On the Evolution of Compact Objects

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This is a course thesis of general relativity lectured by Professor Jia. In this article, we have discussed the theory of the evolution of compact objects and the observation of such compact objects.

I. THE THEORY OF WHITE DWARFS

A. Thermodynamics of Compact Objects

1. Some Fundamental Relations

In thermodynamics system we often use a potential function to describe its properties. Here we will deal with the identical ideal Fermions, therefore the Landau potential $\Omega = -PV$ is a reasonable choice because it can describe a system with variable number of particles and variable energy. From fundamental relation of Thermodynamics we can show that:

$$d\Omega = -SdT - pdV - Nd\mu \tag{1}$$

Now we focus on a single particle state denoted by momentum $\vec{p}$. We can view this state as a subsystem of the whole system, and it has its own Landau potential

$$\Omega_{\mathrm{p}} = -kT\ln\mathrm{Z}_G = -kT\ln\left(1 + e^{-\frac{\varepsilon-\mu}{kT}}\right) \tag{2}$$

where ϵ means the kenetic energy. Use the relation Eq.(1) we can get the mean particle number of these single particle state called Fermi-Dirac distribution

$$n_{\mathrm{p}} = -\left(\frac{\partial\Omega_{\mathrm{p}}}{\partial\mu}\right)_{T,V} = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}} + 1} \tag{3}$$

The Thermodynamics potential Ω must be an extensive quantity, and therefore we can make summation of Ω_p of all single particle states to get the total Ω of the system

$$\Omega = \int \Omega_{\mathrm{p}} \cdot \frac{g}{h^3}\mathrm{d}^3\mathrm{r}\mathrm{d}^3\mathrm{p} = \frac{4\pi g}{h^3}V\int_0^\infty p^2\Omega_{\mathrm{p}}\mathrm{d}p \tag{4}$$

here $g/h^3 = (2s+1)/h^3$ means the state density in phase space. We then can get pressure P through relation $\Omega = -PV$. We can also derive density of particle number and density of energy in position space

$$n = \frac{1}{V}\int n_{\mathrm{p}} \cdot \frac{g}{h^3}\mathrm{d}^3\mathrm{r}\mathrm{d}^3\mathrm{p} = \frac{4\pi g}{h^3}\int_0^\infty p^2 n_{\mathrm{p}}\mathrm{d}p \tag{5}$$

$$\begin{aligned}\rho_E &= \frac{1}{V}\int \sqrt{(\mathrm{p}c)^2 + (mc^2)^2} \cdot n_{\mathrm{p}}\frac{g}{h^3}\mathrm{d}^3\mathrm{r}\mathrm{d}^3\mathrm{p} \\ &= \frac{4\pi g}{h^3}\int_0^\infty p^2 n_{\mathrm{p}}\sqrt{(pc)^2 + (mc^2)^2}\mathrm{d}p\end{aligned} \tag{6}$$

Once we get the energy density ρ_E from Eq.(6) and pressure P from Landau potential, we can derive the implicit relation between ρ_E and P, although it will always depend on the temperature(we assume chemical potential μ will be cancelled).

2. Ideal degenerate Fermi-gas

Since the equation of state always depend on temperature as discussed above, which make the problem complicated, we now discuss a special and simple case of it. At temperature $T = 0$, the Fermi-gas (e.g. the electron gas and neutron gas) will be at the ground state, the degeneracy pressure will be the only source of pressure. Although there will never be a case with $T = 0$, in many cases ideal degenerate Fermi-gas model is a reasonable choice(e.g. the white dwarf and the outer shell of neutron star).

At zero temperature the Fermi-Dirac distribution Eq.(3) becomes a step function located at $\epsilon_F = \mu$ (called Fermi-energy). Then from Eq.(4), Eq.(5) and Eq.(6) we can derive the pressure, density of particle number, density of energy at zero temperature and finally get the equation of state. We begin with equation Eq.(5), let n_P become step function and we get the density of particle number

$$n = \frac{4\pi g}{h^3}\int_0^{p_F} p^2\mathrm{d}p = \frac{x_F^3}{3\pi^2\lambda^3} \tag{7}$$

where $\lambda = \hbar/(mc)$ is the Compton wavelength of the particle, $p_F = x_F mc$ is Fermi momentum, and we have assume $g = 2$ for the further discussion of the electron and neutron. Eq.(7) has clear physical meaning (the following equations can also be interpreted like this), where λ^3 behaves like the volume that a particle occupy (there is just a difference of coefficient). Now for Eq.(6) at zero temperature we get the density of energy

$$\begin{aligned}\rho_E &= \frac{4\pi g}{h^3}\int_0^{p_F} p^2\sqrt{(pc)^2 + (mc^2)^2}\mathrm{d}p \\ &= \frac{mc^2}{\pi^2\lambda^3}\left[\frac{x_F^3}{3} + \int_0^{x_F} x^2\left(\sqrt{1+x^2} - 1\right)\mathrm{d}x\right] \\ &= \frac{mc^2}{24\pi^2\lambda^3}\left(-8x_F^3 + 3\sqrt{1+x_F^2}\left(x_F + 2x_F^3\right) - 3\mathrm{arcsinh}\left(x_F\right)\right)\end{aligned} \tag{8}$$

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his equation looks like a little complicated, we will soon simplify it together with the following equation. And from Eq.(4) we get the pressure of ideal degenerate Fermi-gas

$$\begin{aligned} P &= \frac{4\pi g}{h^3} kT \int_0^\infty p^2 \ln\left(1 + e^{-\frac{\varepsilon-\mu}{kT}}\right) \mathrm{d}p \\ &= \frac{8\pi}{3h^3} \int_0^{p_F} p^3 \frac{\partial \varepsilon}{\partial p} \mathrm{d}p \\ &= \frac{mc^2}{3\pi^2\lambda^3} \int_0^{x_F} \frac{x^4}{\sqrt{1+x^2}} \mathrm{d}x \\ &= \frac{mc^2}{24\pi^2\lambda^3} \left(x_F^2 \sqrt{1+x_F^2} \left(-3 + 2x_F^2\right) + 3\mathrm{arcsinh}\left(x_F\right) \right) \end{aligned} \tag{9}$$

Equations Eq.(7), Eq.(8), Eq.(9) are the final result we want to derive. Then we will use these result to give the equation of state of white dwarf and neutron star.

For white dwarf, the pressure is generated by the degeneracy pressure of electron, while the mass density (we use mass instead of energy density in Newtonion mechanics) is mainly provided by the rest mass of nuclei. We therefore use Eq.(7) and Eq.(9) and replace the density of particle number by the mass density of matter($\rho_M = m_B \nu_e n_e$, where n_e is density of electron number determined by Eq.(7), $\nu_e = n/n_e$, m_β is the rest energy per nucleon) , and then we get the equation of state of white dwarf (implicit function)

$$\begin{aligned} \rho_M &= \frac{m_B \nu_e}{3\pi^2\lambda^3} x_F^3 \\ P &= \frac{mc^2}{3\pi^2\lambda^3} \cdot \frac{3}{8} \left[x_F \sqrt{1+x_F^2} \left(\frac{2}{3} x_F^2 - 1 \right) + \mathrm{arc\,sinh}\left(x_F\right) \right] \end{aligned} \tag{10}$$

Using this equation of state (together with the Newtonion equilibrium equation that we will discuss in Section 2) we can solve the radius and mass of the white dwarf.

However, in the case of neutron star, we use energy density instead of mass density (because the relativistic effect we will discuss latter). We therefore use Eq.(8) and Eq.(9) to give the equation of state of neutron star. According to Oppenheimer's article, it is better to make the variable substitution $t = 4\mathrm{arcsinh} x_F$, and then from Eq.(8) and Eq.(9) we get a simplifying equation of state (implicit function) as following

$$\begin{aligned} \rho_E &= \kappa \left(\sinh t - t\right) \\ P &= \frac{\kappa}{3} \left(\sinh t + 3t - 8 \sinh \frac{t}{2} \right) \end{aligned} \tag{11}$$

where $\kappa = mc^2 / \left(32\pi^2\lambda^3\right)$. This equation of state will be used (together with TOV equation that we will discuss in Section 3) to solve the radius and mass of neutron star.

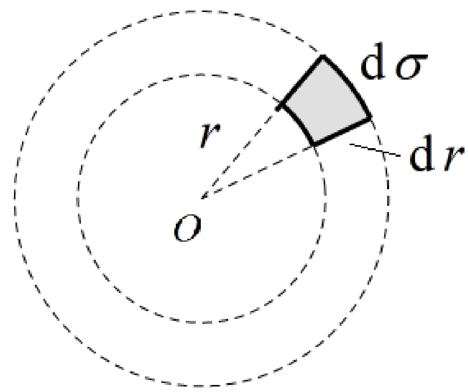

FIG. 1. Sketch of white dwarf

B. Equilibrium of White Dwarf

1. Equilibrium Equation in Newtonion Gravity

For low density compact objects like white dwarf, Newtonion gravity behaves well on solving these system. We assume the white dwarf is spherically symmetric and then use Newtonion gravity to give the equilibrium equation of it. We begin with considering a small shell inside the star. The mass enclosed within r is given by

$$m\left(r\right) = \int_0^r \rho\left(r'\right) \cdot 4\pi r'^2 dr' \tag{12}$$

The gravity and the pressure together make the shell equilibrium. We therefore have

$$d\sigma dP\left(r\right) + \frac{G \cdot \rho\left(r\right) d\sigma dr \cdot m\left(r\right)}{r^2} = 0 \tag{13}$$

By simplifying Eq.(12) and Eq.(13) we will have two equations

$$\frac{dm}{dr} = 4\pi r^2 \rho\left(r\right) \tag{14}$$

$$\frac{dP}{dr} = -\frac{G\rho\left(r\right) m\left(r\right)}{r^2} \tag{15}$$

These two equations determine the equilibrium of the star. But there are three unknown functions in these two equations, we therefore need another equation, usually the equation of state. By solving these two equations together with equation of state, we will finally get the density, pressure distribution within the star, and if $P = 0$ at some $r = r_0$, it means we reach the surface of the star, and thus r_o, $m(r_0)$ will be the radius and mass of it.

2. Mass and Radius of White Dwarf

Therefore, for white dwarf we now have four equations and four unknown functions P, m , ρ , x_F , we list then in the following. In the previous section we have get the equilibrium equation from Newtonian gravity and the definition of mass inside the shell with radius r:

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho \tag{16}$$

$$\frac{dm(r)}{dr} = 4\pi\rho r^2 \tag{17}$$

And other two equations give the implicit relation between density and pressure (here we do not make any further assumptions about the equations of the state, so this is not a polytropic star):

$$\rho = \frac{m_B\nu_e}{3\pi^2\lambda^3}x_F^3 \tag{18}$$

$$P = \frac{mc^2}{3\pi^2\lambda^3}\cdot\frac{3}{8}\left[x_F\sqrt{1+x_F^2}\left(\frac{2}{3}x_F^2-1\right)+\operatorname{arcsinh}(x_F)\right] \tag{19}$$

We bring Eq.(18) and Eq.(19) into Eq.(16) and Eq.(17) , then we get two equations about $x_F(r)$,$m(r)$:

$$\frac{dm^*}{dr^*} = 4\pi {r^*}^2 x^3 \tag{20}$$

$$\frac{dx^*}{dr^*} = -\frac{\sqrt{1+x^2}}{x}\frac{m^*}{{r^*}^2} \tag{21}$$

here we set proper units of m and r, and denote the quantities in our units by '*'. The unit of length is$b = \frac{\pi c}{m_B\nu_e}\sqrt{\frac{3m\lambda^3}{G}} \approx 1.3764\times10^7\text{m}$, the unit of mass is $a = \frac{b^3 m_B\nu_e}{3\pi^2\lambda^3} \approx 5.0643\times10^{30}\text{kg}$, and we choose $\nu_e = 2$ that is proper for most matter. We then set the boundary conditions as $m(0) = 0$, $x(0) = x_0$, where x_0 may be any positive real number corresponding to the density at center of the star, it will finally determine its mass and radius. For any given x_0 , we solve this two equations, and once $x = 0$ at some $r = r_0$, it will be the boundary of the star, then $m(r_0)$ will be the mass of it. By solving this two equations with different x_0 we will get the relations between radius and mass of the star. We may can't solve those equations analytically, but instead we using numerical approach to solve it. Given the initial conditions, we use Euler method to iterate until the boundary is found.

To confirm our result, we can compare it with the analytical one in some special cases. If the density of the white dwarf is low, we can believe that the electron within it is non-relativistic. And if the density is much higher, we may expect the electron to be ultra-relativistic. In these two cases, through approximation of equations Eq.(18) and Eq.(19) we can get the explicit equation of state

$$P = K\rho^\gamma \tag{22}$$

In non-relativistic case $K = \frac{mc^2}{5}\frac{(3\pi^2)^{2/3}}{m_B^{5/3}}\frac{\lambda^2}{\nu_e^{5/3}}$, $\gamma = 5/3$, and in ultra-relativistic case $K = \frac{mc^2}{4}\frac{(3\pi^2)^{1/3}}{m_B^{4/3}}\frac{\lambda}{\nu_e^{4/3}}$, $\gamma = 4/3$.

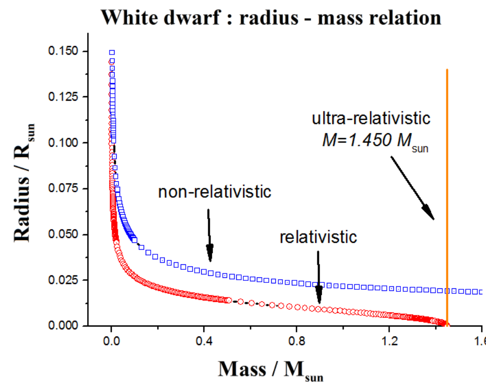

FIG. 2. The radius-mass relation of the white dwarf. Given the central density of the star, we use Euler method to calculate the density distribution of the star from equations Eq.(20), Eq.(21). Vanishing density means the boundary of the star and thus determine the radius and mass of it. Red scattering dots show the numerical result. For comparison, we also plot the result corresponding to non-relativistic (blue scattering dots) and ultra-relativistic (orange line) equation of state. It is clear that there exist upper limit of mass at 1.450 mass of sun (called Chandrasekhar mass).

Now we use Eq.(22) together with Eq.(16) and Eq.(17), and make variable substitution $r = R_0\xi$, $\rho = \rho_0\theta^{1/(\gamma-1)}$, where $R_0 = \left(\frac{\gamma K\rho_0^{\gamma-2}}{4\pi G(\gamma-1)}\right)^{1/2}$, and ρ_0 is the density of the center of the star, then we get the the Lane-Emden equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\xi^2\theta^{1/(\gamma-1)} \tag{23}$$

The corresponding boundary condition is $\theta(0) = 1$, $\theta'(0) = 0$. We can also get the mass and radius of the star in the following way: Solve Lane-Emden equation numerically, and find ξ_0 at which $\theta = 0$ which corresponds the surface of the star, then $R_W = R_0\xi_0$ is the radius of the star, and $M_W = \int_0^{r_0} 4\pi r^2\rho(r)dr = 4\pi R_0^3\rho_0\xi_0^2|\theta'(\xi_0)|$ is the mass of the star. If we cancel the central density ρ_0 from the above two relations, we then get the radius-mass relation of the white dwarf:

$$R_W^3 M_W = \text{const.} = 4\pi\left(\frac{5K}{8\pi G}\right)^3\xi_0^5|\theta'(\xi_0)| \text{ (non-relativistic)}$$

$$M_W = \text{const.} = 4\pi\left(\frac{K}{\pi G}\right)\xi_0^2|\theta'(\xi_0)| \text{ (ultra-relativistic)} \tag{24}$$

FIG.2 show the relation between mass and radius of the star, for comparison we also give the relation of mass and radius that is derived in the non-relativistic and ultra-relativistic cases. From it we can see the fundamental feature of the white dwarf. As the mass of the white dwarf increase, its radius will decrease. That means a more massive white dwarf will much more compact than the

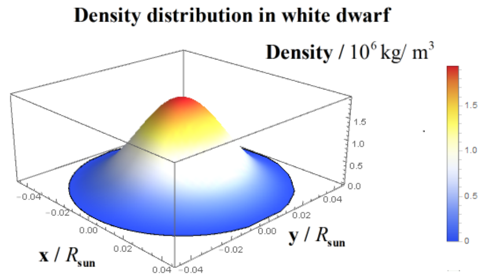

FIG. 3. Density distribution in white dwarf. A spherically symmetric white dwarf is considered here. Using Eq.(19) and Eq.(21) we can solve the density distribution of this star. Here we choose $x_0 = 0.1$ (corresponding to a typical white dwarf with central density $1.942 \times 10^3 \mathrm{kg}/m^3$, radius $0.046 R_\odot$, mass $0.022 M_\odot$), and plot the density on a section passing by the center of the star. The density has a maximum in the center as expected, and vanish in the exterior boundary.

less massive one, so the density of massive white dwarf is very high. We can see that a white dwarf with mass less than $0.1 M_{\mathrm{sun}}$ is "non-relativistic", if we use a non-relativistic equation of state to describe it, the deviation with be small. But once its mass is larger than $0.2 M_{\mathrm{sun}}$ the electron gas within it will be relativistic, we should use relativistic equation of state to describe it. Another feature of white dwarf is that there exist upper limit of its mass (called Chandrasekhar mass), any polytropic white dwarf can't approach this mass unless how high density it would be. The numerical result match the analytically result which use ultra-relativistic equation of state to derive this mass.

FIG.3 show the corresponding density distribution of the star (we just show a section passing the center of the star). We choose the value to make it a typical white dwarf. The density has a maximum value at the center of the star as expected, and it decreases monotonically until reaches the surface of the star.

3. Cooling of White Dwarf

The white dwarf's temperature is high once it comes into being. However, as it burns out its energy source, many process contributes in its cooling. The White dwarf can be viewed as a thin shell (with semi-classical electron gas and thus a relative high conductivity of heat) covering a huge core with degeneracy electron gas and thus being isothermal). We begin with considering the diffusion of photon in the thin shell of the surface of the star, which carries energy of white dwarf to the outer space, and thus cools the star. The luminosity of diffusion of photon is given by

$$L_\gamma = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\mathrm{d}}{\mathrm{d}r} \left(aT^4\right) \tag{25}$$

where ρ is mass density and is opacity, and $a = 7.56 \times 10^{15} \mathrm{erg} \cdot cm^{-3} \cdot K^{-3}$ is radiation constant. When the temperature is relative low, the free-bound process (photon absorbed by bound electron and makes the atom ionized) dominate the opacity given by $\kappa = \kappa_0 \rho T^{-3.5}$. We further assume the star is in equilibrium in a not very long time, so the equation of equilibrium of Newtonion gravity Eq.(14) Eq.(15) is applied to this case. The shell within the surface of the star has low density, so its equation of state should be semi-classical, which mean the following relation

$$P = \frac{\rho}{\nu m_B} \tag{26}$$

where ρ is the mass density as before, $\nu \approx 1.4$ is the mean nucleons per electron. With Eq.(14), Eq.(15) and Eq.(25), Eq.(25) we can solve the relation between pressure and temperature

$$P = \left(\frac{64}{51} \frac{\pi a c G M k}{\kappa_0 L_\gamma \nu m_B}\right)^{1/2} T^{4.25} \tag{27}$$

We now want to find the position at which the semi-classical electron gas become degenerate. So we associate 26, 27 and equation of state of degenerate ideal electron gas 22 (we assume the electron to be non-relativistic, and therefore $\gamma = \frac{5}{3}$), and we therefore get the critical temperature and density

$$T_{\mathrm{critical}} = [\frac{L_\gamma}{2 \times 10^6 M/M_\odot}]^{2/7} (\mathrm{K}) \tag{28}$$

$$\rho_{\mathrm{critical}} = 4.8 \times 10^{-8} (\mathrm{g/cm^3}) T_{\mathrm{critical}}^{3/2} \tag{29}$$

For a typical white dwarf $L = 10^{-3} L_\odot$, $M = M_\odot$, the corresponding temperature and density is $T_{critical} = 8 \times 10^6 \mathrm{K}$, $\rho_{critical} = 1 \times 10^3 \mathrm{g}/cm^3$, which is not much high so that the semi-classical shell is just a thin shell in the surface of the white dwarf.

The critical temperature equation Eq.(28) give the relation between the luminosity and temperature $L_\gamma = CMT^{7/2}$(the core, which occupy the most volume of the star, is isothermal so that the critical temperature can be viewed as the core's temperature), we can use it to determine the cooling of the star. We note that the heat capacity in the core of the star is mainly provided by the ions, because ions here are semi-classical and has large heat capacity while electron here is high degenerate and has nearly negligible heat capacity. So the internal energy of the core here is $E = \frac{3}{2} NkT = \frac{3}{2} kT \frac{M}{m_B A}$, where A is the mass number of the atom and we assume the star is made of single atom molecule. We therefore can get the cooling equation as following

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = L_\gamma \Rightarrow -\left(\frac{3k}{2Am_B C}\right) \frac{dT}{dt} = T^{3/2} \tag{30}$$

We integrate this equation and let $T(t=0) \gg T(t)$, and then get the time evolution of the temperature of the star

$$T(t) = \left(\frac{5Am_B C}{3k} t\right)^{-2/5} \tag{31}$$

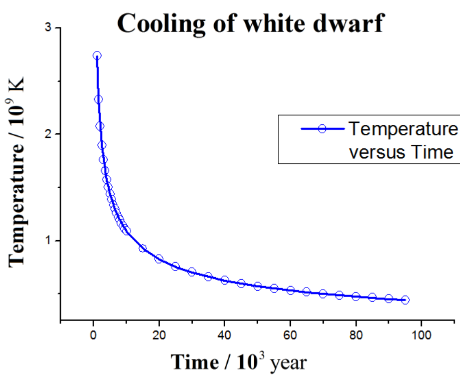

FIG. 4. Cooling of white dwarf. Here we show the temperature – time relation of a typical white dwarf when it is cooling. Parameters here is: mass number $A = 4$ (we assume this star is occupied mainly by helium), and $CM_\odot = 2 \times 10^6$erg/s for a star that bound-free absorption of photon dominate. The temperature $T \propto t^{-2/5}$ so it decreases as time increases. A typical time of cooling of white dwarf is 10^9 year, and here we just plot the time varies from 10^3 to 10^5 year to show the behavior of its cooling.

FIG.4 give the plot of such a process of cooling. The temperature decrease monotonically. If we let the temperature be $T = 8 \times 10^6$K(corresponding luminosity is $L = 10^{-3}L_\odot$ from Eq.(28)), then we get the time $t = 1.8 \times 10^9$year . It is a very long time for human beings!

The white dwarf has relative lower density compared with other compact object, so the Newtonion gravity is suitable for solving the structure of it. However, for other compact object (e.g. neutron star), the strong interaction of nucleons must be considered, so the pressure provided by nucleons will also affect the structure of the star. What's more, as gravity be much stronger, the effect of general relativity must be play import role in determining the structure of the star and problem thus will be much more complicated In the case of neutron star. Therefore white dwarf is a just a very special case of compact objects in our universe. We will consider the neutron star in the following section.

II. THE THEORY OF NEUTRON STARS

A. Equilibrium of Neutron Stars

The constitution of realistic neutron star may be very complex, it has layer structures like FIG.5. It consists of not only neutrons, but also electrons and other quark matters such as protons, pions, even hyperons etc. And there exists strong interaction among quark matters. While the theory of strong interaction is beyond our ability for now,, we neglected the strong interaction and considered neutron star which is completely composed of

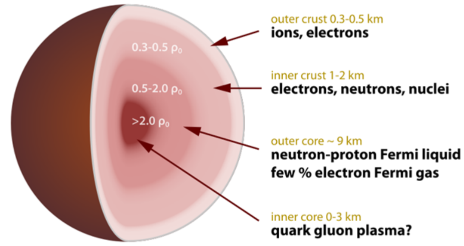

FIG. 5. The possible interior structure of neutron star[a]

[a] cited from Wikipedia

neutron to calculated its equation of state.

1. Equation of Mass Continuity

Due to the spherical symmetry of neutron star, we can easily obtain

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{32}$$

where $m(r)$ represents the mass within a sphere whose radius is r and center is located on the center of neutron star, $\rho(r)$ is the radial distribution of density. It reveals the relationship between $m(r)$ and $\rho(r)$.

2. TOV Equation

Schwarzschild radius $r_S = \frac{2GM}{c^2}$ is a quantity which stands for the strength of the gravity field. When the radius of celestial object $r \ll r_S$, ts gravity field is weak field; while when the radius of celestial object r approaches r_S, its gravity field is quite strong, so the effect of general relativity can't be neglected. Because the mass-radius ratio of neutron star M/R is very large, (for neutron $\frac{r_S}{R} = \frac{2GM}{Rc^2} \approx 0.1$), there is no doubt that general relativity needs to be used.

Starting from static spherical symmetric spacetime metric,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{33}$$

we can acquire the metric tensor $g_{\mu\nu}$ of the spacetime, next $\Gamma^\alpha_{\mu\nu}$, and then $R_{\mu\nu\alpha\beta}$ and so on. Utilizing $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, all $G_{\mu\nu}$ can be calculated out.

Next, we further assume that perfect fluid is in equilibrium, which means that P and ρ do not depend on time, u_μ has only t-components.

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{34}$$

From these equations we can calculate out all $T_{\mu\nu}$.

Using Einstein equation and $\nabla_\mu T^{\mu\nu} = 0$. Eventually, we can derive the TOV equation that determines the state of neutron star,

$$\frac{dP(r)}{dr} = -(\rho_E(r) + \frac{P(r)}{c^2})\frac{G[m(r) + \frac{4\pi r^3 P(r)}{c^2}]}{r(r - \frac{2Gm(r)}{c^2})} \quad (35)$$

when $\rho(r)$ is the pressure at radius r of neutron star, G is the gravitational constant. This equation is quite complex. Note that this equation reduces to Eq.(15).

3. Equation of State

To solve the state function of neutron star, i.e. solve for $\rho(r)$, $m(r)$, $P(r)$, we need another equation $P = P(\rho)$. Pay attention to that the first two equations are almost universal, they are irrelevant to the concrete states and types of matter and so on. Therefore, we need an equation of state that describes neutron star in special. It also is the core part of our neutron star model.

Considering that neutron and electron are both Fermions which obey the Fermi-Dirac statistics, the degeneracy pressure of neutron in neutron stars is similar to the degeneracy pressure of electron in white dwarfs. Imitating the solution of white dwarf, we can obtain the Fermi gas model of neutron star, just by replacing all parameters in white dwarf with parameters in neutron star, such as substituting mass of neutron for electronic mass etc. According to the Fermi gas model of neutron star, we can obtain the expression of pressure,

$$P = \frac{mc^2}{24\pi^2\lambda^3}[x_F^2\sqrt{1+x_F^2}\left(-3+2x_F^2\right) + 3\text{arcsinh}\left(x_F\right)] \quad (36)$$

where

$$\lambda = \frac{\hbar}{m_n c} \quad (37)$$

and P is pressure, p_F is Fermi momentum

$$\rho_E = \frac{mc^2}{24\pi^2\lambda^3}[-8x_F^3 + 3\sqrt{1+x_F^2}\left(x_F + 2x_F^3\right) - 3\text{arcsinh}\left(x_F\right)] \quad (38)$$

We are able to derive the equation of state $P = P(\rho)$ using these relations.

4. The Solution of State Functions of Neutron Star

The ODE set:

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \quad (39)$$

$$\frac{dP(r)}{dr} = -(\rho_E(r) + \frac{P(r)}{c^2})\frac{G[m(r) + \frac{4\pi r^3 P(r)}{c^2}]}{r(r - \frac{2Gm(r)}{c^2})} \quad (40)$$

$$P(r) = P(\rho(r)) \quad (41)$$

with appropriate initial conditions $\rho(0) = 10^{18}$ kg/m^3 determines the solution uniquely. However, due to complexity of this ODE set, we chose to solve it numerically by using Euler–Cromer method. Numerical solutions of $P(r)$, $m(r)$ and $\rho(r)$ are shown in FIG.6.

Having obtained these state functions, we are able to derive the relation of a neutron star's total mass M and its radius R. This relation is important for it can be tested via observation. Our Solution to $M - r$ Relation are shown in FIG.7.

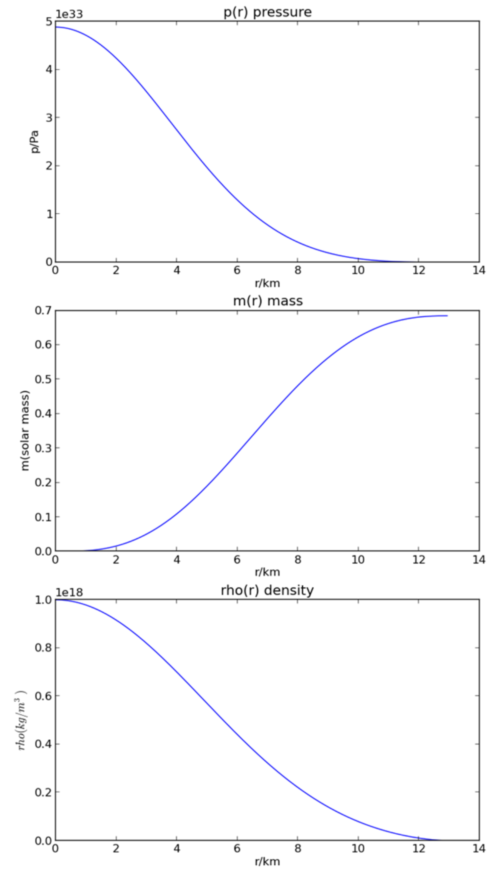

FIG. 6. Numerical Solution of $P(r)$, $m(r)$ and $\rho(r)$

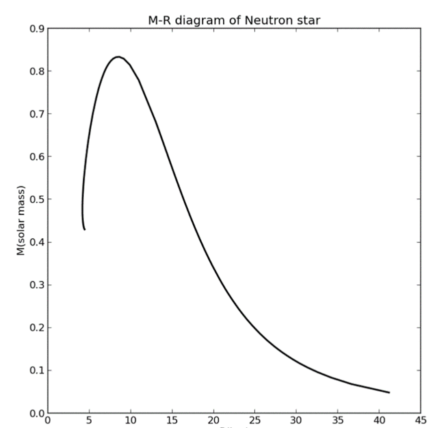

FIG. 7. Our Solution to $M - r$ Relation

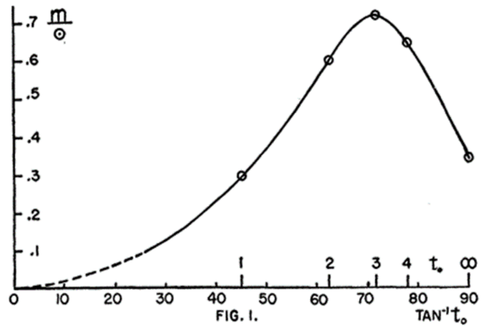

FIG. 8. The solution of Oppenheimer and Volkoff. Note that t is a quantity which is related to radius R. Its concrete expression which is a little complex is negelected here. Reader can refer to their original article

On the right side of single-peaked curve, the radius (or say volume) of neutron star is a decreasing function of its mass, that is normal; while on the left side of single-peaked curve, radius is an increasing function of its mass, which is abnormal! When the increasing of its mass leads to the increasing on its gravity, only if its radius decreases can it produce higher pressure to counteract increasing gravity. However, on the left side of single-peaked curve, radius should be an increasing function of its mass. Such equilibrium is unstable. If there exists a little perturbation, it will collapse or explode rapidly.

Realistic neutron star must be stable, so the peak value point where its radius is about 10km, mass is about 0.85 solar mass, is the limit state of neutron star. Or say the minimal radius of neutron star is about 10 km, the upper limit of mass is about $0.85M_{\odot}$ under this model.

J.R. Oppenheimer and G. M. Volkoff solved the same problem ,the same model by numerical method. Their result is shown in FIG.8. Their conclusion is that the upper limit of mass of neutron star is about $0.75M_{\odot}$. In general, our result is similar to their result. Two curves have the similar shapes and their peak values do not have not large difference. However, the difference between them cannot be ignored, for it cannot be just due to calculation error. This difference may well be explained by following reasoning:
1. Their data points are too few, while our calculation produced far more data points. Hence, our work is more convincing.
2. Their computing method is different from ours.
With all above arguments taken into consideration, we believe our result is more accurate.

5. *The Cooling of Neutron Stars*

Next, we consider the cooling of neutron star. This process includes two periods: neutrino cooling era and photon cooling era. From several days to thousands years, neutrino emission dominated it cooling. After 10^4 years, the star cools via photon emission,

$$\frac{dQ}{dt} = c_V \frac{dT}{dt} = -L_\nu(t,T) - L_\gamma(t,T_s) \tag{42}$$

where N is density of particle, and $x = \frac{p_F}{mc}$.
Neutrino luminosity

$$L_\nu = bT^k \tag{43}$$

There are many processes to emit neutrino, for example modified Urca process. For this process, we can find that cooling time

$$\Delta t(\text{URCA}) \approx 1 \times T_f^{-6} \text{ Yr} \tag{44}$$

Photon luminosity

$$L_\gamma = 4\pi\sigma R^2 T_s^4 \tag{45}$$

where

$$T_s \approx (\alpha T)^\beta \tag{46}$$

$$\alpha = 7 \tag{47}$$

$$\beta = 0.5 \tag{48}$$

Therefore we have

$$L_\gamma = cT^2 \tag{49}$$

where $c = 4\pi \times 49\sigma R^2$.
For this process we have

$$\Delta t(\text{Photon}) \approx 10^3 \text{ Yr} \tag{50}$$

In general, the cooling equation is:

$$\alpha T \frac{dT}{dt} = -bT^k - cT^2 \tag{51}$$

III. EXPERIMENTAL OBSERVATION OF COMPACT OBJECTS

A. Astronomical Observation of White Dwarfs

1. *Historical Introduction*

The first white dwarf identified is *40 Eridani B* which is in a three-body system of *40 Eridani*. The star was discovered by Willam Herschel in 1783, and was later re-observed several times in the next century. It was not until the year of 1910 that *40 Eridani B*'s pecularity had been noticed by astronomers. A group of British astronomers: Henry Norris Russell, Edward Charles Pickering and Williamina Fleming found that *40 Eridani B* is "white" despite the fact that it was considerably dim.

The most famous white dwarf — *Sirius B* was also discovered in early periods. The great mathematician ,astronomer Friedrich Bessel noticed in 1844 that *Sirius* moved in a periodic manner. He further asserted that

Sirius had unseen co-star, and estimated its period to be half a century. Later observations confirmed its existence. In 1915, Walter Adams found that *Sirius B* is also "white" despite the fact that it was considerably dim.

These two white dwarfs, together with *Van Maanen's Star*, were named *classical white dwarfs*, since they were the first three white dwarfs that have been discovered.

It was before long that white dwarfs' extraodinary huge density was noticed by astophysicists. At the beginning of the 20th century, astronomical observations have become accurate. Astrophysicists are able to determine the mass of a star which is in a binary system by investigating its dynamical property. Determining the radii of a star was a little more complicated, but also within our reach. By invoking the following formula:

$$L = 10^{0.4(4.72-M_b)} L_{\odot} \tag{52}$$

one is able to determine total luminosity of any object(Notice that here M_b is the absolute bolometrc magnitude which can be measured on earth). Spectral analysis also enables one to obtain the effective temperature on the star. The combination of these two paramters enable physicists to estimate(*not* determine!) the radii of the star through *Stefan-Boltzmann Law*:

$$L = 4\pi R^2 \sigma T_e^4 \tag{53}$$

In 1916, Ernst Opik estimated that 40 Eridani B's density to be $\rho_{\odot}$. These astonishing findings motivated physicists including Chandraeskhar to seek for a new star model.

The theory of white dwarfs was soon formulated, meanwhile the search for white dwarfs was progressing steadily. By 1999, more than 2000 white dwarfs were found. The project Sloan Digital Sky Survey was a great leap forward, for more than 9000 white dwarfs were found by the project.

2. *Observational Methods and Achievements*

According to the theory of white dwarfs, white dwarfs with definite mass should have definite radii. Almost all the white dwarfs observed have the mass around $M_{\odot}$. Then according to *Stefan-Boltzmann Law*, we should have:

$$L \propto T_e^4 \tag{54}$$

Therefore, white dwarfs are expected to occupy a narrow strap in *H-R diagram*. In fact, they do so as shown in FIG.9. In fact, such tests are named "zero-order tests" of the theory of white dwarfs.

Determination of white dwarfs' radii can be tricky. According to energy conservation in radiation, we have:

$$F_{\nu}(\text{Measured}) = F_{\nu}(\text{Surface}) \frac{R^2}{D^2} \tag{55}$$

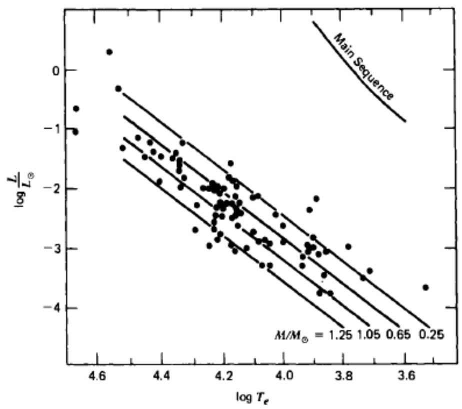

FIG. 9. White Dwarfs on H-R Diagram[a]

[a] cited from Shapiro's book

where F_{ν} is radiation flux. In this expression, F_{ν}(Measured) is measured on earth. F_{ν}(Surface) is calculated in an atmosphere model which requires the measurement of surface temperature and surface gravity of the white dwarf. A detailed description was compiled by Shipman(1979). D is calculated with nearby stars by measuring their parallax. Therefore, we are able to obtain R.

What about mass of white dwarfs? This can be easily done for a white dwarf in binary or triple star system. One simply has to analyze the dynamical behavior. Determination of solitary white dwarf's mass is tricky. General relativity predicts that photons near strong gravitational field are red-shifted. By measuring this red shift effect, one is able to determine the white dwarf's mass. This effect is quantitatively described by:

$$\frac{\Delta\lambda}{\lambda} = \frac{2GM}{Rc^2} \tag{56}$$

In practice, astronomers use the non-LTE core of Hα line to measure this effects, for other lines are easily affected by pressure shifts. We have to be extra careful about the choice of white dwarfs, for the red-shift effects need to be distinguished from the Doppler effects. In order to do that, physicists usually use white dwarfs in wide binaries or common proper-momtion pairs, for these white dwarfs' velocities can be measured accurately.

Typical results obtained by these methods are listed below:

Object	Mass($M_{\odot}$)	Radius($M_{\odot}$)
Sirius B	1.03 ± 0.015	0.0074 ± 0.0007
Stein 2051B	0.48 ± 0.045	0.0111 ± 0.0015
40 Eri B	0.43 ± 0.02	0.0124 ± 0.0005
G107-70AB	0.65 ± 0.15	0.0127 ± 0.002
Procyon B	0.594 ± 0.012	0.0096 ± 0.0005

TABLE I. Typical White Dwarfs' Mass and Radius

As we can see in FIG.10, observations do not agree

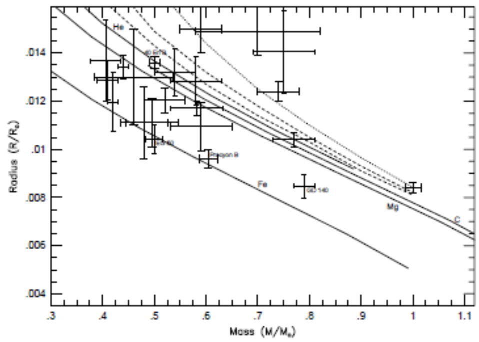

FIG. 10. Mass–Radii Relation[a]

[a] cited from Shipman's paper

with our theory of model perfectly. Such facts indicate the necessity to further modify our model.

B. Astronomical Observation of Neutron Stars

1. *Historical Introduction*

The idea of neutron stars was proposed by Baade and Zwicky in 1934. In the paper they assumed neutron stars to have very high density and small radius, and further suggested that neutron stars would be formed in supernova. After 5 years, Oppenheimer and Volkoff made the first calculation under the assumption that neutron star matter being composed of high density free neutron ideal gas. Desipte the fact that their work was conducted beautifully, the idea of neutron stars was not taken seriously at the time, for other astrophysicists argued that even if neutron star existed, their residual radiation would be too weak to be observed.

In 1962, Giacconi et al. discovered cosmic nonsolar X-ray sources, which stimulated a wide discussion about whether such sources are neutron stars. However, deeper analysis ruled out this possibilty since the sources' mass exceeded the maximum mass of stable neutron stars.

In 1967, Jocelyn Bell and Anthony Hewish discovered the first pulsar. At the time they could not confirm that such an accurate periodic X-ray source was natural, so they named it "*Little Green Man*" which stood for extraterrestrial origin. Soon after the first discovery, another X-ray source at the other part of sky was discovered which ruled out the possibility that this was artificial. In 1968, Gold argued that the observed pulsars were in fact rotating neutron stars which have surface magnetic field $B \sim 10^{12}$ G. He further predicted small increase of the period because of energy lost via electromagnetic radiation. In 1969, Gold showed that the slowdown of the Crab pulsar could be explained by neutron star model. In fact, this discovery also provided strong support on the supernova-origin neutron star formation thoery.

X-ray observation of binary X-ray sources allows physicists to determine neutron star mass. Also, binary neutron star system provides the first indirect proof for gravitational waves.

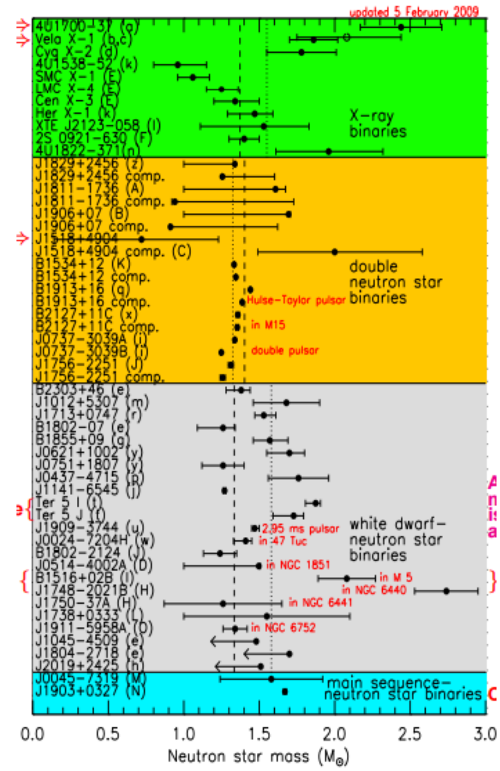

FIG. 11. Neutron Star's Mass[a]

[a] cited from Lattimer's work

By the end of 2015, 2536 pulsars had been discovered.

2. *Observational Methods and Achievements*

Measuring neutron star's mass is very much alike measuring white dwarf's mass. The most trusted results are obtained by measuring the neutron star's dynamical behavior within a binary or triple system. Using this method, astronomers have achieved considerable success till now. Some results are listed below:

Object	Mass($M_\odot$)
$J0337+1715$	1.4378 ± 0.0013
$J0348+0432$	2.01 ± 0.04

TABLE II. Typical Neutron Star

Direct measurement of neutron star's radius do not exist. However, one can still combine the observational data with some assumptions to estimate the radius. Van Pradijs estimated in 1978 that a $1.4M_\odot$ neutron star has a radius ~ 8.5 km.

The observed properties of pulsars are comparatively richer.

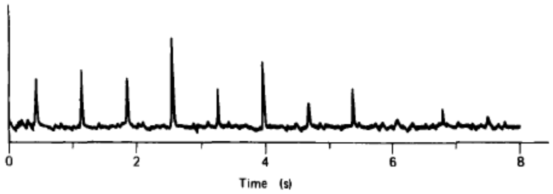

FIG. 12. Pulse Shape[a]

[a] cited from Shapiro's book

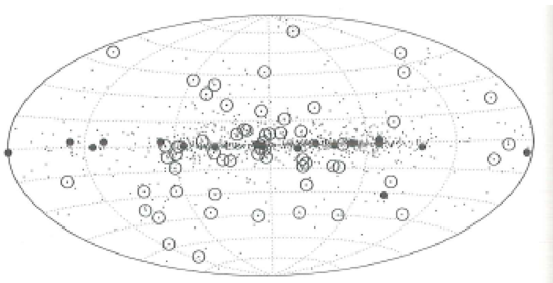

FIG. 13. Galactic Population: Pulsar-supernova remnant associations and millisecond pulsars are shown by the filled and open circles respectively[a]

[a] cited from Lorimer's book

All pulsars' pulse are in the form of periodic pulses. A typical pulse consists of several subpulses which have complicated microstructure. Average over several hundred pulses is remarkably stable(FIG.12).

Galactic distribution is also interesting since it indicates pulsar's origin has relation with supernova.

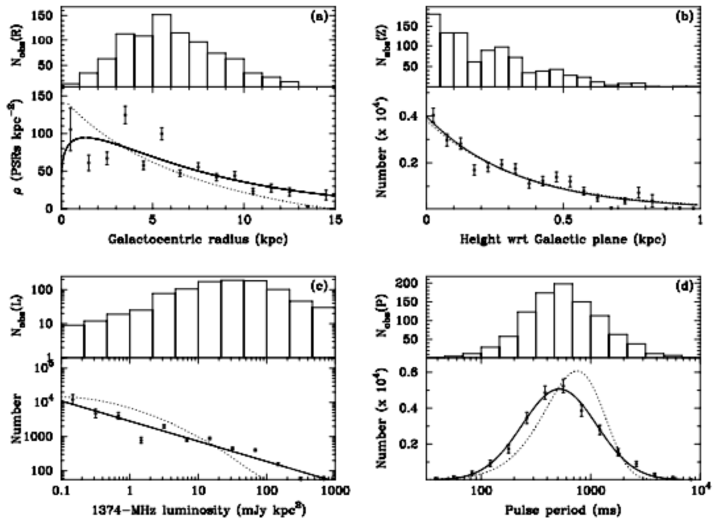

FIG. 14. Observed number distribution from our input sample (upper panels) and derived distributions for model S (lower panels)[a]

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