commutator $[S^{\mu\nu}, \gamma^\rho] = i(\gamma^\mu g^{\rho\nu} - \gamma^\nu g^{\mu\rho})$

$$
\begin{aligned}
[S^{\mu\nu}, \gamma^\rho] &= \tfrac{i}{4}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho] \\
&= \tfrac{i}{4}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu - \underbrace{\gamma^\nu\gamma^\mu\gamma^\rho}_{(\gamma^\mu\gamma^\nu - 2g^{\mu\nu})\gamma^\rho} + \underbrace{\gamma^\rho\gamma^\nu\gamma^\mu}_{\gamma^\rho(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})}) \quad \text{use anticommutator} \\
&= \tfrac{i}{4}(2\gamma^\mu\gamma^\nu\gamma^\rho - 2\gamma^\rho\gamma^\mu\gamma^\nu) \quad \text{add two term } +2\gamma^\mu\gamma^\rho\gamma^\nu - 2\gamma^\mu\gamma^\rho\gamma^\nu \\
&= \tfrac{i}{4}(2\gamma^\mu(\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu) - 2(\gamma^\rho\gamma^\mu + \gamma^\mu\gamma^\rho)\gamma^\nu) \\
&= \tfrac{i}{4}(4\gamma^\mu g^{\nu\rho} - 4\gamma^\nu g^{\mu\rho}) = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})
\end{aligned}
$$

(step 2) use the conclusion of step 1 to prove the original problem:

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{i}{4}\left[S^{\mu\nu},\ \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right]$$

$$= \frac{i}{4}\left\{ [S^{\mu\nu},\gamma^\rho]\gamma^\sigma + \gamma^\rho[S^{\mu\nu},\gamma^\sigma] - \gamma^\sigma[S^{\mu\nu},\gamma^\rho] - [S^{\mu\nu},\gamma^\sigma]\gamma^\rho \right\}$$

$$= \frac{i}{4}\Big\{ i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})\gamma^\sigma + \gamma^\rho i(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma}) - \gamma^\sigma i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}) - i(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma})\gamma^\rho \Big\}$$

$$= \frac{i}{4}\cdot i\Big\{ (\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu)g^{\nu\rho} + (\gamma^\sigma\gamma^\nu - \gamma^\nu\gamma^\sigma)g^{\mu\rho} + (\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho)g^{\nu\sigma} + (\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu)g^{\mu\sigma} \Big\}$$

use definition of $S^{\mu\nu}$, and $S^{\mu\nu} = -S^{\nu\mu}$

$$= i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho} \right)$$

2.2 To verify (3.26) and (3.27), use the γ matrices here

$$\gamma^0 = \begin{pmatrix} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$$

(step 1)
We can directly compute

$$\gamma^0\gamma^i = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}\begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix} = \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}$$

$$\gamma^i\gamma^0 = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} = \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \qquad \gamma^0\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} = \mathbb{1}_{4\times4}$$

$$\gamma^i\gamma^j = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}\begin{pmatrix} & \sigma^j \\ -\sigma^j & \end{pmatrix} = \begin{pmatrix} -i\varepsilon^{ijk}\sigma^k - \delta^{ij} & \\ & -i\varepsilon^{ijk}\sigma^k - \delta^{ij} \end{pmatrix}$$

$$\gamma^j\gamma^i = \begin{pmatrix} & \sigma^j \\ -\sigma^j & \end{pmatrix}\begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix} = \begin{pmatrix} i\varepsilon^{ijk}\sigma^k - \delta^{ij} & \\ & i\varepsilon^{ijk}\sigma^k - \delta^{ij} \end{pmatrix}$$

(step 2)
To get the generator $S^{\mu\nu}$, we must first verify γ^μ satisfy $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$.
From step 1, it is easy to show $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$, $\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij}\mathbb{1}_{4\times4}$, $\gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\mathbb{1}_{4\times4}$
So we have

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times4}$$

(step 3)
Next we compute $S^{\mu\nu}$;

$$S^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = \frac{i}{4}(\gamma^0\gamma^i - \gamma^i\gamma^0) = \frac{i}{4}\left[\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}\right]$$

$$= \frac{i}{4}\begin{pmatrix} -2\sigma^i & \\ & 2\sigma^i \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{i}{4}(\gamma^i\gamma^j - \gamma^j\gamma^i) = \frac{i}{4}\begin{bmatrix} -2i\varepsilon^{ijk}\sigma^k & 0 \\ 0 & -2i\varepsilon^{ijk}\sigma^k \end{bmatrix}$$

$$= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

And because $S^{\mu\nu} = -S^{\nu\mu}$, so S^{00} must equals to 0.

(step 4) We now prove (3.29) $\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12} = \Lambda^\mu{}_\nu\gamma^\nu$ .... ①, where $\Lambda_{\frac12} = e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}$

We have already prove $[S^{\rho\sigma},\gamma^\mu] = i(\gamma^\rho g^{\sigma\mu} - \gamma^\sigma g^{\rho\mu})$ in 2.1 (step 1), we rewrite it into a more compact form

$$[\gamma^\mu, S^{\rho\sigma}] = +i(-\gamma^\rho g^{\sigma\mu} + \gamma^\sigma g^{\rho\mu}) = i(g^{\rho\mu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu)\gamma^\nu$$

$$= (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

where $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = i(g^{\rho\mu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu)$ is generator of 4-vector representation of LG.

Now we can verify the infinitesimal form of ①:

$$\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12} = \Lambda^\mu{}_\nu\gamma^\nu$$

$$\overset{\text{infinitesimal}}{\Longleftrightarrow} \left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho'\sigma'}S^{\rho'\sigma'}\right) = \left[\delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\gamma^\nu$$

$$\Leftrightarrow \gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] + o(\omega^2) = \left[\delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\gamma^\nu$$

$$\Leftrightarrow \gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu = \gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

which is always true.

We can also verify the finite form of ①. Use identity $e^{+A}Be^{-A} = \sum_{k=0}^{\infty}\frac{1}{k!}[A^{(k)},B]$ where $[A^{(k)},B] = [A,[A^{(k-1)},B]]$ and $[A^{(0)},B] = B$. We directly compute

$$\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12} = e^{\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}\gamma^\mu e^{-\frac{i}{2}\omega_{\rho'\sigma'}S^{\rho'\sigma'}}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\left(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)^{(k)},\gamma^\mu\right]$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\left(-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^k\right]^\mu{}_\nu\gamma^\nu$$

$$= \left(e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right)^\mu{}_\nu\gamma^\nu$$

$$= \Lambda^\mu{}_\nu\gamma^\nu$$

$\Longleftarrow$ • $[\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma},\gamma^\mu] = -\frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu,S^{\rho\sigma}]$
$= -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$

• $[(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})^{(2)},\gamma^\mu] =$
$[\frac{i}{2}\omega_{\rho'\sigma'}S^{\rho'\sigma'}, -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu]$
$= (-\frac{i}{2})^2\omega_{\rho'\sigma'}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu[\gamma^\nu, S^{\rho'\sigma'}]$
$= (-\frac{i}{2})^2\omega_{\rho'\sigma'}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu(\mathcal{J}^{\rho'\sigma'})^\nu{}_\alpha\gamma^\alpha$
$= [(-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^2]^\mu{}_\nu\gamma^\nu$
and so on...

2.3. To verify the Dirac eq. and Lagrangian Lorentz invariant, we do a transformation of coordinate $x \to \Lambda x$, and the transformation of field is $\psi(x) \to \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) = \psi'(x)$.

We ought to prove that: if and only if $\psi'(x)$ satisfy the Dirac equation, then the field $\psi(x)$ must also satisfy Dirac equation.

(step 1) verify the Lorentz invariance of Dirac equation:

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi'(x) = 0$$

$$\Leftrightarrow \quad \left(i\gamma^\mu\partial_\mu - m\right)\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) = 0$$

$$\Leftrightarrow \quad \left(i\gamma^\mu\Lambda_{\frac{1}{2}}\cdot(\Lambda^{-1})^\sigma{}_\mu\,\partial_\sigma\psi - \Lambda_{\frac{1}{2}}m\psi\right)_{(\Lambda^{-1}x)} = 0$$

$$\partial_\mu[\psi(\Lambda^{-1}x)] = \frac{\partial(\Lambda^{-1\sigma}{}_\nu x^\nu)}{\partial x^\mu}(\partial_\sigma\psi)(\Lambda^{-1}x) = (\Lambda^{-1})^\sigma{}_\nu\,\delta^\nu{}_\mu(\partial_\sigma\psi)(\Lambda^{-1}x) = (\Lambda^{-1})^\sigma{}_\mu(\partial_\sigma\psi)(\Lambda^{-1}x)$$

$$\Leftrightarrow \quad \Lambda_{\frac{1}{2}}\left(i\,\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}}(\Lambda^{-1})^\sigma{}_\mu\,\partial_\sigma\psi - m\psi\right)_{(\Lambda^{-1}x)} = 0$$

$$\Leftrightarrow \quad \Lambda_{\frac{1}{2}}\left(i\,(\Lambda^{-1})^\mu{}_\nu\gamma^\nu(\Lambda^{-1})^\sigma{}_\mu\,\partial_\sigma\psi - m\psi\right)(\Lambda^{-1}x) = 0$$

$$(\Lambda^{-1})^\mu{}_\nu\gamma^\nu(\Lambda^{-1})^\sigma{}_\mu = \delta^\sigma{}_\nu\gamma^\nu = \gamma^\sigma$$

$$\Leftrightarrow \quad \Lambda_{\frac{1}{2}}\left[\left(i\gamma^\sigma\partial_\sigma - m\right)\psi\right](\Lambda^{-1}x) = 0$$

$$\Leftrightarrow \quad \left[\left(i\gamma^\sigma\partial_\sigma - m\right)\psi\right](\Lambda^{-1}x) = 0$$

Which means if $\psi'(x)$ satisfy Dirac eq. in new coordinate system, then $\psi(x)$ must also satisfy Dirac eq. in old coordinate system.

(step 2) To verify the Lorentz invariance of Lagrangian, we first prove

$$S^{\mu\nu\dagger}\cdot\gamma^0 = \gamma^0 S^{\mu\nu}$$

In Solution 2.2 (step 1) we have verified that $\gamma^0\gamma^i = -\gamma^i\gamma^0$, so we have

$$[\gamma^0, S^{ij}] = \frac{i}{4}[\gamma^0, \gamma^i\gamma^j - \gamma^j\gamma^i] = \frac{i}{4}\begin{pmatrix}\gamma^0\gamma^i\gamma^j - \gamma^0\gamma^j\gamma^i \\ +\gamma^i\gamma^j\gamma^0 - \gamma^j\gamma^i\gamma^0\end{pmatrix}$$

Commute twice

$$= \frac{i}{4}\begin{pmatrix}\gamma^i\gamma^j\gamma^0 - \gamma^j\gamma^i\gamma^0 \\ +\gamma^i\gamma^j\gamma^0 - \gamma^j\gamma^i\gamma^0\end{pmatrix} = 0$$

act dagger to commutator we have

$$[\gamma^0, S^{ij}]^\dagger \underset{\gamma^{0\dagger} = \gamma^0}{=} S^{ij\dagger}\gamma^0 - \gamma^0 S^{ij\dagger} = S^{ij\dagger}\gamma^0 - \gamma^0 S^{ij} = 0$$

$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}$

$S^{ij\dagger} = S^{ij}$

And for S^{0i} we have

commute once

$$\{\gamma^0, S^{0i}\} = \frac{i}{4}\{\gamma^0, \gamma^0\gamma^i - \gamma^i\gamma^0\} = \frac{i}{4}\begin{pmatrix}\gamma^0\gamma^0\gamma^i - \gamma^0\gamma^i\gamma^0 \\ +\gamma^0\gamma^i\gamma^0 - \gamma^i\gamma^0\gamma^0\end{pmatrix} = \frac{i}{4}\begin{pmatrix}\gamma^0\gamma^0\gamma^i - \gamma^0\gamma^i\gamma^0 \\ -\gamma^0\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0\end{pmatrix} = 0.$$

act dagger to anti-commutator we have

$$\{\gamma^0, S^{0i}\}^\dagger = S^{0i\dagger}\gamma^0 + \gamma^0 S^{0i\dagger} = S^{0i\dagger}\gamma^0 - \gamma^0 S^{0i} = 0$$

$S^{0i} = \frac{i}{2}\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix}$

$S^{0i\dagger} = -S^{0i}$

So that we always have

$$S^{\mu\nu\dagger}\gamma^0 = \gamma^0 S^{\mu\nu}$$

(step 3) Now we verify the Lorentz invariance of $\mathcal{L}_{Dirac}$, first, under coordinate transformation

$$\psi'(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)$$

$$\begin{aligned}\bar{\psi}'(x) &= [\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)]^\dagger\gamma^0 = \psi^\dagger(\Lambda^{-1}x)\Lambda_{\frac{1}{2}}^\dagger\gamma^0 \\ &= \psi^\dagger(\Lambda^{-1}x)\left(1 + \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger}\right)\gamma^0 \\ &= \psi^\dagger(\Lambda^{-1}x)\gamma^0\left(1 + \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \\ &= \psi^\dagger(\Lambda^{-1}x)\gamma^0\Lambda_{\frac{1}{2}}^{-1} \\ &= \bar{\psi}(\Lambda^{-1}x)\Lambda_{\frac{1}{2}}^{-1}\end{aligned}$$

Now, the $\mathcal{L}_{Dirac}$ in new coordinate system is

it has been proved in step 1. $(\partial_\mu\psi')(x) = [(\Lambda^{-1})^\sigma{}_\mu\partial_\sigma\psi](\Lambda^{-1}x)$

$$\begin{aligned}\mathcal{L}'_{Dirac}(x) &= \bar{\psi}'(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi'(x) \\ &= \bar{\psi}(\Lambda^{-1}x)\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^\mu(\Lambda^{-1})^\sigma{}_\mu\partial_\sigma - m\right)\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\ &= \bar{\psi}(\Lambda^{-1}x)\left(i(\Lambda^{-1})^\sigma{}_\mu\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}}_{=\Lambda^\mu{}_\nu\gamma^\nu}\partial_\sigma - m\right)\psi(\Lambda^{-1}x) \\ &= \bar{\psi}(\Lambda^{-1}x)\left(i\,\delta^\sigma{}_\nu\gamma^\nu\partial_\sigma - m\right)\psi(\Lambda^{-1}x) \\ &= \bar{\psi}(\Lambda^{-1}x)\left(i\gamma^\sigma\partial_\sigma - m\right)\psi(\Lambda^{-1}x) = \mathcal{L}_{Dirac}(\Lambda^{-1}x)\end{aligned}$$

So $\mathcal{L}_{Dirac}$ is invariant under Lorentz transformation.

Problem 3. Peskin and Schroeder's book, Page 48-49

3.1. Verify (3.59) as a positive energy solution of Dirac eq. (3.31) in momentum space and verify (3.60)

3.2 Verify (3.62) as a negative energy solution of (3.31) in momentum space.

3.3 verify (3.63), (3.64) (3.66) and (3.67).

Solution:

3.1. Dirac equation (3.31) is $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

It's positive energy solution $\psi(x) = u(p)e^{-ip\cdot x}$, $u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix}$

(step 1) We first reformulate Dirac equation into Weyl equations.

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = \left(i\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix}\partial_\mu - m\right)\psi(x)$$

$$= \left[i\begin{pmatrix} 0 & \sigma^\mu\partial_\mu \\ \bar\sigma^\mu\partial_\mu & 0\end{pmatrix} - \begin{pmatrix} m\mathbb{1}_{2\times2} & \\ & m\mathbb{1}_{2\times2}\end{pmatrix}\right]\psi(x)$$

$$= \begin{pmatrix} -m\mathbb{1}_{2\times2} & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & -m\mathbb{1}_{2\times2}\end{pmatrix}\psi(x) = 0$$

This is just Weyl equations.

(step 2) We verify $\psi(x) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix}e^{-ip\cdot x}$ is positive energy ($p^0>0$) solution of Weyl equations.

$$\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi e^{-ip\cdot x} \\ \sqrt{p\cdot\bar\sigma}\,\xi e^{-ip\cdot x}\end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -m\sqrt{p\cdot\sigma}\,\xi e^{-ip\cdot x} + i\sigma^\mu\partial_\mu\sqrt{p\cdot\bar\sigma}\,\xi e^{-ip\cdot x} = 0 \\ i\bar\sigma^\mu\partial_\mu\sqrt{p\cdot\sigma}\,\xi e^{-ip\cdot x} - m\sqrt{p\cdot\bar\sigma}\,\xi e^{-ip\cdot x} = 0\end{cases}$$

$$i\partial_\mu e^{-ip\cdot x} = \hat{P}_\mu e^{-ip\cdot x} = P_\mu e^{-ip\cdot x}$$

$$\Leftrightarrow \begin{cases} -m\sqrt{p\cdot\sigma}\,\xi + \sigma^\mu P_\mu\sqrt{p\cdot\bar\sigma}\,\xi = 0 & ① \\ \bar\sigma^\mu P_\mu\sqrt{p\cdot\sigma}\,\xi - m\sqrt{p\cdot\bar\sigma}\,\xi = 0 & ②\end{cases}$$

$$\xLongleftrightarrow[②\times\sqrt{p\cdot\bar\sigma}]{①\times\sqrt{p\cdot\sigma}} \begin{cases} -m\, p\cdot\sigma\,\xi + \sigma^\mu P_\mu\sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}\,\xi = 0 \\ \bar\sigma^\mu P_\mu\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\,\xi - m\, p\cdot\bar\sigma\,\xi = 0\end{cases}$$

$$(p\cdot\sigma)(p\cdot\bar\sigma) = (E\mathbb{1} - \vec{p}\cdot\vec\sigma)(E\mathbb{1} + \vec{p}\cdot\vec\sigma) = E^2 - (\vec{p}\cdot\vec\sigma)^2 = E^2 - (\vec{p}\cdot\vec{p}) = m^2$$

also $(p\cdot\bar\sigma)(p\cdot\sigma) = m^2$

$$\Leftrightarrow \begin{cases} -m\, p\cdot\sigma\,\xi + \sigma^\mu P_\mu\cdot m\,\xi = 0 \\ \bar\sigma^\mu P_\mu\cdot m\cdot\xi - m\, p\cdot\bar\sigma\,\xi = 0\end{cases}$$

The final equations are always true, so that $\psi(x)$ must be solution of Dirac eq.

(step 3) verify (3.60)

$$\bar{u}^r(p)u^s(p) = (\xi_r^\dagger\sqrt{p\cdot\sigma},\ \xi_r^\dagger\sqrt{p\cdot\bar\sigma})\gamma^0\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s\end{pmatrix} = (\xi_r^\dagger\sqrt{p\cdot\sigma},\ \xi_r^\dagger\sqrt{p\cdot\bar\sigma})\begin{pmatrix}\sqrt{p\cdot\bar\sigma}\,\xi_s \\ \sqrt{p\cdot\sigma}\,\xi_s\end{pmatrix}$$

$(p\cdot\sigma)^\dagger = p\cdot\sigma$, $\sigma^\dagger = \sigma$; $\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}$

$$= \xi_r^\dagger\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\,\xi_s + \xi_r^\dagger\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\,\xi_s$$

$(p\cdot\sigma)(p\cdot\bar\sigma) = m^2$, $(p\cdot\bar\sigma)(p\cdot\sigma) = m^2$ has proved in last step

$$= 2m\,\xi_r^\dagger\xi_s = 2m\,\delta_{rs}$$

$$u^{r\dagger}(p)u^s(p) = (\xi_r^\dagger\sqrt{p\cdot\sigma},\ \xi_r^\dagger\sqrt{p\cdot\bar\sigma})\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}$$

$$= \xi_r^\dagger(\sqrt{p\cdot\sigma})^2\xi^s + \xi_r^\dagger(\sqrt{p\cdot\bar\sigma})^2\xi^s$$

$$= \xi_r^\dagger(E_{\vec p}\mathbb{1} - \vec{p}\cdot\vec\sigma)\xi^s + \xi_r^\dagger(E_{\vec p}\mathbb{1} + \vec{p}\cdot\vec\sigma)\xi^s$$

$$= 2E_{\vec p}\,\xi_r^\dagger\xi^s = 2E_{\vec p}\,\delta_{rs}$$

3.2 Verify (3.62) $\psi(x) = v(p)e^{+ip\cdot x} = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar\sigma}\,\eta\end{pmatrix}e^{ip\cdot x}$ ($p^0>0$) as negative energy solution of Dirac equation. By directly calculating we obtain

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta\cdot e^{ip\cdot x} \\ -\sqrt{p\cdot\bar\sigma}\,\eta\, e^{ip\cdot x}\end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -m\sqrt{p\cdot\sigma}\,\eta e^{ip\cdot x} + i\sigma^\mu\partial_\mu(\sqrt{p\cdot\bar\sigma})\eta e^{ip\cdot x} = 0 \\ i\bar\sigma^\mu\partial_\mu\sqrt{p\cdot\sigma}\,\eta e^{ip\cdot x} + m\sqrt{p\cdot\bar\sigma}\,\eta e^{ip\cdot x} = 0\end{cases}$$

$i\partial_\mu e^{ip\cdot x} = \hat{P}_\mu e^{ip\cdot x} = -P_\mu e^{ip\cdot x}$

$$\Longleftrightarrow \begin{cases} -m\sqrt{p\cdot\sigma}\,\eta + \sigma^\mu P_\mu\sqrt{p\cdot\bar\sigma}\,\eta = 0 & ① \\ -\bar\sigma^\mu P_\mu\sqrt{p\cdot\sigma}\,\eta + m\sqrt{p\cdot\bar\sigma}\,\eta = 0 & ②\end{cases}$$

$$\xLongleftrightarrow[②\times\, p\cdot\bar\sigma]{①\times\, p\cdot\sigma} \begin{cases} -m\, p\cdot\sigma\cdot\eta + \sigma^\mu P_\mu m\,\eta = 0 \\ -\bar\sigma^\mu P_\mu m\,\eta + m\, p\cdot\bar\sigma\,\eta = 0\end{cases} \Leftrightarrow \begin{cases} 0=0 \\ 0=0\end{cases}$$

$(p\cdot\sigma)(p\cdot\bar\sigma) = (p\cdot\bar\sigma)(p\cdot\sigma) = m^2$

which has always true.

And because $v(p)$ and $u(p)$ just differ by a minus sign, which will (not) appear in $\bar{v}^r(p)v^s(p)$ and $v^{\dagger r}(p)v^s(p)$, so the normalization is automatically satisfied: $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$, $v^{\dagger r}(p)v^s(p) = 2E_{\vec p}\delta^{rs}$. We will verify it in 3.3 .

3.3. Verify (3.63) $\begin{cases} \bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs} \\ v^{r\dagger}(p)\, v^s(p) = 2E_{\vec p}\,\delta^{rs} \end{cases}$, (3.64) $\bar{u}^r_{(p)} v^s_{(p)} = \bar{v}^r_{(p)} u^s_{(p)} = 0$

(3.66) $\sum_s u^s(p)\,\bar{u}^s(p) = \gamma\cdot p + m$

(3.67) $\sum_s v^s(p)\,\bar{v}^s(p) = \gamma\cdot p - m$

(step 1) we first verify (3.63) :

$$\bar{v}^r(p)\, v^s(p) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\, -\xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\gamma^0 \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix} = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\, -\xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}-\sqrt{p\cdot\bar\sigma}\,\xi^s \\ \sqrt{p\cdot\sigma}\,\xi^s\end{pmatrix}$$

$\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}$

$$= -\xi^{r\dagger}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\,\xi^s - \xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\,\xi^s$$

$$= -2m\,\xi^{r\dagger}\xi^s = -2m\,\delta^{rs}$$

$(p\cdot\sigma)(p\cdot\bar\sigma) = (p\cdot\bar\sigma)(p\cdot\sigma) = m^2$ has been proved in 3.1

$$v^{r\dagger}(p)\, v^s(p) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\, -\xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}$$

$$= \xi^{r\dagger}\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\,\xi^s + \xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma}\,\xi^s$$

$$= \xi^{r\dagger}(E\mathbb{1} - \vec p\cdot\vec\sigma)\xi^s + \xi^{r\dagger}(E\mathbb{1} + \vec p\cdot\vec\sigma)\xi^s$$

$$= 2E_{\vec p}\,\xi^{r\dagger}\xi^s = 2E_{\vec p}\,\delta^{rs}$$

(step2) we then prove (3.64) :

$$\bar{u}^r_{(p)} v^s_{(p)} = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\, \xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\gamma^0\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix} = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\, \xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}-\sqrt{p\cdot\bar\sigma}\,\xi^s \\ \sqrt{p\cdot\sigma}\,\xi^s\end{pmatrix}$$

$$= \xi^{r\dagger}\left(-\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} + \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\right)\xi^s$$

$$= \xi^{r\dagger}(-m+m)\xi^s = 0$$

$$\bar{v}^r_{(p)} u^s_{(p)} = \underbrace{v^{r\dagger}_{(p)}\gamma^0 u^s_{(p)} = \left(u^{s\dagger}_{(p)}\gamma^0 v^r_{(p)}\right)^\dagger}_{\gamma^{0\dagger}=\gamma^0} = 0^\dagger = 0$$

(step 3) Derive the completeness relation (3.66) and (3.67)

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\, \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\right)\gamma^0$$

$$= \sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\quad \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$\sum_s \xi^s\xi^{s\dagger} = \begin{pmatrix}1 & \\ & 1\end{pmatrix}$

$$= \begin{pmatrix}\sqrt{p\cdot\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix} = \begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & p\cdot\sigma \\ p\cdot\bar\sigma & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\end{pmatrix}$$

$(p\cdot\sigma)(p\cdot\bar\sigma) = (p\cdot\bar\sigma)(p\cdot\sigma) = m^2$

$$= \begin{pmatrix}m & \\ & m\end{pmatrix} + p_\mu\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} = m + p\cdot\gamma$$

$$\sum_{s=1,2} v^s(p)\,\bar{v}^s(p) = \sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\, -\xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\right)\gamma^0 = \sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\left(-\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\, \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$$= \begin{pmatrix}-\sqrt{p\cdot\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}$$

$$= \begin{pmatrix}-m & p\cdot\sigma \\ p\cdot\bar\sigma & -m\end{pmatrix} = -m\,\mathbb{1}_{4\times4} + p_\mu\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} = -m + p\cdot\gamma$$

Reference :

1. Peskin and Schroeder ⟨An Introduction to Quantum Field Theory⟩ Page 35~49
   - The problems are all choosen from this book.
   - Many methods and logics are from this book.
2. Weinberg ⟨The Quantum Theory of Fields⟩ (Vol. 1) P214
   - when proving 2.1, it suggests the tips that we first prove $[S^{\mu\nu}, \gamma^\rho] = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})$
3. My Lec. notes of QFT.

LecXVII Solutions to Homework 2nd   Dec 27, 2012

* Final Exam schedule

Time, Jan 17, 2017, 2:30 PM – 4:30 PM (Tuesday, afternoon)

Place: NOT fixed yet

May be Big classroom 101 in old building of school of Physics

May be elsewhere.

Final Exam: Easy → Basic Concepts, → Basic Calculation

Open test, Open to classnotes & Homeworks.

* Schedule on last class, Jan 3, 2017

Student's talk on their research report: PDF file or PPT, 20 minutes or 30 minutes.

Solution to the Second Homework

Problem 1. Lorentz group & Lorentz Algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left[\eta^{\nu\rho} M^{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right] \quad (1.1)$$

$M^{\mu\nu}$ Total Angular Momentum (Generator), $M^{\mu\nu} = -M^{\nu\mu}$

Sum of orbital and spin

Lorentz group representation

$$D(\Lambda) = \exp\left(-i\frac{\omega_{\mu\nu}}{2}M^{\mu\nu}\right)$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

orbital, spin

$L^{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$

标量表示. 常来轨道

$S^{\mu\nu}$ = Spin Angular Momentum

其它表示, 常来自旋

$W^\pm, Z, \gamma$

= { spin-1 Angular momentum 四矢表示.

$(S_{\mu\nu})^\alpha{}_\beta = i(\eta_{\mu\alpha}\eta_\nu{}^\beta - (\mu\leftrightarrow\nu))$

$\eta_{\mu\nu}$: flat metric

$g_{\mu\nu}$: in General relativity

Complex space time $= \eta_{\mu\nu} + \cdots$

spin-½ Angular momentum

$$S_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

Try to prove $L^{\mu\nu}$

$S^{\mu\nu}$ (spin-1, spin-½)

Satisfying Lorentz Algebra (1.1)

Verify: $[L_{\mu\nu}, L_{\rho\sigma}] = i\left[\eta_{\nu\rho}L_{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right]$

$$L_{\mu\nu} = i(x_\mu\partial_\nu - (\mu\leftrightarrow\nu))$$

∵

$$[L_{\mu\nu}, L_{\rho\sigma}] = i^2\left[x_\mu\partial_\nu - (\mu\leftrightarrow\nu),\ x_\rho\partial_\sigma - (\rho\leftrightarrow\sigma)\right]$$

$$= i\left\{i[x_\mu\partial_\nu, x_\rho\partial_\sigma] - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right\}$$

$$i[x_\mu\partial_\nu, x_\rho\partial_\sigma] = i x_\mu\partial_\nu(x_\rho\partial_\sigma) - i x_\rho\partial_\sigma(x_\mu\partial_\nu)$$

$$= i\eta_{\nu\rho}x_\mu\partial_\sigma - i\eta_{\sigma\mu}x_\rho\partial_\nu$$

$$= (\nu\rho, \mu, \sigma) - (\sigma\mu, \rho, \nu)$$

前两项   后两项

$$= i\left\{i[x_\mu\partial_\nu, x_\rho\partial_\sigma] + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma)\right.$$

$$= i\left\{(\nu\rho, \mu, \sigma) - (\sigma\mu, \rho, \nu) - (\mu\rho, \nu, \sigma) + (\sigma\nu, \rho, \mu) + (\mu\sigma, \nu, \rho) - (\rho\nu, \sigma, \mu) - (\nu\sigma, \mu, \rho) + (\rho\mu, \sigma, \nu)\right\}$$

$$= i\left\{(\nu\rho, \mu\sigma) - (\mu\sigma, \rho\nu) + (\rho\mu, \sigma\nu) + (\sigma\nu, \rho\mu)\right.$$

$$- (\rho\leftrightarrow\sigma) - (\mu\leftrightarrow\nu)$$

Verify: $(J_{\mu\nu})_\alpha{}^\beta = i(\eta_{\mu\alpha}\eta_\nu{}^\beta - (\mu\leftrightarrow\nu))$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left[\eta_{\nu\rho}J_{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right]$$

交换两指标: Index Change

∵

$$[J_{\mu\nu}, J_{\rho\sigma}] = \left[\overset{\eta}{(\nu\rho}, \overset{J}{\mu\sigma)} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right]$$

$$[J_{\mu\nu}, J_{\rho\sigma}]_\alpha{}^\gamma = \left[(J_{\mu\nu})_\alpha{}^\beta (J_{\rho\sigma})_\beta{}^\gamma - \begin{matrix}\mu\leftrightarrow\rho\\ \nu\leftrightarrow\sigma\end{matrix}\right]$$

$$(J_{\mu\nu})_\alpha{}^\beta (J_{\rho\sigma})_\beta{}^\gamma = i(\eta_{\mu\alpha}\eta_\nu{}^\beta - (\mu\leftrightarrow\nu))\, i(\eta_{\rho\beta}\eta_\sigma{}^\gamma - (\rho\leftrightarrow\sigma))$$

$$= i^2\left(\eta_{\mu\alpha}\eta_\nu{}^\beta\eta_{\beta\rho}\eta_\sigma{}^\gamma - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}\right)$$

$$= i^2\eta_{\mu\alpha}\eta_\nu{}^\beta\eta_{\beta\rho}\eta_\sigma{}^\gamma = i^2\eta_{\mu\alpha}\eta_{\nu\rho}\eta_\sigma{}^\gamma$$

→ 交换指标

$$= i^2\eta_{\nu\rho}\eta_{\mu\alpha}\eta_\sigma{}^\gamma = i(\nu\rho, \mu\sigma)$$

→ 交换指标

so $(J_{\mu\nu})_\alpha{}^\beta (J_{\rho\sigma})_\beta{}^\gamma = i\left((\nu\rho, \mu\sigma) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\right)$

now $[J_{\mu\nu}, J_{\rho\sigma}]_\alpha{}^\gamma = i\left[(\nu\rho, \mu\sigma) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\right]$

$- (\mu\leftrightarrow\rho, \nu\leftrightarrow\sigma)$

$= i\big[(\nu\rho, \mu\sigma) - (\mu\rho\ \nu, \sigma) - (\nu\sigma, \mu, \rho) + (\mu\sigma, \nu\rho)$
1 2 3 4

$- (\sigma\mu, \rho, \nu) + (\rho\mu, \sigma, \nu) + (\sigma, \nu, \rho, \mu)$
4 2 3

$- (\rho\ \nu, \sigma, \mu)\big]$
1

$= i\big[(\nu\rho, \mu\sigma) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\big]$

Verify: Spin - $\frac{1}{2}$, $S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}\sigma_{\mu\nu}$

$= \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ - Three approach is totally equivalent.

Approach 1. $[S_{\mu\nu}, S_{\rho\sigma}] = (\frac{i}{4})^2[\gamma_\mu\gamma_\nu - (\mu\leftrightarrow\nu), \gamma_\rho\gamma_\sigma - (\rho\leftrightarrow\sigma)]$

$= (\frac{i}{4})^2\left([\gamma_\mu\gamma_\nu, \gamma_\rho\gamma_\sigma] - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\right)$

where $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu$

$= \gamma_\mu(2\eta_{\rho\nu} - \gamma_\rho\gamma_\nu)\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu$

$= 2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - \gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu$

$= 2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + \gamma_\rho\gamma_\mu\gamma_\nu\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu$

$= 2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\gamma_\rho\gamma_\mu\eta_{\nu\sigma} - \gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu$

$\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu = 2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\nu\sigma}\gamma_\rho\gamma_\mu - 2\eta_{\mu\sigma}\gamma_\rho\gamma_\nu$

How to go on?

Approach 2. $S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ ← antisymmetric. $S_{\mu\nu}$

$= \begin{cases} 0, & \mu = \nu \\ \frac{i}{2}\gamma_\mu\gamma_\nu, & \mu \neq \nu \end{cases}$ $\quad S_{\mu\nu} \overset{\mu\neq\nu}{\Rightarrow} \gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$

$[S_{\mu\nu}, S_{\rho\sigma}] \underset{\substack{\mu\neq\nu\\ \rho\neq\sigma}}{=} (\frac{i}{2})^2(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu) = (\frac{i}{2})^2\big[2\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma$
$+ 2\eta_{\nu\sigma}\gamma_\rho\gamma_\mu - 2\eta_{\mu\sigma}\gamma_\rho\gamma_\nu\big]$

How to go on?

Approach 3. $[S_{\mu\nu}, S_{\rho\sigma}] = \frac{i}{4}[S_{\mu\nu}, \gamma_\rho\gamma_\sigma] - (\rho\leftrightarrow\sigma)$

$= \frac{i}{4}\Big([S_{\mu\nu}, \gamma_\rho]\gamma_\sigma + \gamma_\rho[S_{\mu\nu}, \gamma_\sigma]\Big) - (\rho\leftrightarrow\sigma)$

Commutator

$[S_{\mu\nu}, \gamma_\rho] = \frac{i}{4}\left([\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu, \gamma_\rho]\right) = \frac{i}{4}[\gamma_\mu\gamma_\nu, \gamma_\rho] - (\nu\leftrightarrow\mu)$

$= \frac{i}{4}(\gamma_\mu\gamma_\nu\gamma_\rho - \gamma_\rho\gamma_\mu\gamma_\nu) - (\nu\leftrightarrow\mu)$

$= \frac{i}{4}(-\gamma_\mu\gamma_\rho\gamma_\nu + 2\eta_{\rho\nu}\gamma_\mu - \gamma_\rho\gamma_\mu\gamma_\nu) - (\nu\leftrightarrow\mu)$

$= \frac{i}{4}(\underbrace{-2\eta_{\rho\mu}\gamma_\nu + 2\eta_{\rho\nu}\gamma_\mu}_{\langle\mu\leftrightarrow\nu,\ \text{antisymmetric}\rangle}) - (\nu\leftrightarrow\mu)$

$= \frac{i}{4}\times 2(-2\eta_{\rho\mu}\gamma_\nu + 2\eta_{\rho\nu}\gamma_\mu) = i(-\eta_{\rho\mu}\gamma_\nu + \eta_{\rho\nu}\gamma_\mu)$

$[S_{\mu\nu}, \gamma_\rho\gamma_\sigma] = i(\eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu)\gamma_\sigma$

$+ i\gamma_\rho(\eta_{\sigma\nu}\gamma_\mu - \eta_{\sigma\mu}\gamma_\nu)$

$= i\left(\eta_{\rho\nu}\gamma_\mu\gamma_\sigma - \eta_{\rho\mu}\gamma_\nu\gamma_\sigma + \eta_{\sigma\nu}\gamma_\rho\gamma_\mu - \eta_{\sigma\mu}\gamma_\rho\gamma_\nu\right)$

How to go on.

Final relationship between $\gamma_\mu\gamma_\sigma$ and $S_{\mu\sigma}$

$S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$= \frac{i}{4}(\gamma_\mu\gamma_\nu - (2\eta_{\mu\nu} - \gamma_\mu\gamma_\nu)) = \frac{i}{2}(\gamma_\mu\gamma_\nu - \eta_{\mu\nu})$

$\Rightarrow \gamma_\mu\gamma_\nu = \eta_{\mu\nu} - 2i S_{\mu\nu}$

now $\frac{i}{4}[S_{\mu\nu}, \gamma_\rho\gamma_\sigma] - (\rho\leftrightarrow\sigma)$

$= \frac{i}{4}\Big(i\eta_{\rho\nu}(\eta_{\mu\sigma} - 2i S_{\mu\sigma}) - i\eta_{\rho\mu}(\eta_{\nu\sigma} - 2i S_{\nu\sigma})$
$+ i\eta_{\sigma\nu}(\eta_{\rho\mu} - 2i S_{\rho\mu}) - i\eta_{\sigma\mu}(\eta_{\rho\nu} - 2i S_{\rho\nu})\Big) - (\rho\leftrightarrow\sigma)$

$= \frac{i}{2}\left(\eta_{\rho\nu}S_{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\right) - (\rho\leftrightarrow\sigma)$

now $[S_{\mu\nu}, S_{\rho\sigma}] = i\left(\eta_{\rho\nu}S_{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \left(\begin{smallmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{smallmatrix}\right)\right)$

Note: Result $[S_{\mu\nu}, \gamma_\rho] = i(\eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu)$ is important relation

$= i[\eta_{\rho\nu}\eta_\mu{}^\sigma\gamma_\sigma - \eta_{\rho\mu}\eta_\nu{}^\sigma\gamma_\sigma]$.

$= i[\eta_{\rho\nu}\eta_\mu{}^\sigma - \eta_{\rho\mu}\eta_\nu{}^\sigma]\gamma_\sigma$

$= -i[\eta_{\mu\rho}\eta_\nu{}^\sigma - \eta_{\nu\rho}\eta_\mu{}^\sigma]\gamma_\sigma$

$= -(J_{\mu\nu})_\rho{}^\sigma\gamma_\sigma$

$[\gamma_\rho, S_{\mu\nu}] = (J_{\mu\nu})_\rho{}^\sigma\gamma_\sigma$ very important! we can construct spin-1 from spin-½ algebra.

For vector representation

$v^\alpha = \Lambda^\alpha{}_\beta v^\beta$, $\Lambda^\alpha{}_\beta = \exp(-\tfrac{i}{2}\omega_{\mu\nu}J^{\mu\nu})^\alpha{}_\beta$

$= 1 - \tfrac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta + O(\omega^2)$

$J^{\mu\nu} = -J^{\nu\mu}$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$(J^{\mu\nu})^\alpha{}_\beta = i(\eta^{\mu\alpha}\eta^\nu{}_\beta - \eta^{\nu\alpha}\eta^\mu{}_\beta)$
$\langle\mu\leftrightarrow\nu\rangle$ photon's spin is very strange!

such that: $-\tfrac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta$

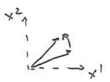
$= -\tfrac{i}{2}\omega_{\mu\nu}\cdot i(\eta^{\mu\alpha}\eta^\nu{}_\beta - (\mu\leftrightarrow\nu))$

$= \omega_{\mu\nu}\eta^{\mu\alpha}\eta^\nu{}_\beta = \omega^\alpha{}_\beta$ ← very easy! just infinitesimal expansion

$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta + O(\omega^2)$

rotation: $\omega_{12} = \theta$, $\omega_{21} = -\theta$, $\omega^1{}_2 = -\theta$, $\omega^2{}_1 = \theta$

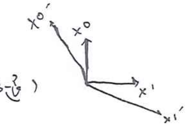
$(\Lambda^\alpha{}_\beta) = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}0&&&\\&0&-\theta&\\&\theta&0&\\&&&0\end{pmatrix}$ 矢量的旋转



boost: $\omega_{01} = \beta$, $\omega_{10} = -\beta$, $\omega^0{}_1 = \beta$, $\omega^1{}_0 = \beta$

$(\Lambda^\alpha{}_\beta) = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}0&\beta&&\\\beta&0&&\\&&0&\\&&&0\end{pmatrix}$

向前的 boost (坐标系后退)



Again, talk about $J^{\mu\nu}$

$J^{01} = \begin{pmatrix}0&i&&\\i&0&&\\&&0&\\&&&0\end{pmatrix}\cdot i$

$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta - \tfrac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta$

$(J^{01})^\alpha{}_\beta = i(\eta^{0\alpha}\eta^1{}_\beta - \eta^{1\alpha}\eta^0{}_\beta) = \begin{cases}\alpha=0, \beta=1 & \cdot i\\ \alpha=1, \beta=0 & i\end{cases}$

$\Lambda^\alpha{}_\beta \overset{\text{boost}}{=\!=} \delta^\alpha{}_\beta - i\omega_{01}(J^{01})^\alpha{}_\beta + O(\omega^2)$ $= \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}0&\beta&&\\\beta&0&&\\&&0&\\&&&0\end{pmatrix}$

$\Lambda^\alpha{}_\beta \overset{\text{rotation}}{=\!=} \delta^\alpha{}_\beta - i\omega_{12}(J^{12})^\alpha{}_\beta + O(\omega^2)$

$(J^{12})^\alpha{}_\beta = i(\eta^{1\alpha}\eta^2{}_\beta - \eta^{2\alpha}\eta^1{}_\beta)$

$= \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}0&&&\\&0&-\theta&\\&\theta&0&\\&&&0\end{pmatrix}$

$= i\begin{cases}\alpha=1 & \beta=2 & (-1)\\ \alpha=2 & \beta=1 & +1\end{cases}$ → anti sym.

$J^{12} = \begin{pmatrix}0&&&\\&0&-1&\\&1&0&\\&&&0\end{pmatrix}\cdot i$

Lemma: $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, γ algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ (not complete,
Key point: Lorentz invariance of
$\gamma_0^\dagger = \gamma_0$, $\gamma_i^\dagger = -\gamma_i$ (anti-Hermitian)

Lemma 1: $\mu\neq\nu$, $S^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu$ $(\mu=\nu,\ S^{\mu\nu}=0)$

Lemma 2: $[\gamma_\rho, S_{\mu\nu}] = (J_{\mu\nu})_\rho{}^\sigma\gamma_\sigma$

(Lorentz algebra)

Lemma 3: $[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\nu\rho}S_{\mu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \binom{\mu\leftrightarrow\nu}{\rho\leftrightarrow\sigma})$

Lemma 4: $S^{0i} = \frac{i}{2}\gamma^0\gamma^i$

$(S^{0i})^\dagger = -\frac{i}{2}(\gamma^0\gamma^i)^\dagger = -\frac{i}{2}(\gamma^i)^\dagger(\gamma^0)^\dagger = \frac{i}{2}\gamma^i\gamma^0 = -S^{0i}$

it is anti Hermitian

Lemma 5. $\{S^{0i}, \gamma^0\} = 0$

$S^{0i}\gamma^0 = -\gamma^0 S^{0i}$
$(S^{0i})^\dagger\gamma^0 = \gamma^0 S^{0i}$
$(S^{0i})^\dagger = -S^{0i}$

两次写错了!

Problem 2,3 Dirac Representation

Lemma 6: $\gamma^0 = \begin{pmatrix}&1_2\\1_2&\end{pmatrix}$ $\gamma^i = \begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}$ $S^{0i} = \frac{i}{2}\begin{pmatrix}&1_2\\1_2&\end{pmatrix}\begin{pmatrix}&\sigma^i\\-\sigma^i&\end{pmatrix} = -\frac{i}{2}\begin{pmatrix}\sigma^i&\\&-\sigma^i\end{pmatrix}$

Lemma 7: $S^{ij} = \frac{i}{2}\gamma^i\gamma^j$

$(S^{ij})^\dagger = -\frac{i}{2}(\gamma^j)^\dagger(\gamma^i)^\dagger = -\frac{i}{2}(-\gamma^j)(-\gamma^i) = -\frac{i}{2}\gamma^j\gamma^i \Rightarrow (S^{ij})^\dagger = S^{ij}$

Lemma 8: $\boxed{[S^{ij}, \gamma^0]} = 0$

since $\gamma^0\gamma^i\gamma^j = \gamma^i\gamma^j\gamma^0$

we have $(S^{ij})^\dagger\gamma^0 = \gamma^0 S^{ij}$, $(S^{ij})^\dagger\gamma^0 = \gamma^0 S^{ij}$ $= \varepsilon^{ijk}\Sigma^k$

Lemma 9: $S^{ij} = \frac{i}{2}\gamma^i\gamma^j = \frac{i}{2}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\sigma^i\sigma^j&\\&-\sigma^i\sigma^j\end{pmatrix} = \varepsilon^{ijk}\frac{1}{2}\begin{pmatrix}\sigma^k&\\&\sigma^k\end{pmatrix}$

$(i\neq j)$ $\sigma^i\sigma^j = i\varepsilon^{ijk}\sigma^k$

Lemma 10: Spin-½ representation

(key point to talk about Dirac invariance)

$$D^{1/2} = \exp(-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu})$$

$$(D^{1/2})^{-1} = \exp(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu})$$

but

$$(D^{1/2})^{\dagger} = \exp\left(+i\left[\omega_{0i}(S^{0i})^{\dagger} + \omega_{ij}(S^{ij})^{\dagger}\right]\right)$$

$$= \exp\left(+i\left(-\omega_{0i}S^{0i} + \omega_{ij}S^{ij}\right)\right)$$

however

$$(D^{1/2})^{\dagger}\gamma_0 = \gamma_0 (D^{1/2})^{-1}$$

$$(D^{1/2})^{\dagger}\gamma_0 = \left[\mathbb{1} + i\left(-\omega_{0i}S^{0i} + \omega_{ij}S^{ij}\right) + \dots\right]\gamma^0$$

$$= \left[\gamma^0 + \gamma_0\, i\left(\omega_{0i}S^{0i} + \omega_{ij}S^{ij}\right) + \dots\right]$$

$$= \gamma_0 (D^{1/2})^{-1}$$

Dirac eq invariance under Lorentz transf:

$$\psi(x) \to D^{1/2}\psi(\Lambda^{-1}x)$$

$$\psi^{\dagger}(x) \to \psi^{\dagger}(\Lambda^{-1}x)(D^{1/2})^{\dagger}$$

$D^{1/2} \Rightarrow$ NOT unitary transformation

$$\bar\psi(x) = \psi^{\dagger}(x)\gamma^0 \longrightarrow \psi^{\dagger}(\Lambda^{-1}x)(D^{1/2})^{\dagger}\gamma^0 = \bar\psi(\Lambda^{-1}x)(D^{1/2})^{-1}$$

$\Rightarrow$ that is why we use $\bar\psi$

$$\bar\psi(x)\psi(x) \leftrightarrow \bar\psi(\Lambda^{-1}x)(D^{1/2})^{-1}D^{1/2}\psi(\Lambda^{-1}x) = \bar\psi(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$$

$\Rightarrow$ Lagrangian Invariance

both important relation of spin-1 and spin-½

Lemma 11.

$$(D_{1/2})^{-1}\gamma^{\mu}D_{1/2} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \quad \Leftarrow \quad [\gamma^{\mu}, S^{\rho\sigma}] = (J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}$$

$$D_{1/2} = \exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)$$

$$\Lambda^{\mu}{}_{\nu} = \exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)^{\mu}{}_{\nu}$$

$$\exp\left(+\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^{\mu}\exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)^{\mu}{}_{\nu}\gamma^{\nu}$$

$$A = +\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}$$

$$B = \gamma^{\mu}$$

Use Baker-Hausdorf formula: $e^{A}Be^{-A} = \sum_{n=0}^{\infty}\frac{C_n}{n!}$

$C_0 = B$

$C_n = [A, C_{n-1}],\ n \geq 1$

Computation:

$$C_0 = B = \gamma^{\mu}$$

$$C_1 = [A, B] = \tfrac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^{\mu}] = -\tfrac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}$$

$$= (-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}$$

$$C_2 = [A, C_1] = \tfrac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma}, (-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}]$$

$$= -\tfrac{i}{2}\omega_{\rho'\sigma'}(J^{\rho'\sigma'})^{\mu}{}_{\nu}(\tfrac{i}{2}\omega_{\rho\sigma})\underbrace{[S^{\rho\sigma}, \gamma^{\nu}]}_{-(J^{\rho\sigma})^{\nu}{}_{\sigma'}\gamma^{\sigma'}}$$

$$= [(-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^2]^{\mu}{}_{\nu}\gamma^{\nu}$$

...

These means:

$$e^{A}Be^{-A} = \left(\mathbb{1} + (-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^{\mu}{}_{\nu} + [(-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^2]^{\mu}{}_{\nu}/2! + \dots\right)\gamma^{\nu}$$

$$= \exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)^{\mu}{}_{\nu}\cdot\gamma^{\nu}$$