§ Manifold

- one of the most fundamental mathematicall object in this course.

  all our spacetimes studied in this couse are manifold.

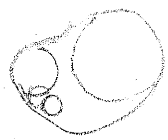
- open ball in $\mathbb{R}^n$ ($\mathbb{R}^n$ is a topological space) 

  an open ball centered around a point $y = (y^1, \cdots, y^n)$ in $\mathbb{R}^n$ is with radius r

$$\left\{ x \in \mathbb{R}^n \,\middle|\, |x-y| = \left(\sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \right)^{\frac{1}{2}} < r \right\}.$$

open set in $\mathbb{R}^n$

set of points expressable as union of open balls.

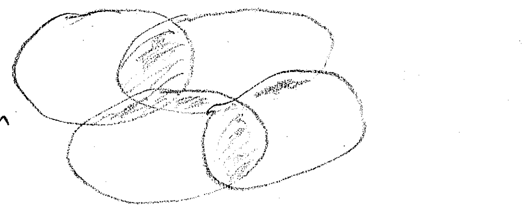


(A) manifold:

simple version: a set made up of pieces "looks like" open subsets of $\mathbb{R}^n$ such that these pieces can be "sewn/glued together" smoothly.

→ suggested to remember

precise version:



an n-dim, C^∞, real manifold M is a set together with a collection of subsets $\{O_\alpha\}$ satisfying:

(1) Each $p \in M$ must lie in at least one $\{O_\alpha\}$. $\{O_\alpha\}$ cover M.

(2) ~~$\forall \alpha$,~~ For each α, ~~there exist~~ there is one-to-one, onto, map

$$\psi_\alpha : O_\alpha \to U_\alpha,$$ where U_α is an open subset of $\mathbb{R}^n$.

(3) for two sets $O_\alpha \cap O_\beta \neq \phi$, the map

$$\psi_\beta \circ \psi_\alpha^{-1}$$ is C^∞.

Now let us look at this definition more closely.

What kind of set?

- M is a set: a collection of objects

$M: \{ ①, ②, ③, \cdots \}$

- Together there should exist $\{O_\alpha\}$

$\{O_\alpha, O_\beta, \cdots\}$

Each $O_\alpha \subseteq M$

- (1) $\{O_\alpha\}$ covers M.

Any object in M must be in ~~at least~~ one or more O_α.

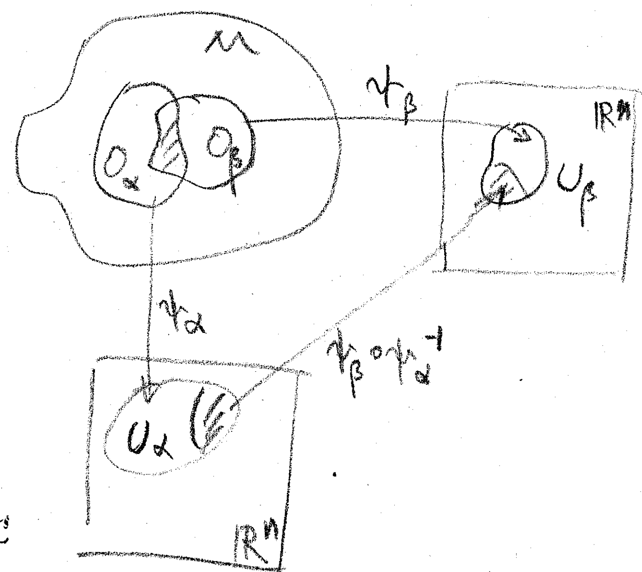
$O_\alpha = \{ ①, ② \}$

$O_\beta = \{ ②, ③, \cdots \}$

$\vdots$

What kind of ~~object~~ subset

- (2) One-to-one and onto mappable to open ~~set~~ subset of $\mathbb{R}^n$.



["one-to-one ~~function~~ ~~mapping~~" & "one-to-one correspondance" are different).

~~Composition is smooth~~

one-to-one + onto = one-to-one correspondance.

one-to-one : let $f: A \to B$ be a function. f is said to be one-to-one if

$\forall x_1, x_2 \in A,$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$

onto : $\forall y \in B, \exists x \in A$ s.t. $f(x) = y$.]

- (3) Composite of $\psi_\beta \circ \psi_\alpha^{-1}$ is smooth.

↳ infinitly differentiable.

(B) Example of manifolds

1. • $\mathbb{R}^n$, $\{0\}$

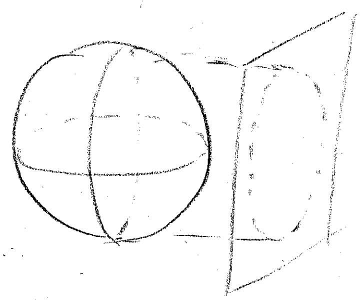
$O = \mathbb{R}^n$, ψ = identity map

• any open subset of $\mathbb{R}^n$.

2. 2-sphere : S^2

$$S^2 = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$$

one ~~map~~ function ψ can not map S^2 to $\mathbb{R}^2$.
And that is also the reason we need a collection of ~~charts~~ ψ_α.



3. (In this course, we view the spacetime as a 4-d manifold.
There are theories with extra-dimension, such as string theory, the view the spacetime as a 10- or 11- dimensional manifold.)

this couse = GR $\Longrightarrow$ space time is 4d manifold

Some other theories (e.g. string) $\Longrightarrow$ spacetime is 10- or 11-d manifold.

Now the definition of manifold is clear. Let us also name other quantities we encountered in this definition.

- The mapping ψ_α are called "chart" or "coordinate system".

mathematician physicists

- The collection of all $\{(O_\alpha, \psi_\alpha)\}$ is called "atlas".

~~Extention~~ product of manifold.

Given two manifold M, M' with dim d, d', form a new manifold

$$p \in M,\ p' \in M',\quad (p, p') \in M \times M' \cancel{\equiv \text{new manifold}}.$$

$$\psi_\alpha : O_\alpha \to U_\alpha,\ \psi'_\beta : O'_\beta \to U'_\beta.$$

$$\psi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta}.$$

$$O_{\alpha\beta} \equiv O_\alpha \times O'_\beta,\quad U_{\alpha\beta} \equiv U_\alpha \times U'_\beta$$

$$\psi_{\alpha\beta}(p, p') = [\psi_\alpha(p), \psi'_\beta(p')].$$

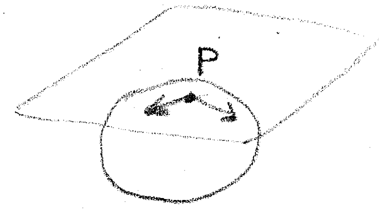
This $\psi_{\alpha\beta}$ and $\{O_{\alpha\beta}\}$ satisfies the definition of a ~~new~~ manifold.

Then $M \times M'$ is a new manifold. It is called the "product manifold".

- Most manifold in this cause $\Rightarrow$ $\mathbb{R}^n \times S^m$, $n + m = 4$.
- Many manifold in string theory are "product manifold" too.

§ Vectors

1. Tangent vectors and tangent space



- Intuitively, when manifold is embedded in $\mathbb{R}^n$
- When embeding is not explicit or possible.

Let $\mathcal{F} = \{$ all C^∞ functions from $M \to \mathbb{R}^1 \}$.

then a tangent vector v at point $p \in M$ is a map $v: \mathcal{F} \to \mathbb{R}^1$, s.t.,

(1) $v(af+bg) = a\,v(f) + b\,v(g) \quad \forall f, g \in \mathcal{F}$; $a, b \in \mathbb{R}$; (tv)

(2) $v(fg) = f(p)\,v(g) + g(p)\,v(f)$ (Leibnitz rule)

The collection of tangent vectors form a vector space, V_p.

Thm: Let M be a n-dim manifold. Let $p \in M$ and V_p denote the tangent space at p. Then $\dim V_p = n$.

The operators $X_\mu : \mathcal{F} \to \mathbb{R}$ defined as

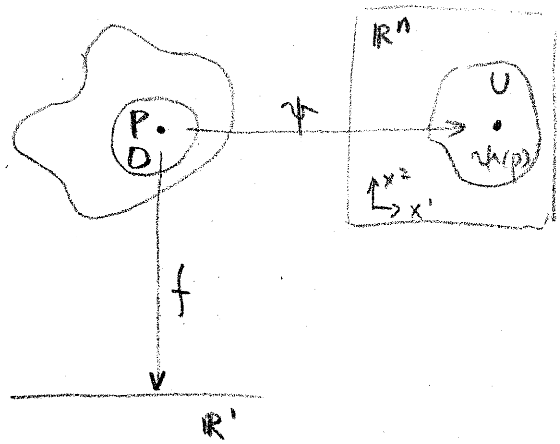
$$X_\mu \circ f \equiv \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}), \quad \mu = 1, \cdots, n$$

span the tangent space V_p and therefore is its basis.

Here $(x^1, x^2, \cdots, x^n)$ are the Cartesian coordinates of $\mathbb{R}^n$; ψ is the chart from M to $\mathbb{R}^n$.

- $\{ X^\mu = \frac{\partial}{\partial x^\mu}, \mu = 1, \cdots, n \}$ is called the "coordinate basis";

(because it has to apply on $f \circ \psi^{-1}$, which means ψ^{-1} has to be used and known. and ψ is called "coordinate system").



(B1)

Proof:

Let $\psi: O \to U \subset \mathbb{R}^n$ be a chart, $p \in O$

Let $f \in \mathcal{F}$, then

$f \circ \psi^{-1}: U \to \mathbb{R}^1$ is C^∞.

then for $\mu = 1, \dots, n$ define $X_\mu : \mathcal{F} \to \mathbb{R}^1$ by

$$X_\mu(f) = \frac{\partial}{\partial x^\mu} (f \circ \psi^{-1}) \Big|_{\psi(p)}$$

we see X_μ, $\mu = 1, \dots, n$ are tangent vectors from definition (tv), and they are linearly independant.

We show that X_μ span V_p.

For any function $\boxed{C^\infty}$, $F: \mathbb{R}^n \to \mathbb{R}$, we can construct $H_\mu(x)$, s.t.

$$F(x) = F(a) + \sum_{\mu=1}^{n} (x^\mu - a^\mu) H_\mu(x)$$

~~Then~~ we have

$x = (x_0^1, x_0^2, \dots, x_0^n)$

$a = (a^1, a^2, \dots, a^n)$

$$H_\mu(a) = \frac{\partial F}{\partial x^\mu} \Big|_{x=a}.$$

Now letting $F = f \circ \psi^{-1}$, $a = \psi(p)$, then for all $q \in O$, acting F on $\psi(q) \equiv x$ ~~yields~~ yields

$$f(q) = f(p) + \sum_{\mu=1}^{n} [x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] H_\mu(\psi(q))$$

here $\underline{x^\mu} \circ \psi(\cdot)$ is the projection operator.

~~Apply~~ Let $v \in V_p$, we show v is a linear combination of X_n. To do this, we apply v on f, and then evaluate on p

$$v(f) = v[f(p)] + \sum_{\mu=1}^{n} \Big\{ [x^\mu \circ \psi(q) - x^\mu \circ \psi(p)] \Big|_{q=p} v(H_\mu \circ \psi)$$
$$+ H_\mu \circ \psi \big|_p \, v[x^\mu \circ \psi - x^\mu \circ \psi(p)] \Big\}$$

Now v acting on constant produce zero, therefore

$$= \sum_{\mu=1}^{n} [H_\mu \circ \psi(p)]\, v(x^\mu \circ \psi)$$

Now $H_\mu \circ \psi(p) = H_\mu(a) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=a} = \left.\frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\right|_{\psi(p)} = X_\mu(f)$,

then

$$v(f) = \sum_{\mu=1}^{n} v^\mu X_\mu(f)$$

where

$$v^\mu \equiv v(x^\mu \circ \psi).$$

This can be done for any f, and therefore an arbitrary vector v can be expressed as a sum of X_μ,

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu.$$

(A) X_μ depend on both ψ and f.

(B) v^μ only depend on ψ.

(C) therefore v depend on f too, but its component does not depend on f. they only depend on ψ.

(D) note that v is indeed a number $\in \mathbb{R}^1$.

X_μ depends on ψ,

$\psi \to \{X_\mu\}$.

If we choose another coordinate system ψ'

$\psi' \to \{X'_\mu\}$.

then chain rule

$$X_\mu = \sum_{\nu=1}^{n} \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\psi(p)} X'_\nu \,, \quad \text{where } x'^\nu \text{ is the } \nu\text{th component of } \psi' \circ \psi^{-1}.$$

For a vector v, the components change according to

$$v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \quad \text{(vector transformation law)}$$

(so they are the same as change of basis in linear algebra, where $\frac{\partial x'^\nu}{\partial x^\mu}$ is the matrix for coordinate change).

2. smooth curve & ~~tangent~~ field.

A <u>smooth curve</u> on a manifold M is a C^∞ map

$$C: \cancel{M \to \mathbb{R}^1}\ \mathbb{R}^1 \to M$$

For any point $p \in M$ lying on the curve C, and $\forall f \in \mathcal{F}$, we associate a tangent vector T_p, defined as

$$T_p(f) \equiv \frac{d(f\circ C)}{dt} = \sum_\mu \frac{\partial}{\partial x^\mu}(f\circ\psi^{-1})\frac{dx^\mu}{dt} = \sum_\mu \frac{dx^\mu}{dt} X_\mu(f)$$

Note here $f\circ C$ is a function from $\mathbb{R}^1$ to $\mathbb{R}^1$; $f\circ\psi^{-1}$ maps from $\mathbb{R}^n$ to $\mathbb{R}^1$.

This tangent vector function apparently depend on the choice of f.

How ever, its components in any coordinate basis, is independent of f & given by

$$T = \sum_{\mu} T^{\mu} X_{\mu}(f), \quad T^{\mu} = \frac{dx^{\mu}}{dt}.$$

3. Tangent field.

- V_p, V_q are different vector space
- If no other structure is given to M, then no natural way to identify $v_p \in V_p$ with $v_q \in V_q$.
- However, we can define a smooth tangent field on all ~~p,q~~ points on M.

(2). Now for $f \in \mathcal{F}: M \to \mathbb{R}$, for each $p \in M$,

$v|_p(f)$ is a number; i.e., $v(f)$ is a function on M.

~~$V(f)$ is called a tangent field on M.~~

~~If for any f, $v(f)$ is smooth, they then~~

(1) A tangent field, v, on a manifold M is an assignment of a tangent vector $v|_p \in V_p$ at each point $p \in M$.

(3) If $v(f)$ is a ~~funct~~ smooth function for any f, then we say the tangent ~~vector~~ field v is smooth.

(4) v is smooth only when v^{μ} is smooth because

$$v = v^{\mu} X_{\mu}(f)$$

and $X_{\mu}(f)$ is smooth because f, ψ^{-1} are smooth.

§ Tensor

(1) Dual vector space

Let V be a finite-dimensional vector space (which could be a tangent space).

Let V^* be collection of linear maps $f: V \to \mathbb{R}^1$.

Define addition and scalar multiplication in this space, and then V^* becomes a linear space. We name it as <u>dual vector space</u> to V.

Any $f \in V$ is called <u>dual vector</u>.

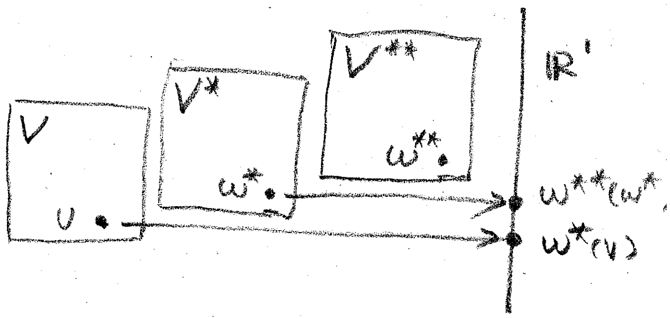
If $v_1, \cdots, v_n$ is a basis of V, ~~we call $v^*_1, \dots, v^*_n \in V$ s.t.,~~ then $v^{1*}, \cdots, v^{n*}$, s.t.,

$$v^{\mu *}(v_\nu) = \delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases},$$

can be proven to be <u>basis of V^*</u>.

Define <u>double dual vector space</u>

$$V^{**} = \{ f \mid f: V^* \to \mathbb{R}^1 \}.$$



Now we do the follow identification

$$w^{**}(w^*) = w^*(v),$$

$$w^{**} \longleftrightarrow v$$

Then the double dual vector space can be identified with the space V.

(2) Tensor

Def: A tensor T of type (k, l) over V is

$$T: \underbrace{V \times V \cdots V}_{K} \times \underbrace{V^* \times \cdots \times V^*}_{L} \longrightarrow \mathbb{R}^1.$$

Define usual addition and scalar multiplication, then

$$\mathcal{T}(k,l) \equiv \{ T \mid T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R}^1 \}$$

- is an n^{k+l} dimensional vector space
- each element T is multilinear ~~on a~~
- any T can be completely known if we know how it acts on the basis $\{v^*\}$ of V^* and $\{v\}$ of V.

$$T \leftarrow T(v^{i_1 *}, v^{i_2 *}, \ldots, v^{i_k *}, v_{j_1}, \ldots, v_{j_l})$$

Two tensor operations:

1° contraction

$$C : \mathcal{T}(k,l) \to \mathcal{T}(k-1,l-1)$$

$$CT = \sum_{\sigma=1}^{n} T(\ldots, \underset{\substack{\uparrow \\ i\text{th}}}{v^{\sigma *}}, \ldots; \ldots, \underset{\substack{\uparrow \\ j\text{th}}}{v_\sigma}, \ldots)$$

Change the vectors at ith, jth position to a pair of basis.

Contraction between ith dual and jth positions.

- independent of change of basis for V, V^*.

2° Outer product: $T \otimes T' \equiv T \cdot T'$ regular times

$$\mathcal{T}(k+k', l+l') \equiv \{ T \otimes T' \mid T \in \mathcal{T}(k,l);\ T' \in \mathcal{T}(k',l'); \}$$

~~$T \otimes T' \equiv T \times T'$~~

~~"regular times"~~

- outer product form a new tensor;
  collection forms a vectorspace too.

- $\{ v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1 *} \otimes \cdots \otimes v^{\nu_l *} \}$ forms a basis for $\mathcal{T}(k,l)$

  Every T of type (k,l) can be expressed

$$T = \sum_{\mu_1, \cdots \nu_l = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, v_{\mu_1} \otimes \cdots \otimes v^{\nu_l *}$$

$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ are called <u>components</u> of T.

3° Contraction & out product in terms of components.

3.1 $$(CT)^{\mu_1 \cdots \mu_{k-1}}{}_{\nu_1 \cdots \nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1 \cdots \sigma \cdots \mu_k}{}_{\nu_1 \cdots \sigma \cdots \nu_{l-1}}$$

3.2. $S = T \otimes T'$

$$S^{\mu_1 \cdots \mu_{k+k'}}{}_{\nu_1 \cdots \nu_{l+l'}} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, T'^{\mu_{k+1} \cdots \mu_{k+k'}}{}_{\nu_{l+1} \cdots \nu_{l+l'}}$$

We will mostly work with components in this course.

4° Applying to tangent space V_p of $p \in M$.

- V_p^* called cotangent space
- $v \in V_p$ called ~~covariant~~ contravariant vectors; ~~compon convention: v^μ~~

  $v^* \in V_p^*$ called covariant vectors; ~~convention~~.

  convention: contravariant vector component use superscript v^μ
  covariant - - - - - subscript v_μ;

- V_p basis : $\partial/\partial x^1, \partial/\partial x^2, \cdots, \partial/\partial x^n$

V^*_p basis denoted as $dx^1, \cdots, dx^n$ (only symbols; defined through $dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_\nu$)

- change of basis

$$v'^{\mu'} = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^{\mu'}}{\partial x^\mu}$$

$$\omega'_{\mu'} = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^{\mu'}}$$

$$T'^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'} = \sum_{\mu_1, \cdots \nu_l = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x'^{\nu_l'}}$$

- tensor field on M = { one tensor from each point on M }.

a tensor field ~~if smooth if~~ is called "smooth" if

$T(\omega^1, \cdots, \omega^k; v_1, \cdots, v_l)$ is smooth

where $\omega^1, \cdots, \omega^k$ are arbitrary k smooth covariant vector field;

$v_1, \cdots, v_l$ - - - - l - - - contra- - - - - .

5° metric tensor

A metric tensor, g, on a manifold M, is a symmetric, nondegenerate tensor of type $(0,2)$.

- symmetric: $g(v_1, v_2) = g(v_2, v_1)$ $v_i \in$ ~~Vect field~~ V_p
- non-degenarate: if $g(v, v_1) = 0$ for all $v \in V_p$, then $v_1 = 0$.

so a metric is a inner product on the tangent space at each point.

notation (B8)

$$g = \sum_{\mu,\nu} g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

Sometimes, write g as ds^2, omitting $\otimes$, so that

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}\, dx^\mu dx^\nu.$$

- $g_{\mu\nu}$ is non-degenarate, therefore diagonalizable to

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{bmatrix}.$$

This is called signature of the metric

positive definite metric is called Riemannian.
Diag$(- + + +)$ is called Lorentzian. (Space time has this signature).

- ~~specialty of $g_{\mu\nu}$~~ g as a map and its inverse

$g_{\mu\nu}: V_p \times V_p \to \mathbb{R}^1$ type $(0,2)$ tensor

For a $v \in V_p$ fixed, then $g(\cdot, v): V_p \to \mathbb{R}^1$,

therefore $g(\cdot, v) \in V^*$;

In another words, $g: V \to V^*$. $\sum_\mu g_{\mu\nu} v^\mu$ is a dual vector for any vector v^μ.

This map is one-to-one correspondant. Then an inverse map exist

$g^{-1}: V^* \to V$ and g^{-1} is type $(2,0)$.

Components are writen as $g^{\mu\nu}$. By definition, we also require

$$\sum_\nu g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma$$

Since $\sum_\mu g_{\mu\nu} v^\mu$ is a vector in dual space,

we use v_ν to denote its components. $v_\nu = \sum_\mu g_{\mu\nu} v^\mu$.

Similarly,

$$g : T(k,l) \longrightarrow T(k-1, l+1).$$

In component notation,

$$\sum_\sigma g_{\sigma\eta} T^{\mu_1 \cdots \mu_i \sigma \mu_{i+2} \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T^{\mu_1 \cdots \mu_i}{}_{\sigma}{}^{\mu_{i+2} \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}.$$

Similaly, ~~ex~~ g^{-1}: $T(k,l) \longrightarrow T(k+1, l-1)$.

And in component notation,

$$\sum_\sigma g^{\sigma\eta} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_i \sigma \nu_{i+2} \cdots \nu_l} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_i}{}^{\eta}{}_{\nu_{i+2} \cdots \nu_l}$$

Notation:

Symmetric tensor built from any tensor of type $(0,l)$

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$$

$$T_{(\mu_1 \cdots \mu_l)} = \frac{1}{l!} \sum_{\pi \in \text{perm.}} T_{\mu_{\pi(1)} \cdots \mu_{\pi(l)}}.$$

perm here is the permutation of $\{1, \cdots, l\}$

Anti-symmetric tensor built from $(0,l)$ type

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$$

$$T_{[\mu_1 \cdots \mu_l]} = \frac{1}{l!} \sum_{\pi \in \text{perm}} \delta_\pi T_{\mu_{\pi(1)} \cdots \mu_{\pi(l)}} \quad \text{where } \delta_\pi = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$$

we can build partially symmetric, anti-symmetric tensors, e.g.,

$$T^{(\mu\nu)\sigma}{}_{[\alpha\beta]} = \frac{1}{4}\left[T^{\mu\nu\sigma}{}_{\alpha\beta} + T^{\nu\mu\sigma}{}_{\alpha\beta} - T^{\mu\nu\sigma}{}_{\beta\alpha} - T^{\nu\mu\sigma}{}_{\beta\alpha}\right].$$

A totally anti-symmetric tensor

$$T_{\mu_1 \cdots \mu_l} = T_{[\mu_1 \cdots \mu_l]}$$

is called a differential l-form.

Einstein summation rule:

~~If two index a~~

For any contraction, or summation over perticular indices, we omit the sum symbol $\sum_{n_1, \cdots n_l}$.

So when two indices repeatly appear in ~~the sum~~ a term, they are summed.

E.g., $T^{\mu\nu} S_{\mu\sigma}{}^{\eta} \equiv \sum_{\mu} T^{\mu\nu} S_{\mu\sigma}{}^{\eta}$.

§3 Curvature

§3.1 Derivative

1° We seek a covariant derivative operator, ∇,

① ∇ : T of type (k, l) ⟶ T of type $(k, l+1)$

② Linearity
$$\nabla_\mu (a T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} + b\, S^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l})$$
$$= a \nabla_\mu T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} + b \nabla_\mu S^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} \qquad a, b \in \mathbb{R}^1$$

③ Leibnitz rule
$$\nabla_\mu (T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} S^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l})$$
$$= (\nabla_\mu T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l}) S^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l} + T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} \nabla_\mu S^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l}$$

④ commutativity with contraction
$$\nabla_\mu (T^{\alpha_1 \cdots \nu \cdots \alpha_l}{}_{\beta_1 \cdots \nu \cdots \beta_k}) = \nabla_\mu T^{\alpha_1 \cdots \nu \cdots \alpha_l}{}_{\beta_1 \cdots \nu \cdots \beta_k}$$

⑤ For ~~scalar~~ field, tangent vector = directional derivative
For all $f \in \mathcal{F}$, $t^a \in V_p$,
$$t(f) = t^a \nabla_a f$$

⑥ Torsion free. For all $f \in \mathcal{F}$, $\nabla_a \nabla_b f = \nabla_b \nabla_a f$

⑦ Inner product of two vectors remain unchanged during parallel-transpor:
$$\nabla_a g_{bc} = 0.$$

under these conditions, there exist a unique ∇_μ:

$$\nabla_\mu T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} = \partial_\mu T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l}$$

$$+ \sum_i \Gamma^{\sigma_i}{}_{\mu\nu} T^{\sigma_1 \cdots \nu \cdots \sigma_k}{}_{\eta_1 \cdots \eta_l} - \sum_j \Gamma^{\nu}_{\mu\eta_j} T^{\sigma_1 \cdots \sigma_k}{}_{\eta_1 \cdots \nu \cdots \eta_l}$$

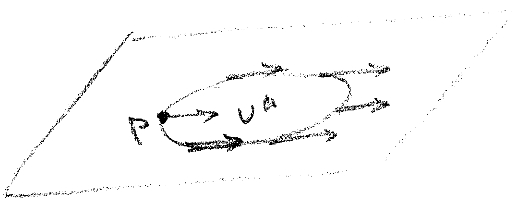
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu},$$

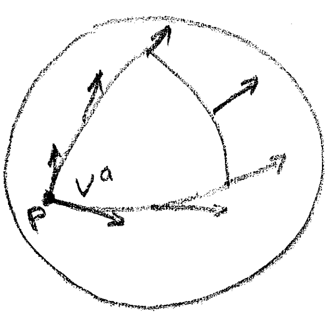
$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \quad \text{Christoffel symbols}$$

- ∇_μ is a generalization of ∂_μ.
  when $g=$ constant, $\Gamma^\rho_{\mu\nu} = 0$, $\nabla_\mu = \partial_\mu$.
- ∇_μ is usually not commutative. $\nabla_\mu \nabla_\nu T <> \nabla_\nu \nabla_\mu T$.
- $\partial_\mu T$ usually do not produce a tensor, because ∇_μ do and $\nabla_\mu \neq \partial_\mu$

  except special cases.

2° parallel transport





- parallel transport (P.T)
  $\approx$ generalization in "curved space" the concept of "keeping the vector constant".
-

- Given a curve $x^\mu(\xi)$, the P.T of a vector V^ν in Euclidian Space is

$$\frac{dV^\nu}{d\xi} = \frac{\partial V^\nu}{\partial x^\mu}\frac{dx^\mu}{d\xi} = 0$$

$\frac{\partial V^\nu}{\partial x^\mu}$ however is not a tensor in curved space, so we should generalize it:

$$t^\mu \nabla_\mu V^\nu = 0 \qquad (*)$$

where $t^\mu = dx^\mu/ds$ is the tangent.

- P.T. of a general tensor

$$t^\mu \nabla_\mu T^{\sigma_1\cdots\sigma_k}{}_{\eta_1\cdots\eta_l} = 0$$

- This equation is an tensor equation. A tensor equation won't be changed under change of coordinate system.

$$T^{\sigma_1\cdots\sigma_k}{}_{\eta_1\cdots\eta_l} = 0$$

Change of coordinate system

$$T'^{\sigma_1'\cdots\sigma_k'}{}_{\eta_1'\cdots\eta_l'} = T^{\sigma_1\cdots\sigma_k}{}_{\eta_1\cdots\eta_l}\frac{\partial x'^{\sigma_1'}}{\partial x^{\sigma_1}}\cdots\frac{\partial x^{\eta_l}}{\partial x'^{\eta_l'}} = 0.$$

§3.2 Curvature

Intuation:

"Curvature can be sensed by parallel transport".

1° Riemann curvature tensor

consider $f \in \mathcal{F}$, $\omega_\sigma \in V^*$, and

$$\nabla_\mu \nabla_\nu (f\omega_\sigma) = \nabla_\mu(\nabla_\nu f \cdot \omega_\sigma + f \nabla_\nu \omega_\sigma)$$
$$= (\nabla_\mu \nabla_\nu f)\,\omega_\sigma + \nabla_\nu f \cdot \nabla_\mu \omega_\sigma + \nabla_\mu f\, \nabla_\nu \omega_\sigma + f\, \nabla_\mu \nabla_\nu \omega_\sigma$$

$$\nabla_\mu \nabla_\nu (f\omega_\sigma) - \nabla_\nu \nabla_\mu (f\omega_\sigma) = f\,(\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma)$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(f\omega_\sigma) = f\,(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\omega_\sigma$$

Now $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\omega_\sigma\big|_p$ depend only on the value of ω_σ at p. *

(home work ?)

Then $f\omega_\sigma \in V^*$, $f(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\omega_\sigma \in \mathcal{T}(0,3)$.

$\forall p \in M$, $\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu : f\omega_\sigma \longrightarrow T(0,3)$.

Its action is of tensor type $(1,3)$.

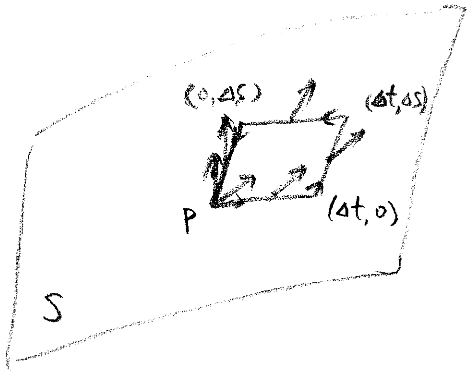
There exist a tensor field $R_{\mu\nu\sigma}{}^{\eta}$, s.t. for ω_σ,

$$(\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma) = R_{\mu\nu\sigma}{}^{\eta}\,\omega_\eta .$$

$R_{\mu\nu\sigma}{}^{\eta}$: Riemann tensor.

- $R_{\mu\nu\sigma}{}^{\eta}$ senses whether the manifold is curved.

2° How $R_{\mu\nu\sigma}{}^{\eta}$ is related to failure of a vector returning to its initial value after P.T. along closed curve, i.e., related to curvature.



Consider P.T. v^μ from point p.

It is convinient if consider variation of $v^\mu w_\mu$, w_μ being arbitrary dual vector field

$$\delta_1 = \Delta t \frac{\partial}{\partial t}(v^\mu w_\mu)\Big|_{(\Delta t/2,0)}$$

$$= \Delta t\, T^\nu \nabla_\nu (v^\mu w_\mu)$$

$T^\nu \equiv \frac{\partial x^\nu}{\partial t}$ is the tangent vector to curve at constant s.

$$= \Delta t\, T^\nu (\nabla_\nu w_\mu) v^\mu \Big|_{(\Delta t/2,0)}$$

$$\delta_1 + \delta_3 = \Delta t \left\{ v^\mu T^\nu \nabla_\nu w_\mu\Big|_{(\Delta t/2,0)} - v^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\Delta t/2,\Delta s)} \right\}$$

$$\rightarrow \Delta t\, \mathcal{O}(\Delta s)$$

Similarly $\delta_2+\delta_4 \Rightarrow \Delta s\, \mathcal{O}(\Delta t)$.

$\delta_1+\delta_3+\delta_2+\delta_4$ is at least a 2nd order infiniterimal.

i.e., P.T. is path-independent at 1st order.

Indeed, we can evaluate the $\delta_1+\delta_3$ to further accuracy

$$\delta_1+\delta_3 = \Delta t \left\{ V^\mu T^\nu \nabla_\nu W_\mu \Big|_{(\Delta t/2,0)} - V^\mu T^\nu \nabla_\nu W_\mu \Big|_{(\Delta t/2,\Delta s)} \right\}$$

$$= -\Delta t\, \Delta s \frac{\partial}{\partial s} \left\{ \qquad \right\}$$

$$= -\Delta t \Delta s \left[(T^\nu \nabla_\nu W_\mu)(S^\alpha \nabla_\alpha V^\mu) + V^\mu S^\alpha \nabla_\alpha (T^\nu \nabla_\nu W_\mu) \right] \Big|_{(\Delta t/2,\, \Delta s/2)}$$

$$= -\Delta t \Delta s\, V^\mu S^\alpha \nabla_\alpha (T^\nu \nabla_\nu W_\mu) \Big|_{(\Delta t/2,\, \Delta s/2)}$$

$$\sim -\Delta t \Delta s\, V^\mu S^\alpha \nabla_\alpha (T^\nu \nabla_\nu W_\mu) \Big|_p + \mathcal{O}(\Delta t^2)\mathcal{O}(\Delta s^2)$$

$$\delta(V^\mu \omega_\mu) = \Delta t \Delta s\, V^\mu \left[T^\alpha \nabla_\alpha (S^\nu \nabla_\nu W_\mu) - S^\alpha \nabla_\alpha (T^\nu \nabla_\nu W_\mu) \right]$$

$$= \Delta t \Delta s\, V^\mu T^\alpha S^\beta (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \omega_\mu$$

$$= \Delta t \Delta s\, V^\mu T^\alpha S^\beta R_{\alpha\beta\mu}{}^\nu \omega_\nu \qquad\qquad T^\alpha \nabla_\alpha S^\nu - S^\alpha \nabla_\alpha T^\nu = 0.$$

Since w_μ is arbitrary, the only possibility is

$$\delta V^\mu = \Delta t \Delta s\, V^\mu T^\alpha S^\beta R_{\alpha\beta\mu}{}^\nu$$

This shows that indeed $R_{\alpha\beta\mu}{}^\nu$ (Riemann tensor) is related to the path-dependence of parallel transport, which is further related to curvature.

3° properties of Riemann tensor

① $R_{\mu\nu\alpha}{}^{\beta} = -R_{\nu\mu\alpha}{}^{\beta}$, $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$

② $R_{[\mu\nu\alpha]}{}^{\beta} = 0$

③ $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$

④ $\nabla_{[\sigma} R_{\mu\nu]\alpha}{}^{\beta} = 0$ Bianchi identity

---

proof. ① definition. $\forall \omega_\mu$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\mu = R_{\alpha\beta\mu}{}^{\nu}\omega_\nu$$

② $\nabla_{[\mu}\nabla_\nu \omega_{\alpha]} = 0$ (*) homework

$\forall \omega_\beta$, $0 = 2\nabla_{[\mu}\nabla_\nu \omega_{\alpha]} = \nabla_{[\mu}\nabla_\nu \omega_{\alpha]} - \nabla_{[\nu}\nabla_\mu \omega_{\alpha]} = R_{[\mu\nu\alpha]}{}^{\beta}\omega_\beta$

$\Rightarrow R_{[\mu\nu\alpha]}{}^{\beta} = 0$

③ Before prove this, note a result similar to the introduction of Riemann tensor:

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu) T^{\sigma_1\cdots\sigma_k}{}_{\eta_1\cdots\eta_l} = -\sum_{i=1}^{k} R_{\mu\nu\alpha}{}^{\sigma_i} T^{\sigma_1\cdots\alpha\cdots\sigma_k}{}_{\eta_1\cdots\eta_l} + \sum_{j=1}^{l} R_{\mu\nu\eta_j}{}^{\alpha} T^{\sigma_1\cdots\sigma_k}{}_{\eta_1\cdots\alpha\cdots\eta_l}$$

$$0 = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu) g_{\alpha\beta} = R_{\mu\nu\alpha}{}^{\sigma} g_{\sigma\beta} + R_{\mu\nu\beta}{}^{\sigma} g_{\alpha\sigma} = R_{\mu\nu\alpha\beta} + R_{\mu\nu\beta\alpha} = 0$$

④ $\forall \omega_\alpha$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \nabla_\alpha \omega_\beta = R_{\mu\nu\alpha}{}^\sigma \nabla_\sigma \omega_\beta + R_{\mu\nu\beta}{}^\sigma \nabla_\alpha \omega_\sigma$$

$$\nabla_\alpha (\nabla_\mu \nabla_\nu \omega_\sigma - \nabla_\nu \nabla_\mu \omega_\sigma) = \nabla_\alpha (R_{\mu\nu\sigma}{}^\eta \omega_\eta) = \omega_\eta \nabla_\alpha R_{\mu\nu\sigma}{}^\eta + R_{\mu\nu\sigma}{}^\eta \nabla_\alpha \omega_\eta$$

Antisymmetrize both equations for μ, ν, α, L.H.S. becomes equal.

$$\underbrace{R_{[\mu\nu\alpha]}{}^\sigma \nabla_\alpha \omega_\beta}_{0 \text{ (property ③)}} + \underline{R_{[\mu\nu|\beta|}{}^\sigma \nabla_{\alpha]} \omega_\sigma} = \omega_\eta \nabla_{[\alpha} R_{\mu\nu]\sigma}{}^\eta + \underline{R_{[\mu\nu|\sigma|}{}^\eta \nabla_{\alpha]} \omega_\eta}$$

cancel

$$\nabla_{[\alpha} R_{\mu\nu]\sigma}{}^\eta = 0$$

4° ~~other~~ Ricci tensor, Ricci scalar, Einstein tensor, Weyl tensor

From properties ①, ②, ③, ④ of Riemann tensor,

there are $\frac{n^2(n^2-1)}{2}$ independent components for n-dim manifold.

Decomposition into "trace part" and "trace free part":

① $R_{\mu\alpha} \equiv R_{\mu\nu\alpha}{}^\nu$ "trace part"

Ricci tensor; symmetric $R_{\mu\alpha} = R_{\alpha\mu}$

From this, define Ricci scalar / curvature

$R \equiv R_\mu{}^\mu$.

② Trace-free-part

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{2}{n-2}\left(g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu}\right) - \frac{2}{(n-1)(n-2)} R\, g_{\mu[\alpha}g_{\beta]\nu}$$

called Weyl tensor / conformal tensor.

② Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

property: $\nabla^{\alpha} G_{\mu\nu} = 0$. (*) homework.

§ 3.2 Geodesics

intuatively, "straightest possible curve"

1° Geodesic equation

P.T. a vector V^μ along a curve with tangent T^ν

$$T^\nu \nabla_\nu V^\mu = 0 .$$

<u>Geodesic</u>: A curve whose tangent vector is parallel propagated along itself.

$$T^\nu \nabla_\nu T^\mu = 0 . \qquad (*1)$$

Supposing $T^\mu = T^\mu(t)$, i.e., the curve is parameterized by t.

& using $\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\alpha} T^\alpha$,

$$\frac{dT^\mu}{dt} + \sum_{\sigma,\nu} \Gamma^\mu_{\sigma\nu} T^\sigma T^\nu = 0. \qquad (*2)$$

using $T^\mu = dx^\mu/dt$, i.e., the curve is $x^\mu = x^\mu(t)$

$$\frac{d^2x^\mu}{dt^2} + \sum_{\sigma\nu} \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{dt} \frac{dx^\nu}{dt} = 0. \qquad (*3)$$

(*i) are the <u>geodesic equations</u>.

- It is a system of second order ordinary differential equations (ODEs). Mathematically, a initial value problem ~~with~~ of $x^\mu(t)$.

Theory of ODE solution uniqueness tells us:

there always exists a unique solution for any given intial $x^\mu(t_0)$ & $dx^\mu/dt|_{t=t_0}$.

i.e.,

Given any $p \in C \subset M$ and a tangent T_p at p, there exists a unique geodesic passing p with tangent T_p.

2° proper length or proper time along the geodesics

- A vector T^μ is called

$$\begin{cases} \text{timelike if} & g_{\mu\nu} T^\mu T^\nu < 0 \\ \text{null} & = 0 \\ \text{spacelike} & > 0 \end{cases}$$

; $g_{\mu\nu} T^\mu T^\nu$ is called norm of T^μ.

$g_{\mu\nu} T^\mu S^\nu$ is called inner product of T^μ & S^ν.

- A vector pair can not change their inner product during P.T.;

A vector can not change its norm during P.T.

proof: For tangent t^μ of any curve, ~~P.T. means~~

$$t^\mu \nabla_\mu (g_{\sigma\eta} T^\sigma S^\eta)$$

$$= (t^\mu \nabla_\mu T^\sigma) g_{\sigma\eta} S^\eta + (t^\mu \nabla_\mu S^\eta) g_{\sigma\eta} T^\sigma + t^\mu (\nabla_\mu g_{\sigma\eta}) T^\sigma S^\eta$$

$$= 0 + 0 + 0 = 0$$

t^μ is arbitrary $\Rightarrow g_{\sigma\eta} T^\sigma S^\eta = $ constant during P.T.

- Then, a geodesic (whose tangent is P.T. along itself), its tangent vectors at all points are either timelike, null, or spacelike; but not change its ~~nature~~ character during P.T..

C12

			Name
A geodesic's tangents (or curve)	timelike	$\Longrightarrow$	timelike geodesic (curve);
	null	$\Longrightarrow$	null " " ;
	spacelike	$\Longrightarrow$	space like " " .

For timelike geodesic/curve, define proper time

$$\tau = \int_{t=t_i}^{t=t_f} (-g_{\mu\nu} T^\mu T^\nu)^{\frac{1}{2}} \, dt .$$

For spacelike geodesics/curve, define proper length

$$l = \int_{t=t_i}^{t=t_f} (g_{\mu\nu} T^\mu T^\nu)^{\frac{1}{2}} \, dt$$

or combined $\quad l = \int_{t_i}^{t_f} |g_{\mu\nu} T^\mu T^\nu|^{\frac{1}{2}} \, dt .$

3. properties of (timelike/null/spacelike) geodesics

3.1 Proper time or proper length is independent on the parameterization of the curve.

Consider $\quad X^\mu(t) \longrightarrow X^\mu(t(s)) = X^\mu(s)$

$$\tau_t = \int (-g_{\mu\nu} T^\mu T^\nu)^{\frac{1}{2}} dt = \int \left(-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right)^{\frac{1}{2}} dt$$

$$= \int \left(-g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right)^{\frac{1}{2}} \frac{ds}{dt} \cdot dt$$

$$= \int (-g_{\mu\nu} S^\mu S^\nu)^{\frac{1}{2}} ds = \tau_s .$$

Essentially, reparameterization is a change of variable for a definite integral.

3.2

- A globally extreme curve (shortest length or greatest proper time) connecting two points, if exists, must be a (spacelike or timelike) geodesic.

A spacelike/timelike geodesic is at least ~~loce~~ a local extreme.

proof: Consider a spacelike curve, fixed end points $t=a$, $t=b$

$$l=\int_a^b\Big(\sum_{\mu,\nu} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\Big)^{\frac{1}{2}}dt$$

$$\delta l=\int_a^b\Big[\sum_{\mu\nu} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\Big]^{-\frac{1}{2}}\sum_{\alpha,\beta}\Big\{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{d(\delta x^\beta)}{dt}+\frac{1}{2}\sum_\sigma\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\Big\}dt$$

l will not be changed by reparameterization, therefore we can set

$$\sum_{\mu\nu} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=1$$

$$=\int_a^b\sum_{\alpha,\beta}\Big\{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{d\,\delta x^\beta}{dt}+\frac{1}{2}\sum_\sigma\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\Big\}dt$$

integral by parts

$$=\int_a^b\sum_{\alpha\beta}\Big\{-\frac{d}{dt}\Big(g_{\alpha\beta}\frac{dx^\alpha}{dt}\Big)-\frac{1}{2}\sum_\lambda\frac{\partial g_{\alpha\lambda}}{\partial x^\beta}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}\Big\}\delta x^\beta dt$$

δx^β arbitrary

$$-\sum_\alpha g_{\alpha\beta}\frac{d^2x^\alpha}{dt^2}-\sum_{\alpha\lambda}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\frac{dx^\lambda}{dt}\frac{dx^\alpha}{dt}+\frac{1}{2}\sum_{\alpha\lambda}\frac{\partial g_{\alpha\lambda}}{\partial x^\beta}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}=0$$

reshuffling dummy indices

$$-g_{\alpha\beta}\frac{d^2x^\alpha}{dt^2}+\frac{1}{2}[\partial_\beta g_{\alpha\lambda}-\partial_\lambda g_{\alpha\beta}-\partial_\alpha g_{\lambda\beta}]\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}=0$$

multiply $g^{\beta\gamma}$

$$-\frac{d^2x^\gamma}{dt^2}+\frac{1}{2}g^{\beta\gamma}[\partial_\beta g_{\alpha\lambda}-\partial_\lambda g_{\alpha\beta}-\partial_\alpha g_{\lambda\beta}]\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}=0$$

geodesic equation.

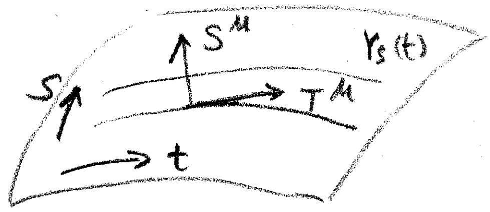
$$\frac{d^2x^\gamma}{dt^2}+\Gamma^\gamma{}_{\alpha\lambda}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}=0$$

4. Collection of geodesics

In flat manifold, initially parallel geodesics remain parallel forever.

In curved manifold, they will not necessarily do this.

We study how ~~two~~ geodesics deviate from each other.



- To be concerete, consider a

~~one-family~~ one-parameter family

of geodesics, $\gamma_s(t)$. I.e., for each $s \in \mathbb{R}$

$$\gamma_s(t) : t \to C \subset M.$$

Here t is the affine parameter.

— Then these geodesics define a 1+1 dimensional surface embedded in M.

Its coordinates can be chosen as s & t:

$$M = \{ x^\mu (s, t) \}.$$

— Two natural vector fields

$$T^\mu = \frac{\partial x^\mu}{\partial t} \quad \text{tangent vector}$$

$$S^\mu \equiv \frac{\partial x^\mu}{\partial s} \quad \text{let us call it "deviation vector"}$$

- Then how fast the S^μ changes can characterize how fast geodesics deviate from each other. Define "relative velocity of geodesics"

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$$

Further more, how fast this "velocity" change becomes "acceleration"?

Define "relative acceleration of geodesics"

$$a^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu$$

The names are just names, but these quantities are well defined.

- We would like to show a^μ is related to the curvature of the manifold.

$$a^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$

① $T^\rho \nabla_\rho S^\mu - S^\rho \nabla_\rho T^\mu \equiv [T, S]^\mu$ commutator of vectors! (homework)

$[T, S]^\mu = 0$ if T^μ, S^μ are basis vectors.

$$= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

②

$$= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

③

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

④

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma - (S^\sigma \nabla_\sigma T^\rho)\nabla_\rho T^\mu$$

⑤

$$= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

In ③, commutator of covariant derivatives is used;

In ⑤, geodesic condition $T^\rho \nabla_\rho T^\mu = 0$ is used.

therefore: geodesics will accelerate towards or away from each other if and only if $R^\mu{}_{\nu\alpha\beta} \neq 0$.

Initially parallel geodesics will not be parallel again if & only if $R^\mu{}_{\nu\alpha\beta} \neq 0$.

§ Motivation & special relativity. (S.R.)

1. Now we motivate the introduction of general relativity, during this we also do a short review of S.R..

- why we study tensor calculus?

  All physical laws can be expressed as a tensor equation.

  All physical measurements are either $\underbrace{\text{scalar or components of vectors, tensors}}_{\text{tensors of different type}}$.

- physical laws should not depending on the frame or any particular vector/tensor fields.

  I.e., laws formulated in different frames should predict the same physics.

  More generally, the metric is the only quantity associated with spacetime that will appear in physics laws.

  $\Leftarrow$ general covariance principle (GCP).

- In prerelativity physics, laws also follow special covariance principle (S.C.P.)

  physical laws (written in components form) remain unchanged under metric rotation and translation; plus time reversal and space parity.

2 Special Relativity

2.1 The spacetime in S.R. is an $\mathbb{R}^4$ manifold.

The mapping Spacetime $\longrightarrow \mathbb{R}^4$ is called a <u>global inertial coordinate system</u>. (I.C.S.)

The infinitesimal spacetime interval is

$$(ds)^2 = -(c\,dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

Remember the norm (or "interval") between two points on a manifold is

$$(ds)^2 = g_{\mu\nu}\, dx^\mu dx^\nu .$$

This suggest metric in S.R. is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \begin{array}{c|c} \mu = 0\ 1\ 2\ 3 & \nu = \\ \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$\Rightarrow$ Christoffel symbol $\Gamma^\mu_{\nu\alpha} = 0$;

$\Rightarrow$ covariant derivative $\nabla_\mu = \partial_\mu$;

Spacetime in SR is an 4-d manifold with metric $\eta_{\mu\nu}$. This spacetime is called Minkowski space.

2.2 Poincare transforms

S.R. also assert that the speed of light in vacuum in any inertial coordinate system is the same.

Combining with the isotropic properties (or assumption) of spacetime, one can prove that the spacetime interval is invariant in different I.C.S. (homework) 中文 郭硕鸿老师书

Also, from the relativity principle, I.C.S can only be connected by linear transforms.

Now linear transforms should be

$$[x'^{\mu}] = \Lambda [x^{\mu}] + [A^{\mu}]$$

① $[A^{\mu}]$ part corespond to translation :

$$x'^{\mu} = x^{\mu} + a^{\mu} \qquad \mu = 0, \cdots, 3.$$

② $\Lambda [x^{\mu}]$ part correspond to "generalized rotation".

To have spacetime intervals invariant

$$(ds)^2 = [dx^{\mu}]^T [\eta_{\mu\nu}] [dx^{\nu}] = [dx'^{\mu}]^T \eta_{\mu\nu} [dx'^{\nu}]$$

$$= [dx^{\mu}]^T \Lambda^T [\eta_{\mu\nu}] \Lambda [dx^{\nu}]$$

$$[\eta_{\mu\nu}] = \Lambda^T [\eta_{\mu\nu}] \Lambda$$

$$\eta_{\rho\sigma} = \Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} \eta_{\mu\nu}$$ $[\Lambda]$ called Lorentz transforms

First kind Λ : conventional rotations

$$[\Lambda^{\mu}_{\ \nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ rotation in x-y plane by angle θ

Second kind Λ : boost

$$[\Lambda^{\mu}_{\ \nu}] = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ boost along x-direction

why this is a boost ?

consider the $[x'^{\mu}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ point in the $[x^{\mu}]$ coordinate system

$$\begin{cases} t' = t\cosh\phi - x\sinh\phi \\ x' = -t\sinh\phi + x\cosh\phi \end{cases}$$

The $t'=0, x'^{1,2,3}=0$ piont correspond to $(t, x^{1,2,3})$ point s.t

$$\frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi \xrightarrow[\phi = \tanh^{-1} v]{} v$$

so we went from a I.C.S of velocity zero to velocity v. $\Rightarrow$ **boost**

$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases} \qquad \gamma = 1/\sqrt{1-v^2}$$

$\Rightarrow$ length contraction and time dilation

There are 6 independent rotations and boosts. They form a proper Lorentz group, $SO(3,1)$
Non-abelian.

proper Lorentz transforms, together with 4 translations, form a ~~ten~~ parameter group, called poincare group. Non-abelian either.

Third kind Λ: discrete transform

$$[T] = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad [P] = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

time reverse & parity transform

The proper Lorentz transform plus time reverse & parity forms a full Lorentz group, $O(3,1)$.

2.3 physical laws in S.R.

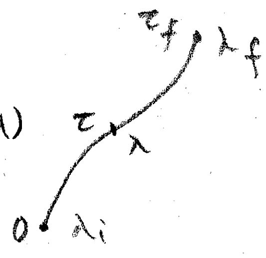
All quantities & laws has to be written as tensors to garantee inveriance in different I.C.S.

- 4-velocity:

A timelike curve C parameterized by its proper time τ.

4 velocity $u^\mu \equiv \frac{dx^\mu(\tau)}{d\tau}$

property: $u^\mu u_\mu = -1$.



[Proper time is defined as for any parametrization $x^\mu(\lambda)$

$$\tau = \int_{\lambda_i}^{\lambda} (-g_{\mu\nu} T^\mu T^\nu)^{\frac{1}{2}} d\lambda$$

where $T^\mu = \frac{dx^\mu(\lambda)}{d\lambda}$.

~~prove~~ $\cancel{|g_{\mu\nu} u^\mu u^\nu| = 1}$. prove $-g_{\mu\nu} u^\mu u^\nu = 1$. (homework).]

- 4 momentum:

material particles have an attribute called "~~rest~~ mass" m.

4-momentum:

$$p^\mu = m u^\mu.$$

- Energy and energy-momentum tensor

Energy measured by a moving observer with 4-velocity v^μ of an particle with 4-velocity u^μ is defined as

$$E = -p_\mu v^\mu = -m g_{\mu\nu} u^\mu v^\nu$$

when observer is at rest w.r.t particle, $v^\mu = u^\mu$

$$E = mc^2, \quad \text{where } c = 1.$$

Generalize this idea, the energy-momentum tensor of a continuous matter (stress-energy tensor) is denoted as $T_{\mu\nu}$.

$T_{\mu\nu} U^\mu U^\nu \Longrightarrow$ energy density.

i.e., the mass-energy per unit volume measured by an U^μ observer.

conservation of matter: $\partial_\mu T^{\mu\nu} = 0$

2.4 Different matter

- perfect fluid: a fluid that is isotropic in its rest frame & having no stress

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(\eta_{\mu\nu} + u_\mu u_\nu)$$

$$= (\rho + P) u_\mu u_\nu + P \, \eta_{\mu\nu}$$

In rest frame $[T_{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{bmatrix}$

ρ: energy density in rest frame

P: pressure in rest frame

U^μ: 4-vel. of fluid.

For pressureless dust in rest frame:

$$[T_{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Conservation: $\partial^\mu T_{\mu\nu} = 0$.

- [In the limit of nonrelativistic, $P \ll \rho$, $U^\mu = (1, \vec{v})$, $|\vec{v}| \frac{dP}{dt} \ll |\vec{\nabla} P|$, derive the Euler's equations of fluids.

(homework)].

- Electromagnetic field

$$T_{\mu\nu} = \frac{1}{4\pi} \left\{ F_{\mu\alpha} F^{\alpha}{}_{\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right\}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called field strength ;

A_μ is the vector potential.

Electric vector field $E_\mu = F_{\mu\nu} U^\nu$

Magnetic vector field $B_\mu = -\frac{1}{2} \epsilon_{\mu\alpha}{}^{\beta\gamma} F_{\beta\gamma} U^\alpha$

$\epsilon_{\alpha\beta\mu\nu}$ is the totally anti-symmetric tensor with $\epsilon_{0123} = 1$.

Maxwell's equation

$$\begin{cases} \partial^\mu F_{\mu\nu} = -4\pi j_\nu \\ \partial_{[\mu} F_{\alpha\beta]} = 0 \end{cases}$$

j_ν is the electric charge density current.

Lorentz force law on a particle with 4-velocity u^μ :

$$u^\mu \partial_\mu u^\nu = \frac{q}{m} F^\nu{}_\mu u^\mu$$

(verify yourself. Homework)

[~~Homework : show the energy-momentum~~

D8

- Scalar ~~tensor~~ field described by Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right)$$

The energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\phi\,\partial_\alpha\phi + m^2\phi^2\right)$$

prove that for any observer, the E.M.T for perfect fluid, electromagnetic field & scalar field satisfies the condition

$$T_{\mu\nu}U^\mu U^\nu \geq 0$$

This is called (weak) energy condition.

§ General Relativity & Einstein Equation (G.R.)

1. Motavition

We start from basic principles and attempt to argue that they lead naturally to an almost unique physical theory.

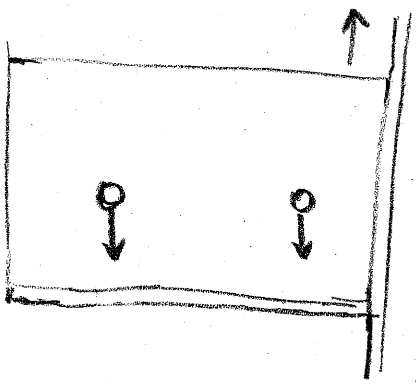
- Weak equivalance principle (WEp)

$$\vec{f} = m_i \vec{a} \qquad m_i : \text{ inertial mass}$$

$$\vec{f_g} = m_g \vec{\nabla} \phi \qquad m_g : \text{ gravitational mass}$$

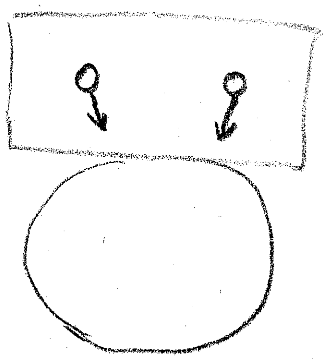
ϕ : gravitational potential

WEp : m_i / m_g is the same for anybody.



- Einstein E.P.

→ The above box should be small for the argument to work.



→ After S.R., the equivalance of "mass" becomes less meaningful.

After all, mass can be converted to other format.

This motivated Einstein for his E.E.P.: (the principle of equivalence)

In small enough regions, the law of physics reduces to those of S.R.
All bodies are influenced by gravity, and all bodies fall precisely the same way in a gravitational field.

E.E.P $\Longrightarrow$ Spacetime should be considered as a curved manifold.

Gravity

- no global inertial frame because gravity is inescapable.
  the establishment of global inertial frame in S.R. implicitly assummed the existence of a force-free reference.
  Now this is impossible.

~~instead~~

"The acceleration due to gravity" becomes an imprecise statement because we don't have a frame for measurement.

- instead we say bodies under influence of gravity (only), do "free falling". (along their geodesics in a curved spacetime)
  and is not accelerated.

~~The concept of "gravitational force" become of little use because force is the thing cause acceleration but in gravity, we say there is no acceleration.~~

⇓ inertial frames have to become "local"

when the region is small enough, we expect the influence of gravity on the 'box (frames)' is universal.

S.R. should be restored here.

⇓ the best we can do is associate one "local inertial frame" with each particles moving freely in gravity (free falling).

Einstein then tried to find an equation that describes the influence of gravity on matter. [Note the E.E.P. is a postulate: it can not be proven but be falsified.

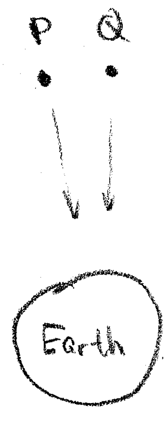
2. Einstein equation.

We start from an ~~classical~~ equation in classical mechanics, which connects the spacetime geometry with matter distribution, and then use the above principles and intuitions as guidelines.

Study the description of tidal acceleration in Newtonian mechanics & G.R. I.e.,

~~In NM~~:

Consider the free fall of two test particles:



In N.M., the free falling is described by

$$\left(\frac{d^2 x_i}{dt^2}\right)_P = -\left(\frac{\partial \phi}{\partial x^i}\right)_P , \quad \left(\frac{d^2 x_j}{dt^2}\right)_Q = -\left(\frac{\partial \phi}{\partial x^j}\right)_Q$$

Φ : gravitational potential.

Then the separation of P & Q, changes as free falls. ~~When~~ $\vec{\chi} = (x_j)_P - (x_j)_Q$.
when the separation is itself small

$$\frac{d^2 \vec{x}_j}{dt^2} = -\left(\frac{\partial^2 \phi}{\partial x^j \partial x^k}\right) \vec{\chi}_k$$

(Taylor expansion can prove this).

I.e., the "relative acceleration of geodesics"

$$a^\mu = -(\partial_\beta \partial^\mu \phi)\, \chi^\beta .$$

In G.R. / Differential geometry, page (C15),

$$a^\mu = -R_{\alpha\beta\gamma}{}^{\mu} U^\alpha \chi^\beta U^\gamma$$

- Guess $\quad R_{\alpha\beta\gamma}{}^{\mu} U^\alpha U^\gamma \longleftrightarrow \partial_\beta \partial^\mu \phi$

$\nabla^2 \phi = 4\pi \rho$ poisson's equation

~~$\rho \leftrightarrow T_{\alpha\beta} U^\alpha U^\beta$~~ ρ : mass (energy) density of matter

$$\rho \longleftrightarrow T_{\alpha\gamma} U^\alpha U^\gamma$$

$T_{\alpha\gamma}$: E.M. tensor

U^μ : velocity of free falling fluid

- Suggest:

$$R_{\alpha\mu\gamma}{}^{\mu} U^\alpha U^\gamma = 4\pi T_{\alpha\gamma} U^\alpha U^\gamma \qquad (*)$$

$$R_{\alpha\gamma} \overset{?}{=} 4\pi T_{\alpha\gamma} \qquad (**)$$

This was first postulated by Einstein.

- However from Bianchi identity   from E.M. tensor conservation

$$\nabla^\alpha (R_{\alpha\gamma} - \tfrac{1}{2} g_{\alpha\gamma} R) = 0 \qquad \nabla^\alpha T_{\alpha\gamma} = 0 \qquad (***)$$

so applying ∇^α to (**), we get

$$\nabla_\alpha R = 0$$

$$R = \text{const}. \text{ ~~for any matter~~}$$

$$T^\mu{}_\mu = \text{const for any matter}.$$

This is apparently unphysical.

- Now we want a relation satisfies (***) and (*).

Such an equation is found by Einstein in 1915

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (\dot{*})$$

The LHS tensor is actually named after Einstein, the Einstein tensor.

(1) This is the celebrated Einstein equation.

(2) From now on, we shift our focus of the course to the solution of this equation.

- First let us see that ~~(**) and~~ (*) is indeed satisfied.

From (*),

$$R = -8\pi T$$

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right)$$

when observer is roughly at rest with fluid $v^\mu v_\mu = -1$

$$T \approx -\rho = -T_{\alpha\beta} v^\alpha v^\beta$$

Then

$$R_{\mu\nu} v^\mu v^\nu = 8\pi\left(T_{\mu\nu} v^\mu v^\nu\right) + 4\pi g_{\mu\nu} T_{\alpha\beta} v^\alpha v^\beta v^\mu v^\nu$$

$$= 4\pi T_{\mu\nu} v^\mu v^\nu.$$

This recovers (*).

3. physical laws in G.R.

- general covariance
- locally reduce to S.R.

(1) In G.R., physical quantities are still described by tensors.

Motion of particle described by timelike curve

perfect fluid velocity u^μ, e.m.t $T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$.

E.m field: $F_{\mu\nu}$

(2) physical laws are still described by tensor equations.

"Two principles requires equation being invariant under any coordinate transform."

Two rules should be applied to equations in S.R.

(a)
$$\begin{array}{ccc} \eta_{\mu\nu} & \longrightarrow & g_{\mu\nu} \\ \downarrow & & \downarrow \\ \text{diag}(-1,1,1,1) & & \text{general} \end{array}$$

(b) $\partial_\mu \longrightarrow \nabla_\mu$ "minimal substitution"

A particle: momentum $p^\mu = m u^\mu$

perfect fluid: E.O.M.

$$\nabla^\mu T_{\mu\nu} = 0$$

$$\Rightarrow \begin{cases} u^\mu \nabla_\mu \rho + (\rho + p) \nabla^\mu u_\mu = 0 \\ (\rho + p) u^\mu \nabla_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu) \nabla^\mu p = 0 \end{cases}$$

Scalar field

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla_\sigma\phi\nabla^\sigma\phi + m^2\phi^2)$$

D16

Electromagnetic field

$$T_{\mu\nu} = \frac{1}{4\pi}\{F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\}$$

Maxwell's equation

$$\nabla^\mu F_{\mu\nu} = -4\pi j_\nu$$

(3) However, these two rules "$\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu \to \nabla_\mu$" are only guide lines.

It will not recover terms that will only appear due to curvature.

For example, Maxwell's equations for vector gauge field A_μ (in the Lorentz gauge)

$$\nabla^\mu\nabla_\mu A_\nu - R^\mu{}_\nu A_\mu = -4\pi j_\nu \quad (*1)$$

However, in S.R., the corresponding equation was

$$\partial^\mu\partial_\mu A_\nu = -4\pi j_\nu \quad (*2)$$

$$\nabla^\mu\nabla_\mu A_\nu = -4\pi j_\nu$$

If "minimal substitution" was used, the $-R^\mu{}_\nu A_\mu$ term won't be constructed. However (*1) is favors over (*2) because it satisfies current conservation $\nabla_\nu j^\nu = 0$

So, the true physical equation really ~~depend on~~ is determined from physics.

(In this case, the one with $-R^\mu{}_\nu A_\mu$), not the "m.s" rules.

However, they are good guide lines.

In our course, all equations will be the one obtained from physical consideration.

§ Linear gravity

~~1. The Newtonian limit~~

- G.R. is a new description of gravity.

  It should reduce to Newton's gravity in where it works:

  weak gravity & low speed.

1. Linearize G.R.

   Consider metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

$$|\gamma_{\mu\nu}| << 1$$

then

$$g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}.$$

the Christoffel symbol

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu \gamma_{\nu\beta} + \partial_\nu \gamma_{\mu\beta} - \partial_\beta \gamma_{\mu\nu})$$

Ricci tensor $R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}_{\alpha\nu}$

$$= \partial^\alpha \partial_{(\nu}\gamma_{\mu)\alpha} - \frac{1}{2}\partial^\alpha \partial_\alpha \gamma_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu \gamma$$

here $\gamma = \gamma^{\alpha}{}_{\alpha}$.

Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$

$$= \partial^\alpha \partial_{(\nu}\gamma_{\mu)\alpha} - \frac{1}{2}\partial^\alpha \partial_\alpha \gamma_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu \gamma - \frac{1}{2}\eta_{\mu\nu}(\partial^\alpha \partial^\beta \gamma_{\alpha\beta} - \partial^\alpha \partial_\alpha \gamma)$$

Let us define $\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma$, then Einstein Equ.

$$G_{\mu\nu} = -\frac{1}{2}\partial^\alpha \partial_\alpha \bar{\gamma}_{\mu\nu} + \partial^\alpha \partial_{(\nu}\bar{\gamma}_{\mu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha \partial^\beta \bar{\gamma}_{\alpha\beta} = 8\pi T_{\mu\nu}$$

Let us do an infinitesimal gauge transform with vector field ξ^α

$$x'^\alpha = x^\alpha + \xi^\alpha \, , \qquad |\xi^\alpha / x^\alpha| \ll 1$$

then $g'_{\mu\nu} = g_{\alpha\beta} \dfrac{\partial x'^\alpha}{\partial x^\mu} \dfrac{\partial x'^\beta}{\partial x^\nu}$

$$= (\eta_{\alpha\beta} + \gamma_{\alpha\beta})(\delta^\alpha_\mu + \partial_\mu \xi^\alpha)(\delta^\beta_\nu + \partial_\nu \xi^\beta)$$

$$\eta_{\mu\nu} + \gamma'_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} + \partial_\nu \xi_\mu + \partial_\mu \xi_\nu + \mathcal{O}$$

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

and $\gamma' = \gamma'^\alpha_\alpha = \gamma^\alpha_\alpha + 2\partial^\alpha \xi_\alpha = \gamma + 2\partial^\alpha \xi_\alpha$,

$$\bar{\gamma}'_{\mu\nu} = \gamma'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma'$$

$$= \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(\gamma + 2\partial^\alpha \xi_\alpha)$$

$$= \bar{\gamma}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\alpha \xi_\alpha \, ,$$

$$\partial^\nu \bar{\gamma}'_{\mu\nu} = \partial^\nu \bar{\gamma}_{\mu\nu} + \partial^\nu \partial_\mu \xi_\nu + \partial^\nu \partial_\nu \xi_\mu - \partial_\mu \partial^\alpha \xi_\alpha$$

$$= \partial^\nu \bar{\gamma}_{\mu\nu} + \partial^\nu \partial_\nu \xi_\mu \, .$$

Since ξ^α is the gauge, we have the freedom to choose it s.t.,

$$\partial^\nu \bar{\gamma}_{\mu\nu} = -\partial^\nu \partial_\nu \xi_\mu \, ,$$

then $\partial^\nu \bar{\gamma}'_{\mu\nu} = 0$.

(This is an analog of the "Lorentz gauge" condition in E&M theory.)

Now the E.E under the gauge transform becomes

$$-\frac{1}{2}\partial^\alpha \partial_\alpha \bar{\gamma}'_{\mu\nu} = 8\pi T'_{\mu\nu}$$

Dropping the "$'$" for the simplicity sake,

$$\partial^\alpha \partial_\alpha \bar{\gamma}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad \text{(L.E.E)}$$

2. The Newtonian limit

- In the Newtonian limit, the ~~e.m.t of particles or~~ fluids should have

$$\rho >> P, \qquad \text{(rest energy dominates)}$$

$$\text{4-velocity } u^\mu = (1, 0, 0, 0). \qquad \text{(slow speed condition)}$$

then their e.m.t becomes

$$T_{\mu\nu} \approx \rho u_\mu u_\nu .$$

- More generally, for other matter form in the Newtonian limit, we assume

$$[T_{\mu\nu}] = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & O \end{array}\right]$$

the ignorance of "$0i$" $i=1,2,3$ components mean velocity is small

———— "ij" ———— streess is small.

- We therefore expect the metric to very slowly.

We seek solutions with $\partial_0 \gamma_{\mu\nu} = 0$.

Then the L.E.E becomes

① $\nabla^2 \bar{\gamma}_{00} = -16\pi\rho$ $\nabla^2 = \partial^1\partial_1 + \partial^2\partial_2 + \partial^3\partial_3$ is the

② $\nabla^2 \bar{\gamma}_{ij} = 0$ Laplacian operator on space index

The solution to ② and satisfying the good asymptotic flatness is

$$\bar{\gamma}_{ij} = 0$$

The solution to ① is the following

$$\bar{\gamma}_{00} = -4\phi$$

where ϕ satisfies $\nabla^2\phi = 4\pi\rho$. (phi.sol)

Therefore $\bar{\gamma}_{\mu\nu} = -4\phi t_\mu t_\nu$, $\bar{\gamma} = 4\phi$.

~~Then the following $\gamma_{\mu\nu}$ solves the~~

Then the perturbatie metric becomes

$$\begin{aligned} \gamma_{\mu\nu} &= \bar{\gamma}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\gamma \\ &= \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma} \\ &= -(4t_\mu t_\nu + 2\eta_{\mu\nu})\phi . \end{aligned}$$

The ~~equation of~~ motion of a test particle is given by its geodesic

$$\frac{d^2x^\mu}{d\tau^2} + \sum_{\rho,\sigma} \Gamma^\mu{}_{\rho\sigma}\left(\frac{dx^\rho}{d\tau}\right)\left(\frac{dx^\sigma}{d\tau}\right) = 0$$

Now in Newtonian limit, particle is slow moving

$$[dx^\mu/d\tau] = [1, 0, 0, 0]$$

$$\tau \simeq t$$

Then the geodesics eq.

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{00}$$

For $\mu=1,2,3$, using $\gamma_{\mu\nu}$ we get after neglecting time derivative

$$\Gamma^\mu_{00} = -\frac{1}{2}\frac{\partial\gamma_{00}}{\partial x^\mu} = \frac{\partial\phi}{\partial x^\mu}$$

Finally, we get

$$\frac{d^2x^\mu}{dt^2} = -\partial_\mu\phi \qquad \mu=1,2,3.$$

$$\text{i.e.,} \quad \vec{a} = -\vec{\nabla}\phi \qquad (gf)$$

We then recognize that

- ϕ is the gravitational potential
- the (gf) equation is the Newton's law under gravity
- the (phi.sol.) is the poison equation

Newtonian limit is recovered.

(A homework can be given about

how E&M field interact with a neutral test particle through gravity in the Newtonian limit.
More precisely, relation between
$\vec{a}$ & $\vec{E}$, $\vec{B}$)

3. Gravitational Radiation

3.1 The Coulomb gauge

The linearized E.E. in a source free spacetime is

$$\partial^\alpha \partial_\alpha \bar{\gamma}_{\mu\nu} = 0 . \qquad \text{(L.E.E)}$$

It should also satisfy the Lorentz gauge condition

$$\partial^\mu \bar{\gamma}_{\mu\nu} = 0 . \qquad \text{(Lor. cond)}$$

In obtaining these, we made the gauge transform

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}^{(\text{original})} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

and ξ^μ should satisfy

$$\partial^\mu \partial_\mu \xi_\nu = 0 . \qquad \text{(C1)}$$

Now we would like to show that there exist wave solutions to the L.E.E.

The key is Lorentz gauge condition does not fix the gauge completly.

In addition to (C1), we can set the condition for ξ^μ at a initial surface $t = t_0$:

$$2\left(-\frac{\partial \xi_0}{\partial t} + \vec{\nabla}\cdot\vec{\xi}\right) = -\gamma ,$$

$$2\left(-\vec{\nabla}^2 \xi_0 + \vec{\nabla}\cdot\left(\frac{\partial \vec{\xi}}{\partial t}\right)\right) = -\frac{\partial \gamma}{\partial t} ,$$

$$\frac{\partial \xi_\mu}{\partial t} + \frac{\partial \xi_0}{\partial x^\mu} = -\gamma_{0\mu} \quad (\mu = 1, 2, 3) ,$$

$$\vec{\nabla}^2 \xi_\mu + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \xi_0}{\partial t}\right) = -\frac{\partial \gamma_{0\mu}}{\partial t} \quad (\mu = 1, 2, 3) .$$

This is an first order PDE system of $\{\partial_\mu \xi_\nu \,,\ \mu,\nu=0,1,2,3\}$.

We assert that with these conditions on ξ_μ,

$$\gamma = 0$$

$$\gamma_{0\mu} = 0 \qquad (\mu = 1,2,3)$$

can be achieved throughout the source free spacetime.

Then substitute back into (Lorentz cond) (homework ? need to work this out myself first.)

$$\frac{\partial \gamma_{00}}{\partial t} = 0$$

and into (L.E.Z)

$$\vec{\nabla}^2 \gamma_{00} = 0.$$

the only sensable solution there is $\gamma_{00} = 0$. (constant can be gauged away)

In short, the following can be achieved by just manipulating the gauge

$$\gamma = 0\,, \qquad (gc2)$$

$$\gamma_{0\mu} = 0\,. \qquad (gc3)$$

the combination of (Lorentz gauge) & (gc2), (gc3) forms the radiation (Coulomb) gauge condition.

Now we seek plane wave solutions of the LEZ. of the form

$$\gamma_{\mu\nu} = H_{\mu\nu} \exp(i k_\alpha x^\alpha)$$

where $H_{\mu\nu}$ = constant field.

Then the LEZ's solution exists if and only if

$$k^\mu k_\mu = 0 \quad . \qquad (r0)$$

the Coulumb gauge condition

$$k^\mu H_{\mu\nu} = 0 \quad , \qquad (r1)$$

$$H_{0\nu} = 0 \quad , \qquad (r2)$$

$$H^\mu{}_\mu = 0 \quad . \qquad (r3)$$

There are $4+4+1-1$ equations,

↑

(r1), (r2) have 1 redudancy: $H_{0\nu} k^\nu = 0$

and there are 10 free components of $H_{\mu\nu}$.

Therefore (r0) – (r3) suggest that the gravitational wave has 2 independent (= 10 − 8) polarization states;

~~and (r0) suggest the wave is massless and travel along null geodesics.~~

The wave is still waiting to be detected.

§ The Schwarzschild Solution

Now we would like to put G.R. to work:

to predict something beyond the Newtonian gravity.

Mathematically, put E.E. to work:

wherever gravity is important & manifest

& yet $T_{\mu\nu}$ is not too complicated.

⇒ Exterior space:

(a) gravity of our solar system ⇒ Schwarzschild Solution

(b) or some other exotic objets → e.g. black holes

(c) or even larger space. e.g. our universe ⇒ cosmology.

We do the (a) → (c) $\xrightarrow{\text{time?}}$ (b) order.

§ The sphereciallly symmetric metric

Apparently, the E.E will be much simpler if the metric has some symmetry.

The simplest : Minkovski metric $g_{\mu\nu} = \eta_{\mu\nu} = [^{-1}\ _1\ _1\ _1]$

rotational symmety + translational symmetry

The next simplest : rotational symmetry

$\Rightarrow$ spherically symmetric S^2.

¶ A handwaving argument

- Symmetries in differential geometry are described by Killing vectors.
- There exist a Frobenius's theorem which assert that :

a spherically symmetric manifold can be foliated into spheres.

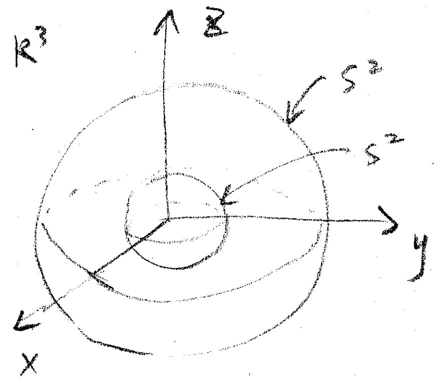
By foliation, we mean :

~~4 d manifold = 2 d manifold × S^2~~

if we have an n-dimensional manifold foliated by m-dimensional submanifolds, we can use a set of m coordinate functions u^i on the m-d submanifold to tell us where we are on this submanifold and use n-m coordinate functions v^J to tell us which m-d submanifold we are on.

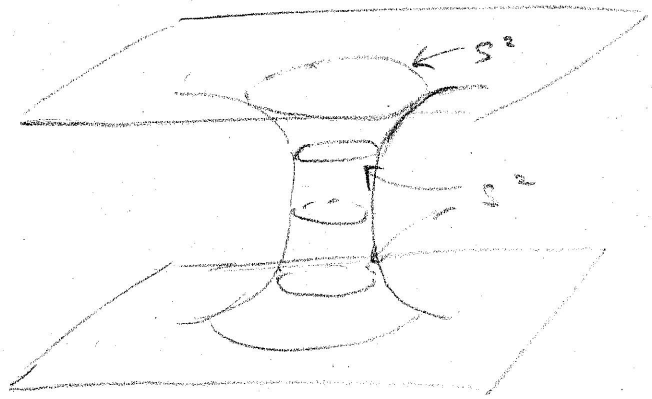
Foliation by S^2:

- $\mathbb{R}^3$ Foliation by S^2 with an origin



- A different manifold with an $R \times S^2$ topology "Warmhole"

Here we suppressed one dimension of s^2 and use a circle to represent it.



There is no point that looks like an "origin".

[Imagine yourself being labeled on such an S^2,
as you walk in radial direction, you experience a peirod of smaller radius and then it become large again;]

All of these are possible because we do not have a flat spacetime like $\eta_{\mu\nu}$ any more, we are allow any 4-d manifold to be a possible candidate of spacetime until we exclude them by experiments.

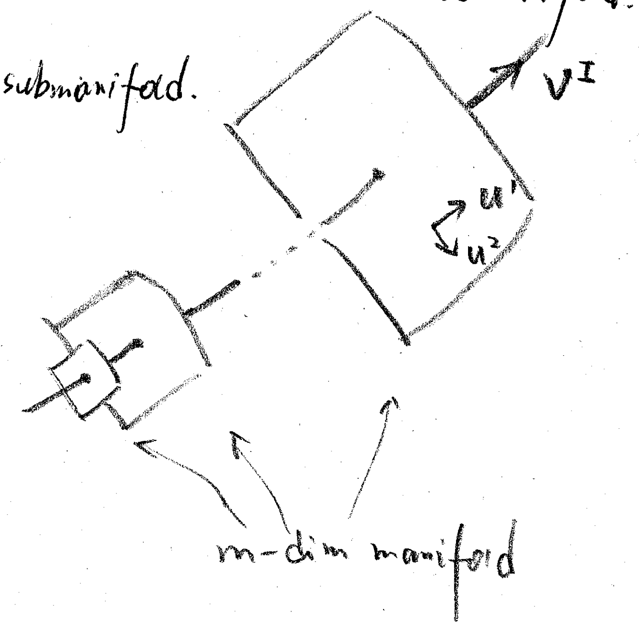
More precisely, an n-d manifold foliated by m-d submanifold has a metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{IJ}(U)\, dU^I dU^J + f(U)\, \gamma_{ij}(u)\, du^i du^j \qquad \text{(gm)}$$

Here U^I, $I = 1, \cdots, n-m$ are the coordinates that fixing which submanifold

u^i, $i = 1, \cdots, m$ are the coordinates on the m-dim submanifold.

$\gamma_{ij}(u)$ is the metric of the submanifold.



Two things to notice:

① no cross terms $dU^I du^j$

② both $g_{IJ}(U)$ and $f(U)$ are functions of U^I alone.

2. The spherecally symmetric metric in 4-d space time

- Realize this theorem to our foliation by S^2,

then our m-dim submanifold is an 2-d sphere.

Its metric <u>can always</u> be chosen as

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

where (θ, ϕ) are the two coordinates (same angles as you learned in spherecal cordinates in Calculus).

E5

Then the metric of our 4-d spacetime becomes from equ. (gm)

$$ds^2 = g_{aa}(a,b)\,da^2 + g_{ab}(a,b)\,(da\,db + db\,da) + g_{bb}(a,b)\,db^2$$
$$+ r^2(a,b)\,d\Omega^2$$

I.e., we use a, b to denote the rest 2 coordinates,

and $r^2(a,b)$ to denote the $f(v)$ function in eq. (gm).

- we are free to do coordinate transform from (a,b) to (a,r)

  i.e., a reparameterization of b by r.

(~~Condition~~: assuming r depends on b; otherwise do $(a,b) \to (r,b)$;

even otherwise, $\frac{\partial r}{\partial a} = 0 = \frac{\partial r}{\partial b}$.)

So the metric becomes ~~db(r)~~ $db(r) \to \frac{\partial b}{\partial r} dr$

$$ds^2 = g_{aa}(a, b(r))\,da^2 + g_{ab}(a, b(r))\left(da \frac{\partial b}{\partial r} dr + \frac{\partial b}{\partial r} dr\, da\right)$$
$$+ g_{bb}(a,b) \left(\frac{\partial b}{\partial r}\right)^2 dr^2 + r^2 d\Omega^2$$

$\downarrow$ define new functions & rename

$$ds^2 = g_{aa}(a,r)\,da^2 + g_{ar}(a,r)\,(da\,dr + dr\,da) + g_{rr}(a,r)\,dr^2$$
$$+ r^2 d\Omega^2 \qquad \text{(m1)}$$

- Now we seek another transform of coordinates so that the crossing term $da\,dr + dr\,da$ can be killed.

I.e., we seek a transform $(a, r) \longrightarrow (t(a,r), s(a,r))$ (gt)

s.t., $ds^2 = g_{tt}\,dt^2 + g_{ss}\,ds^2 + r^2(s,t)\,d\Omega^2$

The transform (gt) is the most general one we can do.

however it spoiled the $r^2 d\Omega^2$ term that we tried to put r as one of the coordinates.

Then, let us try to see whether we can find a transform T

$$T: (a, r) \longrightarrow (t(a,r), r)$$

s.t. ① kill cross term ② keep r as a coordinate.

$$ds^2 = g'_{tt}\,dt^2 + g'_{rr}\,dr^2 + r^2 d\Omega^2 .$$

Does this coordinate transform exist?

If yes, then since

$$dt = \frac{\partial t}{\partial r}dr + \frac{\partial t}{\partial a}da$$

$$dt^2 = \frac{\partial^2 t}{\partial r^2}dr^2 + \frac{\partial t}{\partial a}\frac{\partial t}{\partial r}(dr\,da + da\,dr) + \frac{\partial^2 t}{\partial a^2}da^2$$

then

$$ds^2 = \left(g'_{tt}\frac{\partial^2 t}{\partial a^2}\right)da^2 + g'_{tt}\frac{\partial t}{\partial a}\frac{\partial t}{\partial r}(dr\,da + da\,dr)$$

$$+ \left(\frac{\partial^2 t}{\partial r^2}g'_{tt} + g'_{rr}\right)dr^2 + r^2 d\Omega^2$$

Comparing to m1, this require

$$g_{aa} = g'_{tt} \frac{\partial^2 t}{\partial a^2} \qquad \text{eq.1}$$

$$g_{ar} = g'_{tt} \frac{\partial t}{\partial a} \frac{\partial t}{\partial r} \qquad \text{eq.2}$$

$$g_{rr} = g'_{tt} \frac{\partial^2 t}{\partial r^2} + g'_{rr} \qquad \text{eq.3}$$

Three unknown functions $t(a,r)$, $g'_{tt}(t(a,r),r)$, $g'_{rr}(t(a,r),r)$

and three equations.

I.e., for any g_{aa}, g_{ar}, g_{rr}, we can find such $t(a,r)$, g'_{tt}, g'_{rr}

that eq.1 → eq.3 are satisfield.

⇒ The transform T alway exists.

using this transform (without having to ~~knowing~~ exactly its form),

the metric with spherical symmetry can be written as (dropping ')

$$ds^2 = g_{tt}(t,r)\,dt^2 + g_{rr}(t,r)\,dr^2 + r^2 d\Omega^2 \qquad \text{gf1}$$

Claim: Any spherically symetric metric in 4-d spacetime

can be written into gf1 form.

- Signature of spacetime in G.R. $(- + + +)$

therefore one of $g_{tt}(t,r)$ and $g_{rr}(t,r)$ is negative

and the other should be positive.

we choose $\mathrm{sign}(g_{tt}) = -1$, $\mathrm{sign}(g_{rr}) = +1$. for now.

Then the metric sometimes is written as

$$ds^2 = -f(r,t)\,dt^2 + h(r,t)\,dr^2 + r^2 d\Omega^2 . \qquad (gf2)$$

$f, h > 0$

or

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 . \qquad (gf3)$$

The metrics (gf1), (gf2), (gf3) are the best we can do to a spherecally symmetric spacetime.

2. Vacaum

The metric of a simplest spacetime (except Minkovski) is known now.

On the other side of the E.E.

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

the simplest form of $T_{\mu\nu}$ is Vacuum, defined by $T_{\mu\nu} = 0$.

see inserted page (E9').

3. Solution of the vacuum E.E for a spherically symmetric spacetime

- published January 13, 1916 by Karl Schwarzschild
- written during W.W.I when he is in the Russian front.
- Einstein's G.R. was known November 1915.
- First exact solution of E.E and reminds the most important one.

  Einstein:

  "I had not expected that one could formulate the exact solution of the problem in such a simple way."

- K. Schwarzschild died shortly after his paper was published in 1916 because of the disease he caught at Rusian frontline. (May 11, 1916)

- In vacuum, the E.E. can be re-written.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Taking trace

$$R - \frac{1}{2} \cdot 4 R = 8\pi T \qquad [T \equiv T^{\mu}{}_{\mu}]$$

$$R = -8\pi T$$

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

In vacuum, becomes

$$\cancel{R_{\mu\nu}} \quad R_{\mu\nu} = 0.$$

- Reminder: ⓐ $T_{\mu\nu}$ does not have to be zero everywhere in spacetime.

  As long as it is zero in a large chunk,

  we sometimes call it a "vacuum" solution.

  ⓑ The word "vacuum" here is really a local word;

  meaning $T_{\mu\nu}$ only need to be zero locally.

  ⓒ E.E is a local equation, like Maxwell's equations in its differential form.

  Everywhere $T_{\mu\nu}=0$, the equation $R_{\mu\nu}=0$ has to be satisfied.

  While for regions $T_{\mu\nu} \neq 0$, then $R_{\mu\nu}$ ~~only~~ need to satisfy the $G_{\mu\nu} = 8 T_{\mu\nu}$ at those region.

Start from

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(t, r, θ, φ) being coordinates

Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. ($\Rightarrow$ We need only compute $\mu \le \nu$ cases.)

$$[g_{\mu\nu}] = \begin{bmatrix} -e^{2\alpha(t,r)} & & & \\ & e^{2\beta(t,r)} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix}$$

$$[g^{\mu\nu}] = \begin{bmatrix} -e^{-2\alpha(t,r)} & & & \\ & e^{-2\beta(t,r)} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$

(t, r, θ, φ) indices $\Longleftrightarrow$ $(0, 1, 2, 3)$

I will only work out the Christoffel symbols on board.

The rest: Riemann tensor, Ricci tensor, Ricci scalar, Einstein tensor. You can ~~work~~ work them out (using Maple).

We increase ν first, then increase μ, finally increase ρ

$$\Gamma^0_{00} = \frac{1}{2} g^{0\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00})$$

$$= \frac{1}{2}(-e^{-2\alpha})(-e^{2\alpha})2\cdot\partial_0\alpha$$

$$= \partial_0\alpha$$

$$\Gamma^0_{01} = \frac{1}{2} g^{0\lambda}(\partial_0 g_{1\lambda} + \partial_1 g_{\lambda 0} - \partial_\lambda g_{01})$$

$$= \frac{1}{2}(-e^{-2\alpha})(-e^{2\alpha})2\cdot\partial_1\alpha$$

$$= \partial_1\alpha$$

$$\Gamma^0_{11} = \frac{1}{2} g^{0\lambda}(\partial_1 g_{1\lambda} + \partial_1 g_{\lambda 1} - \partial_\lambda g_{11})$$

$$= \frac{1}{2}(-e^{-2\alpha})(-e^{2\beta})\partial_0\beta\cdot 2$$

$$= e^{2(\beta-\alpha)}\partial_0\beta$$

E11

The rest $\Gamma^0_{\mu\nu} = \frac{1}{2} g^{0\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$

$\nu \geq 2, \quad \mu \leq \nu$

$\lambda = 0$

$$= \frac{1}{2} g^{00} (\partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu})$$

$\mu = 0$, $\nu \geq 2$

$$= \frac{1}{2} g^{00} (\partial_\nu g_{00}) = 0$$

$\mu = \nu \geq 2$

$$= \frac{1}{2} g^{00} (-\partial_0 g_{\nu\nu}) = 0 \qquad \text{(not summed)}$$

$$\Gamma^1_{\mu\nu} = \frac{1}{2} g^{1\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$= \frac{1}{2} g^{11} (\partial_\mu g_{\nu 1} + \partial_\nu g_{1\mu} - \partial_1 g_{\mu\nu})$$

$$\Gamma^1_{00} = \frac{1}{2} g^{11} (-\partial_1 g_{00}) = \frac{1}{2} (e^{-2\beta})(+e^{+2\alpha}) 2 \partial_1 \alpha$$

$$= e^{2(\alpha-\beta)} \partial_1 \alpha$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11} (\partial_0 g_{11}) = \frac{1}{2} (e^{-2\beta})(e^{2\beta}) 2 \partial_0 \beta$$

$$= \partial_0 \beta$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} (\partial_1 g_{11}) = \frac{1}{2} (e^{-2\beta})(e^{2\beta}) 2 \partial_1 \beta$$

$$= \partial_1 \beta$$

$$\Gamma^1_{\mu 2} = \frac{1}{2} g^{11} (\partial_\mu g_{21} + \partial_2 g_{1\mu} - \partial_1 g_{\mu 2})$$

$$\Gamma^1_{02} = \Gamma^1_{12} = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} (-\partial_1 g_{22}) = \frac{1}{2} (e^{-2\beta})(-2r)$$

$$= -re^{-2\beta}$$

$$\Gamma^1_{\mu 3} = \frac{1}{2} g^{11} (\partial_\mu g_{31} + \partial_3 g_{1\mu} - \partial_1 g_{\mu 3}),$$

$$\Gamma^1_{03} = \Gamma^1_{13} = \Gamma^1_{23} = 0,$$

$$\Gamma^1_{33} = \frac{1}{2} (e^{-2\beta}) (-2r \sin^2\theta)$$

$$= -r \sin^2\theta \, e^{-2\beta} .$$

$$\Gamma^2_{\mu\nu} = \frac{1}{2} g^{2\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$= \frac{1}{2} g^{22} (\partial_\mu g_{\nu 2} + \partial_\nu g_{2\mu} - \partial_2 g_{\mu\nu})$$

$$\Gamma^2_{\mu 0} = \Gamma^2_{\mu 1} = 0$$

$$\Gamma^2_{\mu 2} = \frac{1}{2} g^{22} (\partial_\mu g_{22} + \partial_2 g_{2\mu} - \partial_2 g_{\mu 2}) = \frac{1}{2} g^{22} (\partial_\mu g_{22})$$

$$\Gamma^2_{02} = 0 = \Gamma^2_{22}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} (\partial_1 g_{22})$$

$$= \frac{1}{2} \cdot \frac{1}{r^2} \cdot 2r = \frac{1}{r}$$

$$\Gamma^2_{\mu 3} = \frac{1}{2} g^{22} (\partial_\mu g_{32} + \partial_3 g_{22} - \partial_2 g_{\mu 3})$$

$$\Gamma^2_{03} = \Gamma^2_{13} = \Gamma^2_{23} = 0$$

$$\Gamma^2_{33} = \frac{1}{2} \cdot \frac{1}{r^2} \cdot (\cancel{\phi} - 2r^2 \sin\theta \cos\theta) = -\sin\theta\cos\theta$$

$$\Gamma^3_{\mu\nu} = \frac{1}{2} g^{3\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$= \frac{1}{2} g^{33} (\partial_\mu g_{\nu 3} + \partial_\nu g_{3\mu} - \partial_3 g_{\mu\nu})$$

$$= \frac{1}{2} g^{33} (\partial_\mu g_{\nu 3} + \partial_\nu g_{3\mu})$$

$$\Gamma^3_{\mu 0} = 0 = \Gamma^3_{01} = \Gamma^3_{11} = \Gamma^3_{02} = \Gamma^3_{12} = \Gamma^3_{22} = \Gamma^3_{03}$$

$$\Gamma^3_{13} = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} 2r \sin^2\theta = \frac{1}{r}$$

$$\Gamma^3_{23} = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} 2r^2 \sin\theta \cos\theta = \frac{\cos\theta}{\sin\theta}$$

$$\Gamma^3_{33} = 0.$$

Riemann tensor

...

Ricci tensor

(zero terms & terms obtainable by symmetry are not written out)

① $R_{00} = [\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\,\partial_0\beta] + e^{2(\alpha-\beta)}[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\,\partial_1\beta + \frac{2}{r}\partial_1\alpha]$

② $R_{01} = \frac{2}{r}\partial_0\beta$

③ $R_{11} = -[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\,\partial_1\beta - \frac{2}{r}\partial_1\beta] + e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\,\partial_0\beta]$

④ $R_{22} = e^{-2\beta}[r(\partial_1\beta - \partial_1\alpha) - 1] + 1$

⑤ $R_{33} = R_{22}\sin^2\theta$

E.E. for vacuum : $R_{\mu\nu} = 0$

- ② $\Rightarrow \partial_0\beta = 0$
- ∂_0 ④ $= \partial_0 R_{22} = e^{-2\beta}.r.(-\partial_0\partial_1\alpha) = 0$

These requires

$$\beta = \beta(r) \qquad (*1)$$

$$\partial_\phi \alpha = h(r)$$

$$\Rightarrow \alpha = \int h(r)\, dr + g(t)$$

$$\alpha = f(r) + g(t) \qquad (*2)$$

Aside:

The metric is then

$$ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

We can do another coordinate transform

$$dt \to e^{-g(t)} dt$$

then the metric becomes

$$ds^2 = -e^{2f(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2.$$

We call this a static metric — independent of x^0.

And the above procedure proved:

Any spherically symmetric vacuum metric can be transformed into a static metric.

Therefore sometimes in literature, a spherically symmetric & vacuum metric is often assumed to take the form

$$ds^2 = -a(r)\, dt^2 + b(r)\, dr^2 + r^2 d\Omega^2.$$

Continue our solution to ① → ⑤:

- $e^{2(\beta-\alpha)} \cdot ① + ③ = \frac{2}{r}(\partial_1 \alpha + \partial_1 \beta) = 0$

$$\Rightarrow \partial_1 \alpha + \partial_1 \beta = 0$$

- $④ = e^{-2\beta}[r 2 \partial_1 \beta - 1] + 1 = 0$

$$\Rightarrow (1 - 2r\partial_1\beta)e^{-2\beta} = 1$$

$$\partial_1(re^{-2\beta}) = 1$$

Solution: $e^{-2\beta} = 1 + \frac{\mu}{r}$ μ is arbitrary constant (*3)

$\alpha = -\beta(r) + h(t)$ (*4)

The metric becomes

$$ds^2 = -e^{-2\beta(r)} e^{2h(t)} dt^2 + (1 + \frac{\mu}{r})^{-1} dr^2 + r^2 d\Omega^2$$

↓ Change of coordinates

$$= -e^{-2\beta(r)} dt^2 + (1 + \frac{\mu}{r}) dr^2 + r^2 d\Omega^2$$

$$ds^2 = -(1 + \frac{\mu}{r}) dt^2 + (1 + \frac{\mu}{r})^{-1} dr^2 + r^2 d\Omega^2 \quad \text{(S.M)}$$

This is the Schwarzschild solution of the vacuum Einstein equation for an spherically symmetric spacetimes.

• Fix the constant μ

→ The s.m is asymptotically flat, i.e.,

$$g_{\mu\nu} \xrightarrow{r\to\infty} \eta_{\mu\nu}$$ Minkowski metric in spherical coordinates with r being the radius from origin

Therefore we interpret s.m as the exterior gravitational field of an isolated body.

And we also call the "r" is sm as the "radius" and the "t" as the "time".

→ μ can be fixed by comparing the s.m with the metric of the weak field of a ~~mass~~ body with mass M.

In previous section, such a metric was obtained on P20.

$$\begin{cases} \bar{\gamma}_{00} = -4\phi & \text{where } \phi = -\frac{GM}{r} \text{ is the gravitational potential} \\ \bar{\gamma}_{ij} = \bar{\gamma}_{i0} = 0 \end{cases}$$

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma} = \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(4\phi)$$

$$[\gamma_{\mu\nu}] = \begin{bmatrix} -4\phi+2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{bmatrix} = \begin{bmatrix} -2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{bmatrix}$$

$$\gamma_{\mu\nu} = -2\phi\,\delta_{\mu\nu}$$

then we have

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \, \star .$$

In spherecal coordinates

$$g_{00} = -(1+2\phi)$$

$$g_{rr} = (1-2\phi)$$

Comparing with s.m, we get $\mu = -2GM$.

Then finally we arrive at the celebrated Schwarzschid metric

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega$$

Comments:

(a) Birkhoff's theorem: S.M. is the unique spherically symmetic vacuum Solution.

By "unique", all other solutions can be reached from the SM by coordinate transformation.

(b) The source dose not need to be static.

E.g. a collapsing star,

a supernova explosion

as long as the process is spherically symmetric.

(c) This last point (b) is like in e&m, where a radial redistribution of ~~total~~ charge dose not affect the electric field ~~far away~~ outside.

In particular, ~~little~~ radiation is generated.

Similarly, since the gravititional field is not much changed during redistribution, little gravititional wave will be generated.

§ Singularity in the SM.

Singularity? what is it.

Mathematically, a (set of) points at which some given ~~mathematical~~ mathematical object becomes not defined or not "well-behaved".

E.g., infinite or not-differentiable.

Complex analysis: $\frac{1}{x^n} \to$ order n singularity;

$$f(z) = \frac{\sin(z)}{z} \qquad z=0 \quad \text{removable singularity}$$

1. Here $\quad ds^2 = -(1-\frac{2M}{r})\,dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2$

Two superfacial singularities of the spacetime

① $r=0$

② $r=2M$

Since the metric becomes poorly behaved.

But is this a ~~good~~ precise enough reason to say they are singularities?

Answer: No.

True physical singularities of spacetime should be defined from the behavior of some physical process, not simply from the metric.

In particular, metric $g_{\mu\nu}$'s components depend on the coordinate system you chose for the manifold.

E.g., in polar coordinates, the metric of $\mathbb{R}^2$

$$ds^2 = dr^2 + r^2 d\theta^2$$

At point $r=0$, $g_{rr}=0$; but we know that point is perfectly ok.

So, we need to study coordinate independent quantities to characterize singularities.

Here is our definition:

Singularity:

Points at which any scalar constructed from the metric becomes infinite.

- By "scalar", $(0,0)$ type tensor.
- singularities defined this way are sometimes called "curvature singularities".
- many times we also associate a physical condition to the definition: These points has to be reachable ~~in a finite~~ by travelling a finite distance along a geodesic.
- other definitions exist.

Scalars we usually consider

$$R_{\alpha\beta\sigma\rho}\, g^{\alpha\sigma} g^{\beta\rho} \qquad = R \qquad \text{Ricci scalar}$$

$$R_{\alpha\beta\sigma\rho} R_{\mu\nu\lambda\eta}\, g^{\alpha\mu} g^{\beta\nu} g^{\sigma\lambda} g^{\rho\eta} \qquad = R^{\mu\nu\lambda\eta} R_{\mu\nu\lambda\eta}$$

$$R_{\alpha\beta\sigma\rho} R_{\mu\nu\lambda\eta}\, g^{\alpha\sigma} g^{\mu\lambda} g^{\beta\nu} g^{\rho\eta} \qquad = R^{\mu\nu} R_{\mu\nu}$$

$$R_{\mu\nu\rho\sigma} R^{\rho\sigma\lambda\tau} R_{\lambda\tau}{}^{\mu\nu}$$

$\vdots$

even high orders in $R_{\mu\nu\sigma\rho}$.

Non-singularity ?

- check all scalars at these points
- Geodesics behave well at these points.

2. Analysis of $r=0$ and $r=2M$

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48 G^2 M^2}{r^6}$$

Then $r=0$ is a singularity point, since the above blows up.

While the $r=2M$ manifold is not singular from the point of view of $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$.

Indeed, you can check other curvature scalars, at $r=2M$, they are not singular.

proof that $r=2M$ is not singular:

find a coordinate system near $r=2M$ s.t. the metric is well-defined and well-behaved.

Consider the coordinate transform

$$(t,r) \longrightarrow (v,r)$$

where

$$v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right| \qquad (\text{*}1)$$

The metric then will take the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2 \qquad (\text{*}2)$$

This metric behaves completely fine at $r=2M$.

- There exist (many) other coordinate transforms that will make the metric well behaved at $r=2M$.

Homework: ① show that (*1) leads to (*2)

② find another coordinate transform that can transform the s.m. to a well behaved one when $r \to 2M^+$ & $r \to 2M^-$.

- Though the $r=2M$ surface is not singular, we will show later on that it is an very very interesting surface, which has astonishing properties.

§ Interior solution

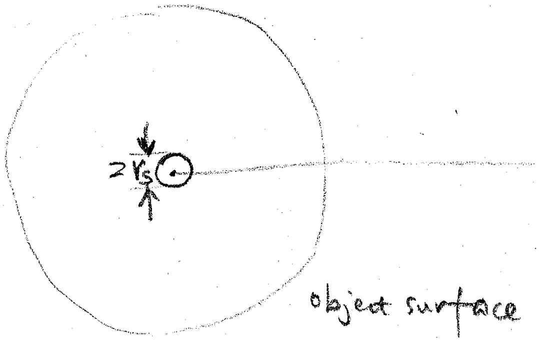
The manifold of $r = 2M$ happens for an object with mass M at

$$r_s \equiv 2M = \frac{2GM}{c^2} \simeq 2.95\left(\frac{M}{M_\odot}\right)\,\text{km} \qquad \text{Schwarzschild radius}$$

For our sun, $r_s = 2.95\,\text{km} \ll R_{sun}$

earth, $r_s = 2.95 \times 3.0\times10^{-6}\,\text{km} = 9\,\text{mm} \ll R_{earth}$

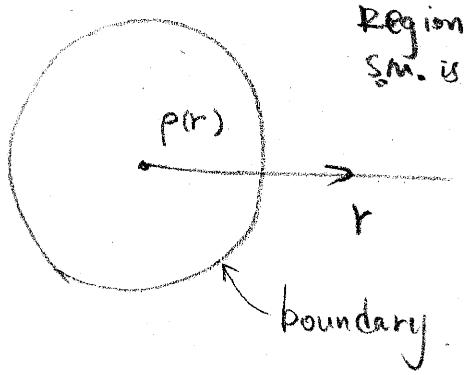
Therefore for most objects, the Schwarzschild radius is well inside the object, where the S.M solution is not valid anyway.



1. E.E with spherically symmetric fluid

We wish to ① study the gravitational field inside an sphereical object ② compare with Newtonian limit & ③ see how to join with the exterior Schwarzschild vacuum solution.

- Suppose that the <u>equation of state</u> of the interior matter is
$$P = P(\rho).$$
- The interior matter density $\rho = \rho(r)$;
- Interior reached equilibrium & therefore static.



We use a metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 .$$

E.E. becomes

(a) $G_{00} = \frac{1}{r^2} e^{2\alpha} \frac{d}{dr}[r(1-e^{-2\beta})]$

(b) $G_{rr} = -\frac{1}{r^2} e^{2\beta}(1-e^{-2\beta}) + \frac{2}{r}\frac{d\alpha}{dr}$

(c) $G_{\theta\theta} = r^2 e^{-2\beta}\left[\frac{d^2\alpha}{dr^2} + (\frac{d\alpha}{dr})^2 + \frac{1}{r}\frac{d\alpha}{dr} - \frac{d\alpha}{dr}\cdot\frac{d\beta}{dr} - \frac{d\beta}{dr}\cdot\frac{1}{r}\right]$

(d) $G_{\varphi\varphi} = \sin^2\theta \, G_{\theta\theta}$

Assumming the interior is a perfect fluid, then

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + P g^{\mu\nu}$$

Here U^μ is the four-velocity field of the fluid satisfying $U^\mu U_\mu = -1$.

Since the star is static, U^μ only has -t component, so

$$U^0 U^0 g_{00} = -1,$$

$$U^0 = e^{-\alpha(r)}, \quad U_0 = -e^{\alpha(r)}; \quad U^i = 0.$$

These produce a $T^{\mu\nu}$:

① $T_{00} = \rho e^{2\alpha}$,

② $T_{rr} = P e^{2\beta}$,

③ $T_{\theta\theta} = r^2 P$,

④ $T_{\varphi\varphi} = \sin^2\theta \, T_{\theta\theta}$.

And they have to satisfy the conservation law

$$\nabla_\mu T^{\mu\nu} = 0 .$$

Computing the Chiristofel symbols, then the covariant derivitive,

we get the only non-vanishing component $\nu = r$,

$$⑤ \qquad (\rho + p) \frac{d\alpha}{dr} = - \frac{dp}{dr}$$

Now, we have all components collected in our hands, we need to solve them.

$$\begin{cases} (a) = ① \cdot 8\pi \\ (b) = ② \cdot 8\pi \\ (c) = ③ \cdot 8\pi \\ (d) = ④ \cdot 8\pi \qquad \text{(same as above)} \\ ⑤ \end{cases}$$

we have ~~4~~ functions $\alpha(r)$, $\beta(r)$, $\rho(r)$, $p(r)$;

we need to know $\alpha(r)$ and $\beta(r)$. To do this, perhaps we need to know

~~→ we nee~~ as an input:

→ both $\rho(r)$ and $p(r)$; or

→ only one of $\rho(r)$ and $p(r)$ and the other can be fixed; or

→ perhaps both $\rho(r)$ and $p(r)$ have to have a fixed form to let $\alpha(r)$ and $\beta(r)$ have a solution.

Which is the case, let us try solve the system.

First let us trade-off $\beta(r)$ by $m(r)$

$$m(r) \equiv \frac{1}{2} r \left(1 - e^{-2\beta(r)}\right)$$

Then the $(0,0)$ component of the E.E becomes

$$\frac{d\,m(r)}{dr} = 4\pi r^2 \rho(r) \qquad \text{(A)}$$

while $(1,1)$ component becomes

$$\frac{d\alpha(r)}{dr} = \frac{m(r) + 4\pi r^3 p(r)}{r\,(r - 2m(r))} \qquad \text{(B)}$$

using this in the conservation eqn (5), we get

$$\frac{dp}{dr} = -(\rho + p)\,\frac{m + 4\pi r^3 p}{r\,(r-2m)} \qquad \text{(C)}$$

The $(2,2)$ component equation, which is the longest, can be checked to be automatically satisfied using (A), (B), (C) and definition of $m(r)$.

~~Comments before we continue the solution process:~~

~~(1) In Newtonian limit, $p \ll \rho$~~

~~$p\,r^3 \ll \rho r^3 \approx m$~~

So effectively, four functions $m(r)$, $\alpha(r)$, $p(r)$, $\rho(r)$ and three equations.

Only when one of them is fixed, the other three can be solved.

Ⓐ has solution of $m(r)$ in $\rho(r)$

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx + a .$$

This allow us to express the g_{rr} component

$$g_{rr} = e^{2\beta} = 1 - \frac{2m(r)}{r}$$

$$= 1 - \frac{2\int_0^r 4\pi x^2 \rho(x) dx + 2a}{r}$$

$r =$ constant should be a spacelike surface for a static field, therefore $g_{rr} > 0 \Rightarrow 2m(r) \leq r$.

$$g_{rr}(r \to 0) \to 1 - \frac{2a}{r}$$

We want a smooth metric at $r=0$ and therefore $a=0$.

I.e., $m(r) = \int_0^r 4\pi x^2 \rho(x) dx$

- Then assuming $m(r)$ is known, the next simplest equ is Ⓒ.

  After solving $P(r)$ from Ⓒ and putting into Ⓑ, we can solve $\alpha(r)$.

- So, all of $m(r)$, $P(r)$ and $\alpha(r)$ can be fixed up to some integral constants by knowing $\rho(r)$.

- unfortunately, we can not solve Ⓒ generally without knowing some particular form of $\rho(r)$.

Therefore we study some simple $\rho(r)$ next.

Before doing that, some comments:

(1) The formula of $m(r)$ is the same as the mass formula in Newtonian gravity.

However, it is not exact the mass in G.R., because in G.R., the proper volume element is $\sqrt{|{}^{3}g|}\, dx^3 = e^{-\beta(r)} r^2 \sin\theta\, dr\, d\theta\, d\varphi$.

Mass is

$$M(x) = \int_0^r 4\pi x^2 \rho(x) \left[1 - \frac{2m(x)}{x}\right]^{-\frac{1}{2}} dx$$

$$> m(r) \text{ since } 2m(x) \leq x.$$

There difference $\Delta M \equiv M(x) - m(x)$ is the gravitational binding energy.

~~I.e.~~

(2) In Newtonian limit, $\rho \gg P$, $m(r) \sim \rho(r) r^3 \gg P r^3$, $m(r) \ll r$

then eq. (B) becomes

$$\frac{d\alpha(r)}{dr} \approx \frac{m(r)}{r},$$

which is the Poisson's equation for Newtonian gravity potential.

(3). Eq. (C) is call Tolman-Oppenheimer-Volkoff equation,

It states a relation between P and ρ that a equilibrium and static fluid should satisfy in a spherical gravity field.

In Newtonian limit, it becomes

$$\frac{dP}{dr} = -\frac{\rho\, m(r)}{r^2}, \qquad (C')$$

which is Newtonian hydrostatic equilibrium equation.

We can solve these potentially extreme points

$(\star 1) \Rightarrow \quad r^2 = R^2(9 - \frac{4R}{M}) \qquad (\star 2)$

$(\star 1) \Rightarrow \quad r = 0$

boundaries $\quad r = 0, \quad \underline{r = \infty} \nearrow$ gone.

① when $9 - \frac{4R}{M} < 0$, $(\star 2)$ has no real solution.

Extreme happens at the center of the star

$$P(r=0) = \rho_0 \left[\frac{1-(1-\frac{2M}{R})^{\frac{1}{2}}}{3(1-\frac{2M}{R})^{\frac{1}{2}} - 1} \right]$$

$$9 - \frac{4R}{M} < 0 \Rightarrow 3(1-\frac{2M}{R})^{\frac{1}{2}} - 1 > 0$$

Therefore $P(r)$ is bounded.

② when $9 - \frac{4R}{M} \geq 0$, $(\star 2)$ has real ~~solution~~ positive solution

$$r = R(9 - \frac{4R}{M})^{\frac{1}{2}}$$

$\leq R$ since $r \geq 2m(r)$

$R \geq 2M$ for r=constant surface to be static & spacelike.

I.e., $P(r = R(9-\frac{4R}{M})^{\frac{1}{2}})$ will blow up.

(E-28)

2. pressure, density & stability of stars.

(2.1) Now we have to specify a form of $\rho(r)$.

We consider an incompressible fluid of density ρ_0

$$\rho(r) = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

Then immediately from (A)

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r \leq R \\ \frac{4}{3}\pi R^3 \rho_0 \equiv M & r > R \end{cases}$$

(2.2) Then from (C), now $P(r)$ is solvable

$$P(r) = \begin{cases} \rho_0 \left[\dfrac{\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} - \left(1-\frac{2M}{R}\cdot\frac{r^2}{R^2}\right)^{\frac{1}{2}}}{\left(1-\frac{2M}{R}\cdot\frac{r^2}{R^2}\right)^{\frac{1}{2}} - 3\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}} \right] & r \leq R \\ 0 & r > R \end{cases}$$

There was an integral constant that we used to set the boundary condition $P(R) = 0$.

We would like $P(r)$ to be bounded at all $r \geq 0$:

$$|P(r)| < \infty \quad \text{for} \quad r \geq 0$$

The extreme of $P(r)$ occur at points

$$\begin{cases} \left(1-\frac{2M}{R}\cdot\frac{r^2}{R^2}\right)^{\frac{1}{2}} - 3\left(1-\frac{2M}{R}\right)^{\frac{1}{2}} = 0, & (※1) \\ \text{or at } r = 0, \ r = \infty \\ \text{or at } \dfrac{dP(r)}{dr} = 0. & (※1) \end{cases}$$

Summarizing ① & ②, we get ① has to be the case for a stable star.

That is $M < \frac{4R}{9}$

or $M < \frac{4}{9} \cdot \frac{1}{(3\pi\rho)^{\frac{1}{2}}}$ since $\frac{4}{3}\pi R^3 \rho = M$.

physical statement: A object with low density but large mass can not be stable.

E.g., A gas planent can not be too heavy if it's stable.

Comments.

① Is this statement due to G.R.? What about Newtonian case?

In Newtonian limit, ~~eq~~ poisson's equ. (C') lets us solve

$$P(r) = \begin{cases} \frac{2}{3}\pi\rho_0^2(R^2 - r^2) & r \leq R \\ 0 & r > R \end{cases}$$

This is everywhere finite.

Therefore no stability statement can be said about the star.

② What if $\rho \neq$ constant? Results:

(2.1) For $\frac{d\rho}{dr} \leq 0$, and the star has a fixed radius R,

one can always prove the maximum stable mass is

$$M_{max} = \frac{4}{9}R.$$

(2.2) For $\frac{dp}{dr} \le 0$, $|p(r)| \le \rho_0$ and $\frac{d P(\rho)}{d\rho}$ small at small ρ,

there always exist an upper limit ~~for stab~~ of ~~Mass~~ for stable star.

• Along this line, many interesting and grand conclusions can be drawn about stars.

Consult some modern astronomy books.

(2.3) The equation (B) for ρ = constant case can be solved analytically.

The result is quite long so ~~we~~ will not write it out.

But beyond $r = R$, one can verify that the solution become

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \qquad r \geq R.$$

I.e., the two part of the metric jion smoothly.

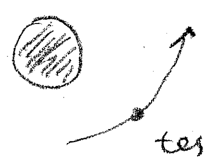
§ Geodesics in the Scharszchild spacetime

The interest here is to study how the geodesics behave in the vacuum part of the spacetime ~~in~~ schwarzschild

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

The reason to study geodesics is apparent:

all matter, including sattlite/moons around planets, planets around stars or space travellers, follow their geodesics.



We would like to know what happens to them.

1. The general geodesic equation and its simplification.

Geodesic eq. $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

where λ is a affine parameter.

For timelike geodesics, ~~$\lambda = \tau$ is the pro~~ λ is chosen as propertime.

For spacelike geodesics, it can be chosen as proper space.

For null geodesics, it is some affine parameter.

Also, we have the normalization condition

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu} = -K \qquad K = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$$

The Christoffel symbols: (non-zero)

$$\Gamma^0_{01} = \frac{M}{r(r-2M)}$$

$$\Gamma^1_{00} = \frac{M}{r^3}(r-2M)$$

$$\Gamma^1_{11} = -\frac{M}{r(r-2M)}$$

$$\Gamma^1_{22} = -(r-2M)$$

$$\Gamma^1_{33} = -(r-2M)\sin^2\theta$$

$$\Gamma^2_{12} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}$$

- 4-components of the Geodesic equation:

(t) $$\frac{d^2t}{d\lambda^2} + \frac{2M}{r(r-2M)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0$$

(r) $$\frac{d^2r}{d\lambda^2} + \frac{M}{r^3}(r-2M)\left(\frac{dt}{d\lambda}\right)^2 - \frac{M}{r(r-2M)}\left(\frac{dr}{d\lambda}\right)^2 - (r-2M)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

(θ) $$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

(φ) $$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

Normalization condition

(k) $$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 + r^2\sin^2\theta\left(\frac{d\theta}{d\lambda}\right)^2 = -k$$

$k=+1$ timelike ; $k=0$ null ; $k=-1$ spacelike.

Simplification :

Now this is a system of ode's.

- Solution will depend on initial position and 1-st derivitive.

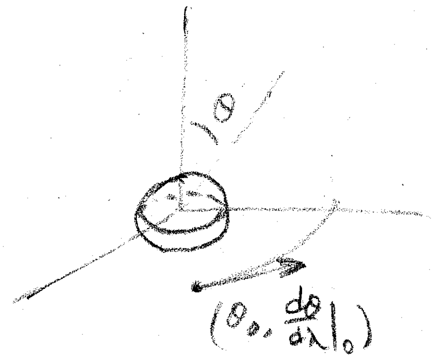
Notice $\theta \to \pi - \theta$ is a symmetry of the system (and the metric),

then if initiall position and tangent vector lies in the plane $\theta = \frac{\pi}{2}$,

then the test particle will remain in this plane,

because it should not deviate from

this plane to any direction.



On the other hand,

any intial position and tangent vector

can be brough to this plane through rotation of coordinate system.

Therefore, without losing any generality, we can set

$$\theta(\lambda) = \frac{\pi}{2} \text{ for all } \lambda.$$

And this apparent solves θ.

- Then using this in eq $\varphi \cdot r^2$, we relaize its a total derivitive

$$\frac{d(r^2 \frac{d\phi}{d\lambda})}{d\lambda} = 0$$

i.e.,

$$r^2 \frac{d\phi}{d\lambda} = L . \qquad \varphi'$$

• The (t) equation can also be tackled, treating $r(\lambda)$ and $\frac{dr}{d\lambda}$ as known functions, multiplying by $(1-\frac{2M}{r})$, we see

$$(t) \Rightarrow \quad \frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}\right] = 0$$

$$\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} = E \qquad (t')$$

So, once $r(\lambda)$ is known, using (φ') & (t'), we can get $t(\lambda)$ and $\varphi(\lambda)$.

• The (K) equation, after multiplying $(1-\frac{2M}{r})$, & using (φ') & (t'), becomes

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+K\right) = 0 \qquad \cancel{(K)}$$

~~Finally, the longest equation (r), after replacing $\frac{dt}{d\lambda}$, $\frac{d\varphi}{d\lambda}$, becomes~~

$$\cancel{\frac{1}{2}\left(\frac{dr}{d\lambda}\right) + V(r) = \frac{1}{2}E^2} \qquad (r')$$

~~where~~ $\cancel{V(r) = \frac{1}{2}K - \frac{M}{r}K + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}}$.

This can also be written as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \qquad (K')$$

where
$$V(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)K + \frac{L^2}{2r^2}\left(1-\frac{2M}{r}\right)$$
$$= \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(K+\frac{L^2}{r^2}\right)$$

• Finally the longest equation (r), after substituting $\frac{d\varphi}{dr}$, $\frac{dt}{dr}$, becomes

$$\frac{d}{d\lambda}\left[\left(\frac{dr}{d\lambda}\right)^2 \Big/ \left(1-\frac{2M}{r}\right) + \frac{L^2}{r^2} - \frac{E^2}{1-\frac{2M}{r}}\right] = 0$$

(see "引力论与宇宙论" eq. 8.4.12)

From this we get

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V'(r) = \frac{1}{2}E^2$$

where $V'(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+c\right)$ for any constant c.

Then comparing with (K'), we see that $c = K$ in order to have solutions.

⇒ Summary: We finally have 3 equations, (t'), (φ') and (K')

with $r(\lambda)$, $\varphi(\lambda)$ and $t(\lambda)$ undetermined.

⇒ Once $r(\lambda)$ is determined, the rest can be fixed easily.

⇒ The (K') equation is the same as a unit mass particle of energy $E^2/2$ moving in 1-d effective potential $V(r)$.

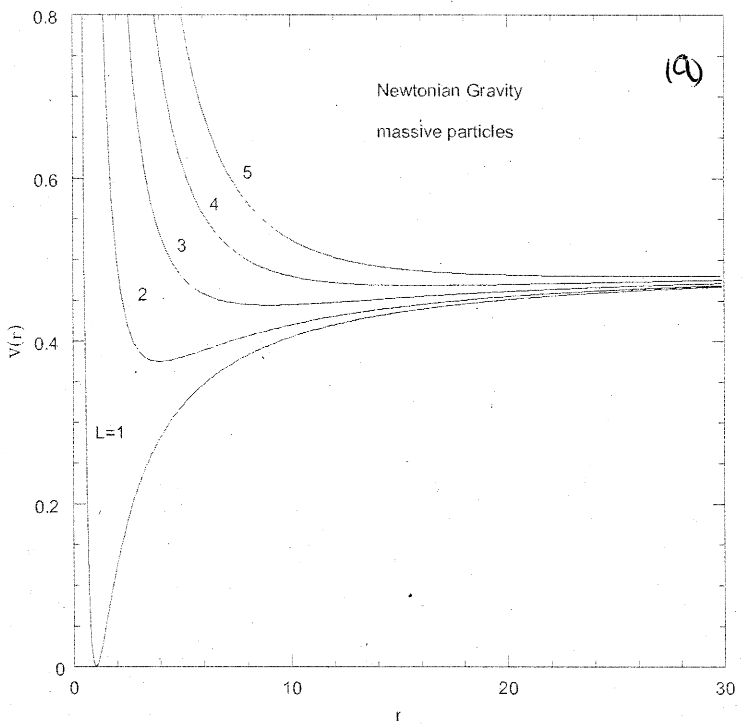
⇒ The integral constants L and E corresponds to the test particle's angular momentum and energy per unit mass respectively. They are conserved because they are constants of the motion.

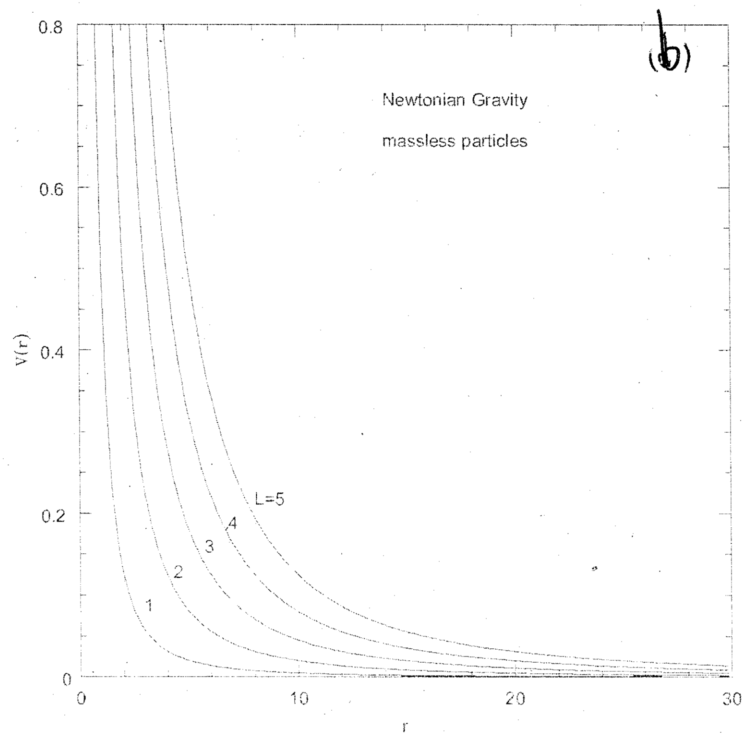
⇒ In Newtonian gravity, the equation of motion would be the same except the potential $V_{Newton} = \frac{1}{2}\left(1-\frac{2M}{r}\right)K + \frac{1}{2}\frac{L^2}{r^2}$.

I.e., without the $-\frac{ML^2}{r^3}$ term.

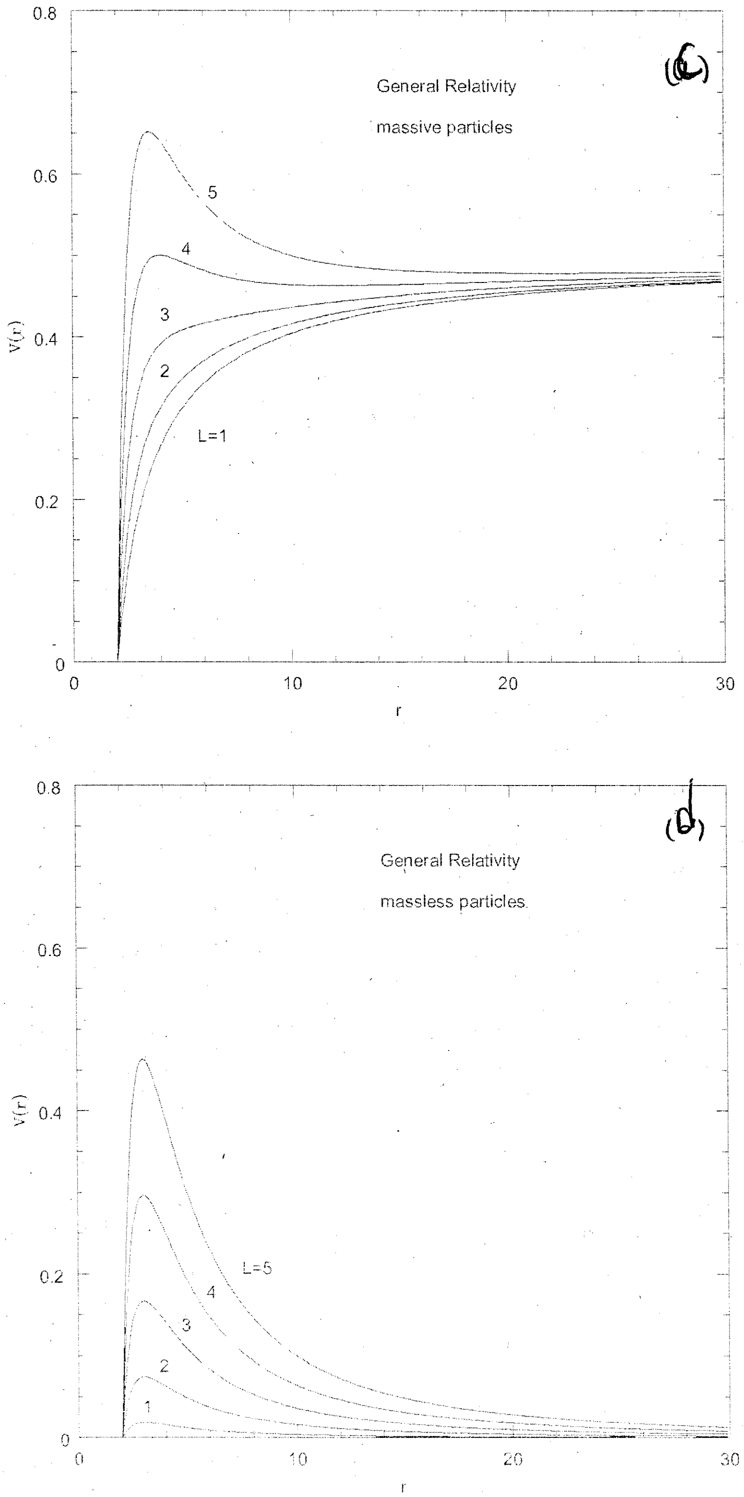
~~So a combined notation~~ $\underline{V(r) = \frac{1}{2}}$

E37a ~~E37b~~


(a)
Newtonian Gravity
massive particles
5
4
3
2
L=1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r


(b)
Newtonian Gravity
massless particles
L=5
4
3
2
1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r

(E37b) ~~E37a~~


(c)
General Relativity
massive particles
5
4
3
2
L=1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r
(d)
General Relativity
massless particles
L=5
4
3
2
1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r

§ Analysis of the motions

We compare the motion of massive and massless test particles in G.R. and Newtonian gravity, hoping to find some detetable effects that can confirm the validity of G.R.

First let us point out some general conclusions that are not specific to G.R. or Newtonian gravity, but to any central force potential

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \text{Constant}$$

(a) If $V(r)$ allows a stable circular motion for a test particle ($\frac{dr}{d\lambda}=0$) they $V(r)$ should have local minima. I.e.,

$$\frac{dV(r)}{dr} = 0 \quad \text{has solutions and} \quad \frac{d^2V(r)}{dr^2} > 0 \quad \text{at that } r.$$

In our case,

$$\frac{dV}{dr} = \frac{KMr^2 - L^2 r + 3ML^2\delta}{r^4} = 0,$$

where $K = \begin{cases} 1 & \text{massive} \\ 0 & \text{massless} \end{cases}$ and $\delta = \begin{cases} 1 & \text{G.R.} \\ 0 & \text{Newtonian} \end{cases}$.

The solution then is

$$r_c = \begin{cases} 3M\delta & K=0, \ \delta=1 \\ \dfrac{L^2 \pm \sqrt{L^4 - 12M^2L^2\delta}}{2M} & K=1. \end{cases}$$

They are potential locations of stable circular motion.

(b) If the potential has a local minimum, then there can exist bound orbits. The bound orbits which are not circular will oscillate around radius of stable circular motion.

(E39)

1. Newtonian potential, massive particle.

- $k=0,\ \delta=0$

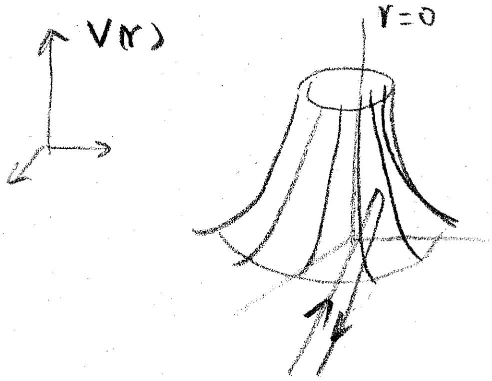
  There is not a solution for r_c.

  no Circuler motion, no bound orbits.

  see Fig (E37a) page, figure (b). $\Longrightarrow$ no minimum.

- stright line in $\mathbb{R}^3$ since in Newtonian gravity, gravitational force is zero because of zero mass.

- ~~Larger~~ smaller L means closer initial shooting direction.



- Imagion a Massless pareicle hitting a potential vocano like this.

↗ Slowing down as approaching, then bounce back, all along straight lines.

↗ If you have higher energy, you can pass straightly.

↗ If you have closer aimming direction, you have an lower version of the potential, i-e., a lower mountain. and you can come closer to $r=0$.

ξ 40

2. Newtonian potential, massive particle

- $k=1$, $\delta=0$, there exist a local minimum

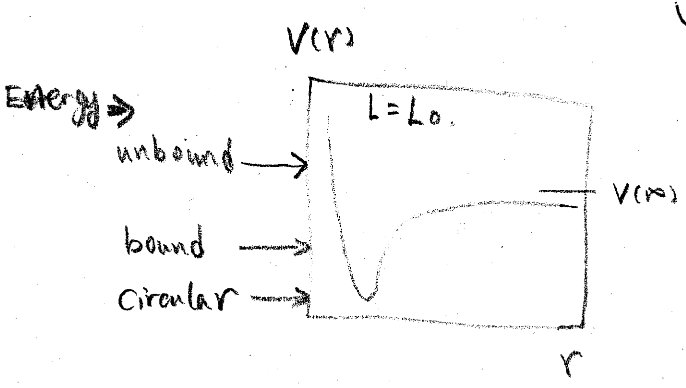
  stable circular orbit at $r_c = \frac{L^2}{M}$

  bound orbit around this radius

- If your energy $E > V(r=\infty)$, you have unbound orbit.

  (From classical mechanics, bound ~~orbits~~: ellipse

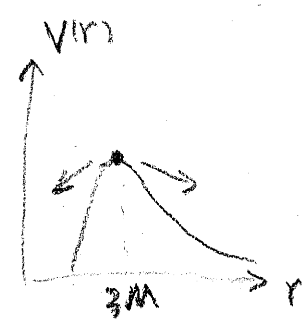
  unbound orbits: para-bola or hyper-bola).



3. G.R geodesic motions

The difference in potentials is the $\sim \frac{1}{r^3}$ term, therefore it will be manifest when r is small.

3. G.R. massless particle



- $k=0$, $\delta=1$.

  $r_c = 3M$ is an maximum

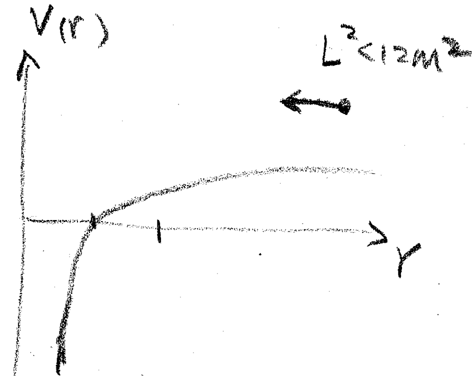
  no stable bound orbit: either fly in or out.

4. G.R massive particle

- $K=1,\ \delta=1$

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$$

- If $L^2 < 12M^2$, r_c is not real.



V(r) has no minimum or maximum

(It do has a V(r)=0 point at $r=2M$.)

From the (K') eq., indeed one can show that for particles with $\frac{dr(\text{initial})}{d\lambda} \leq 0$ and $L^2 < 12M^2$, the particle will fall directly ~~tot~~ the $r=2M$ surface and enter it.

- If $L^2 > 12M^2$, then

$$r_{c-} \equiv \frac{L^2 - \sqrt{L^4 - 12L^2M^2}}{2M}$$ ~~$r_+ =$~~ is a maximum

while ~~the~~ $r_{c+} \equiv \frac{L^2 + \sqrt{L^4 - 12L^2M^2}}{2M}$ is a local minimum.

So stable circular motion is possible at r_{c+}.

Bound orbit exist if ~~$V(\infty) < E < V(r_-)$~~

$$\frac{1}{2}E^2 < V(\infty) = \frac{1}{2}$$

Unband orbit exist if $V(\infty) < \frac{1}{2}E^2 < V(r_-)$.

- Because $L^2 > 12M^2$,

$$r_{c+} > 6M,$$

$$3M < r_{c-} < 6M,$$

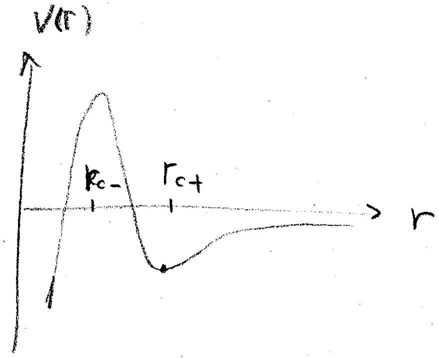
where both $6M$ are obtained when $L^2 = 12M^2$

and the $3M$ for r_{c-} is when $L^2 \to \infty$.

Noting in Newtonian case, $\cancel{r_c = 3M}$ $r_c = \frac{L^2}{M} = r_{c+}(L^2 \to \infty)$. Therefore the $L \to \infty$ limit in G.R. corresponds to the Newtonian limit.

§ Obeservational effects of G.R.

1. precession of perihelia



- ~~At~~ The radial geodesic was

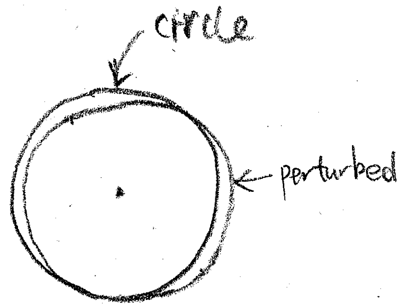
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2.$$

We showed that when $L^2 > 12M^2$,

there exist a r_{c+} for massive particle.

If $\frac{1}{2}E^2 = V(r_{c+})$, then the test particle

will do circular motion.



- when $\frac{1}{2}E^2$ is slightly larger than $V(r_{c+})$,

  the $r(\lambda)$ will oscillate around r_{c+}.

using $\quad r = r_{c+} + \cancel{\delta(\lambda)} \; \delta \sin(\omega\lambda + c)$,

$\frac{1}{2}E^2 = V(r_{c+} + \delta)$,

we solve the frequency of oscillation to be

$$\omega_r^2 = \left.\frac{d^2V}{dr^2}\right|_{r=r_{c+}} = \frac{M(r_{c+} - 6M)}{r_{c+}^3(r_{c+} - 3M)} \qquad \omega_r$$

where we used r_{c+} to substitute L^2.

- The above was all about radial geodesics.

  We had two simple equations (t') and (φ') that we need to solve too.

From (φ') when $r \approx r_{c+}$,

$$\omega_\varphi^2 = (d\phi/d\lambda)^2 = \left(\frac{L}{r^2}\right)^2 = \left(\frac{L}{r_{c+}^2}\right)^2 + O(\delta)$$

$$\omega_\phi^2 = \frac{M}{r_{c+}^2(r_{c+}-3M)} \quad . \qquad (\omega_\phi)$$

(a) In the limit of Newtonian gravity

$$L \to \infty,$$

then $r_{c+} \gg M$.

Apparently $\omega_r^2 \approx \omega_\varphi^2$. ~~Both r and ϕ return to their initial value simutaneously~~

~~Indeed, in Newtonian gravity, not only near circular orbit~~

Both r and ϕ return to their initial values simoutinously.

orbit is closed.

(Ineed in N.G., $\frac{\omega_r}{\omega_\phi} = \frac{n}{m}$ for n, m integers).

(b) In G.R., $\omega_r \neq \omega_\varphi$ cause a precession of angle at which a minimum r is achieved.

These ~~is called~~ minimum r points are called "perihelia".

The precession due to G.R is

$$\omega_p \equiv \omega_\phi - \omega_r$$

$$= \frac{M}{r_{c+}^2(r_{c+}-3M)} - \frac{M(r_{c+}-6M)}{r_{c+}^3(r_{c+}-3M)}$$

$$\Big\downarrow \text{lowest order in } \frac{M}{r_c^+}$$

$$= \frac{3M^{\frac{3}{2}}}{r_{c+}^{\frac{5}{2}}}.$$

It was proven that for elliptical orbit with semi major axis a and eccentricity e, the precession to lowest order is

$$\omega_p \simeq \frac{3M^{\frac{3}{2}}}{(1-e^2)a^{\frac{5}{2}}}.$$

* This result was applied to the precession of Mecury around the Sun. Einstein at 1916 showed that the G.R. explained it well. And this ~~f~~ success was an important factor for the early acceptance of G.R by people.

2. Bending of light ray

In last Section/class, we used the φ' equation for timelike geodesics to slove the frequency of angular motion.

We now look at the φ' equation for null geodesics.

- The K' equ. allows us to solve

$$\dot{r} = \left[E^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{\frac{1}{2}}$$

while φ' can be written as

$$\dot{\phi} = \frac{L}{r^2}$$

combining

$$\frac{d\phi}{dr} = \frac{L}{r^2}\left[E^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-\frac{1}{2}} \qquad (\phi r)$$

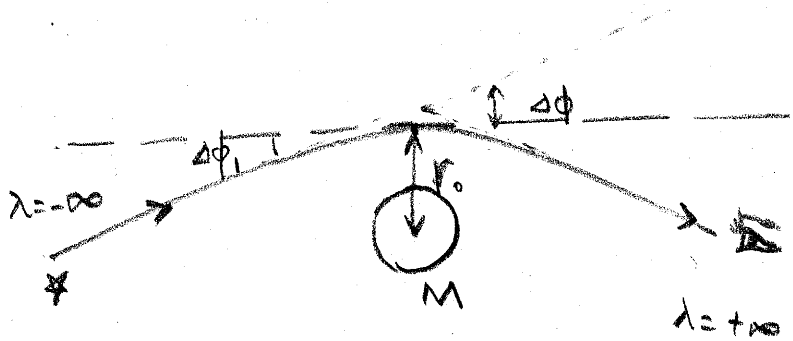
We would like to consider

$$\Delta\phi \equiv \phi(\lambda = +\infty) - \phi(\lambda = -\infty)$$

Graphically:

At $\lambda = \pm\infty$, the trajectories are almost straight lines.

Because at $r \to \infty$, the metric is Newtonian-like.



- For the ~~distance~~ distance $r(\lambda)$, there is a turning point where r stops decreasing and starts increasing.

That is when $\frac{dr}{d\lambda} = 0$.

From (k') eq., that is when

$$V(r) = \frac{1}{2}E^2$$

$$\frac{L^2}{2r^2} - \frac{ML^2}{r^3} = \frac{1}{2}E^2$$

$$r^3 - \left(\frac{L}{E}\right)^2 (r - 2M) = 0$$

The solution, denoted by r_0, is

$$r_0 = \frac{2b}{\sqrt{3}} \cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{3^{\frac{3}{2}}M}{b}\right)\right], \quad b \equiv \frac{L}{E} \qquad (r_0)$$

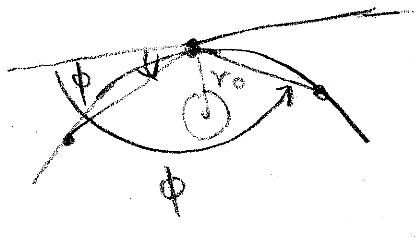
b is called "apparent impact parameter".

- Then the angle of the tangent vector accumulated from $r = \infty$ to $r = r_0$ is given ~~by~~ using eq (ϕr)

$$\Delta\phi_1 = \int_\infty^{r_0} \frac{L}{r^2}\left[E^2 - \frac{L^2}{r^3}(r-2M)\right]^{-\frac{1}{2}} dr$$

$$= \int_\infty^{r_0} \frac{dr}{\left[r^4 b^{-2} - r(r-2M)\right]^{1/2}}$$

while the whole bending angle, by symmetry (or by defining ϕ = angle from baseline, and then $\pi - \phi$ on the other half), is two time $\Delta\phi_1$.



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$$\Delta\phi = |2\Delta\phi_1| = 2\int_0^{\frac{1}{r_0}} \frac{du}{(b^{-2}-u^2+2Mu^3)^{\frac{1}{2}}}$$ after changing of variable $u=\frac{1}{r}$. $(\Delta\phi)$

This depends on two parameters b and M; note $r_0 = r_0(b, M)$.

- This is an integral quantity, the Newtonian limit is in $M=0$

$$\Delta\phi(M\to 0) = 2\int_0^{\frac{1}{b}} \frac{du}{(b^{-2}-u^2)} = \pi$$

$$r_0(M\to 0) = b$$

Completly in agreement that ~~the~~ geodesics go stright in Newtonian gravity.

- In G.R., we would like to see the bending of light passing ~~points~~ with ~~at~~ some known distance from the center, ~~not~~ light with particular apparent impact factor.

Then we replace b^2 in eq. $(\Delta\phi)$ by $b^2 = \frac{r_0^3}{r_0-2M}$ from eq. (r_0).

$$\Delta\phi = 2\int_0^{\frac{1}{r_0}} \frac{du}{(r_0^{-2}-2Mr_0^{-3}-u^2+2Mu^3)^{\frac{1}{2}}}$$

$\downarrow$ small M limit

$$= \Delta\phi(M=0) + \left.\frac{\partial(\Delta\phi)}{\partial M}\right|_{M=0} \cdot M + \mathcal{O}(M^2)$$

$$= \pi + \frac{4}{b}$$

- 1919 Eddington, Arthur measured this bending and find good agreement with G.R..

First major prediction confirmed for G.R.

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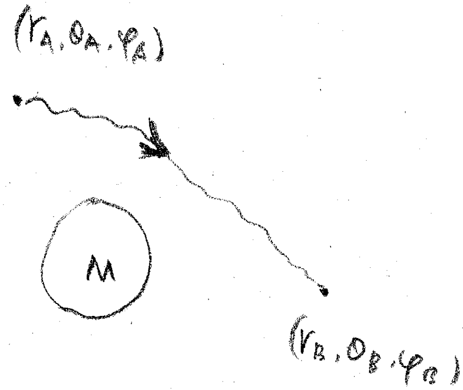
3. Gravitational Redshift

We have been working with the radial and angular geodesic eqs.

Now we also use the t-component of the geodesics to derive another effect of G.R.

- Consider a light ray was emitted at time $t'(\lambda=\lambda_i)$ from point $(r_A, \theta_A, \varphi_A)$ and was received at time $t'(\lambda=\lambda_f)$ by someone at point $(r_B, \theta_B, \varphi_B)$.



So its null geodesic is described by

$$[x'^\mu] = [t'(\lambda),\ r'(\lambda),\ \theta'(\lambda), \varphi'(\lambda)]$$

$$\begin{cases} r'(\lambda_i) = r_A, & r'(\lambda_f) = r_B \\ \theta'(\lambda_i) = \theta_A, & \theta'(\lambda_f) = \theta_B \\ \varphi'(\lambda_i) = \varphi_A, & \varphi'(\lambda_f) = \varphi_B \end{cases}$$

Consider another light ray, emitted at some later coordinate time

$$t' = t(\lambda) + \underset{\uparrow\ \lambda \text{ independent constant}}{\underline{\underline{\Delta t}}},$$

with same initial r, θ, φ and four velocity.

Then from ⓣ, ⓡ, ⓪, ⓟ and ⓚ, it is not hard to see ~~they~~ are also satisfied by

$$[\bar{x}'^\mu] = [\ t'(\lambda) + \Delta t,\ r'(\lambda),\ \theta'(\lambda), \varphi'(\lambda)]$$

~~will describe the also satisfied~~

~~that~~. I.e., the above is the geodesic solution of light ray.

Then at ~~time~~ coordinate time $t'(\lambda_f) + \Delta t$, the second ray will reach $(r(\lambda_f), \theta(\lambda_f), \varphi(\lambda_f))$ which is $(r_B, \theta_B, \varphi_B)$.

That is, the light ray emitted Δt coordinate time after from the same source, will be received Δt coordinate time later by the receiver at the same position.

- The source fixed ~~at~~ at $(r_A, \theta_A, \varphi_A)$ and receiver fixed at $(r_B, \theta_B, \varphi_B)$ usually are massive materials. Therefore ~~they follow~~ their world lines are timelike.

Their tangent vectors should satisfy the normalization condition

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1.$$

Denoting the source's proper time as λ_A, and that of the receiver as λ_B. Their 4-components as ~~x^μ_A and x^μ_B~~ $[x^\mu_A] = (t_A, r_A, \theta_A, \varphi_A)$ and $[x^\mu_B] = (t_B, r_B, \theta_B, \varphi_B)$. Then

$$-\left(1-\frac{2M}{r_A}\right)\left(\frac{dt_A}{d\lambda_A}\right)^2 = -1. \qquad (r_A, \theta_A, \varphi_A) \text{ are fixed.}$$

$$-\left(1-\frac{2M}{r_B}\right)\left(\frac{dt_B}{d\lambda_B}\right) = -1. \qquad (r_B, \theta_B, \varphi_B) \text{ are fixed.}$$

I.e., the proper time and coordinate time ~~for A~~ has relation

for A: $$d\lambda_A = \left(1-\frac{2M}{r_A}\right)^{\frac{1}{2}} dt_A$$

for B: $$d\lambda_B = \left(1-\frac{2M}{r_B}\right)^{\frac{1}{2}} dt_B.$$

Integrating both sides, noting r_A, r_B does not depend on t_A, t_B

$$\Delta\lambda_A = \left(1-\frac{2M}{r_A}\right)^{\frac{1}{2}} \Delta t_A$$

$$\Delta\lambda_B = \left(1-\frac{2M}{r_B}\right)^{\frac{1}{2}} \Delta t_B.$$

Now if ~~two~~ light ray was emitted ~~and received~~ by a time difference Δt_A and received by a coordinate time difference Δt_B,

we showed $\Delta t_A = \Delta t_B$.

Then $$\frac{\Delta \lambda_A}{\Delta \lambda_B} = \left(\frac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}} \right)^{\frac{1}{2}}$$

In terms of frequency, the emitting and receiving frequency would be

$$\frac{\omega_A}{\omega_B} = \frac{\Delta \lambda_B}{\Delta \lambda_A} = \left(\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}} \right)^{\frac{1}{2}}$$

For $r_B > r_A > 2M$ (the usual case: r_A is the surface of a star, r_B is the earth-star distance),

$\cancel{\omega_B > \omega_A}$ $\omega_B < \omega_A$,

or $\lambda_B > \lambda_A$ due to constancy of light speed.

$\Longrightarrow$ A redshift.

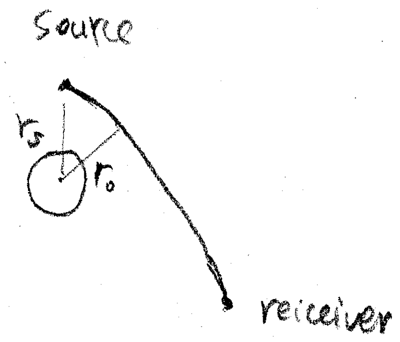
- First conclusive verification: 1959 the Pound-Rebka experiment.

4 Time delay of light signal

Now we try to use the (t') equation of null geodesic to show another measurable effect of G. R. — Time delay of light signal.

- The (t') equation again

$$\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}=E$$



and the (K') equation

$$\frac{dr}{d\lambda}=\left[E^2-\left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}\right]^{\frac{1}{2}}$$

Then, we get

$$\frac{dt}{dr}=\frac{\frac{dt}{d\lambda}}{\frac{dr}{d\lambda}}=\left(1-\frac{2M}{r}\right)^{-1}\left[1-\left(1-\frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-\frac{1}{2}}, \quad b=\frac{L}{E}.$$

Then the cooridnate time taken ~~from~~ for the light from a source at r_s to ~~a receiver at r_r is~~ its bending point at r_0 is

$$\Delta t\Big|_{r_0}^{r_s}=\int_{r_0}^{r_s}\left(1-\frac{2M}{r}\right)^{-1}\left[1-\left(1-\frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-\frac{1}{2}}dr$$

- The result depends on b, which is not easy to measure, therefore we, ~~we, as~~ as before, use $b=r_0\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$ to replace it.

We also work on the first order in M, the result is

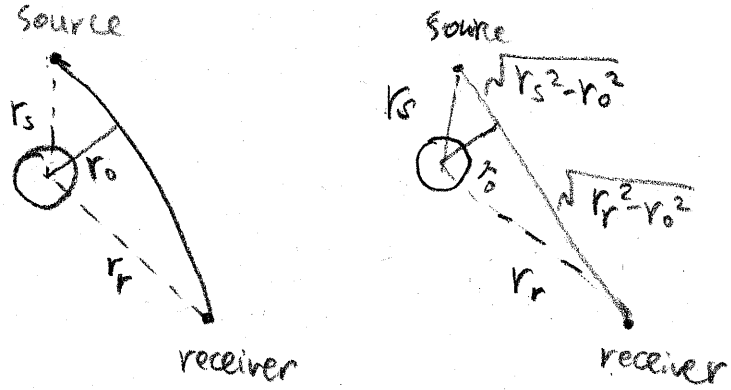
$$\Delta t\Big|_{r_0}^{r_s}=\sqrt{r_s^2-r_0^2}+2M\ln\left(\frac{r_s+\sqrt{r_s^2-r_0^2}}{r_0}\right)+M\left(\frac{r_s-r_0}{r_s+r_0}\right)^{\frac{1}{2}}$$

The time ~~from send a~~ from sending until receiving by an receiver at r_r, is then

$$\Delta t = \Delta t \Big|_{r_0}^{r_s} + \Delta t \Big|_{r_0}^{r_r}$$

$$= \sqrt{r_s^2 - r_0^2} + \sqrt{r_r^2 - r_0^2} + 2M\left[\ln\left(\frac{r_s + \sqrt{r_s^2 - r_0^2}}{r_0}\right) + \ln\left(\frac{r_r + \sqrt{r_r^2 - r_0^2}}{r_0}\right)\right]$$

$$+ M\left[\left(\frac{r_s - r_0}{r_s + r_0}\right)^{\frac{1}{2}} + \left(\frac{r_r - r_0}{r_r - r_0}\right)^{\frac{1}{2}}\right]$$



- The proper time at the receiver radius r_r is related to the proper time by

$$\Delta \lambda = \left(1 - \frac{2M}{r_r}\right)^{\frac{1}{2}} \Delta t$$

$\downarrow$ First order of M

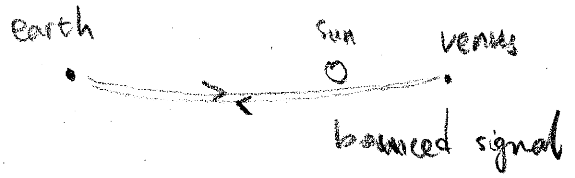
$$\boxed{\cancel{= \Delta t - \frac{2M}{r_r}}}$$

$$= \Delta t - \frac{M}{r}\left(\sqrt{r_s^2 - r_0^2} + \sqrt{r_r^2 - r_0^2}\right).$$

$\Rightarrow$ The $\sqrt{r_s^2 - r_0^2} + \sqrt{r_r^2 - r_0^2}$ is the Newtonian limit result.

$\Rightarrow$ The rest terms are G.R. corrections.

$\Rightarrow$ This effect is first noticed by Irwin. I. Shapiro. First confirmed ~~by~~ in 1966



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§ Cosmology

§1. Homogeneity and Isotropy.

1. Homogeneity: A spacetime is said to be homogenous if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime, and for each t and any points $p, q \in \Sigma_t$, there exits an distance-preserving map which takes p into q.

About the obeservational support of the homogeneity of universe, there are some data but not very conclusive.

Isotropy: A spacetime is said to be isotropic around a point p, if ~~there~~ for two unit spatial tangent vectors in the tangent space of p: $v_1 \in T_p$, $v_2 \in T_p$, there exist an distance-preserving map that can takes v_1 to v_2.

Observational support of isotropy around earth:

Cosmic microwave background first observed 1964 – 1965.

Current CMB temperature

$$T = 2.72548 \pm 0.00057 \text{ Kelvin}$$

- The homogeneity and isotropy implies that the space time of the universe ~~has a metric~~ can be foliated into $R \times \Sigma_t$. Its metric takes the form

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}(u)\,du^i du^j \qquad \text{(U.m 1)}$$

① the "$-dt^2$" term might contain an coefficient fuction of t; but we can always scale t to make it 1.

② that $a(t)$ is called <u>scale factor</u>; γ_{ij} is the metric on Σ_t. $i,j=1,2,3$.

③ the coordinates that make the metric free of $dt\,du^i$ term and spacelike part $du^i du^j$ only depend on a single fuction of $a(t)$, is called "comoving coordinates"

④ an observer has constant u^i is called a "comoving observer".

2. Robertson - Walker metric

- Now let us see what else ~~eff~~ requirement the homogeneity and istropy can put on the metric.

Look at $\underline{{}^{(3)}R_{ij}{}^{kl}}$ at any point p.

$\uparrow$ Riemann tensor calculated from the metric γ_{ij} of the 3-d hyper surface.

For a tensor in the tensor space of rank (0,2), T_{kl}, we have at point p,

$$({}^{(3)}R_{ij})^{kl}\, T_{kl} = T'_{ij}$$

is another element in the (0,2) rank tensor space.

So $({}^{(3)}R_{ij})^{kl}$ can be thought as an linear map from

rank (0,2) tensor space $\longrightarrow$ rank (0,2) ~~tensor~~ space.

In linear algebra, you learned

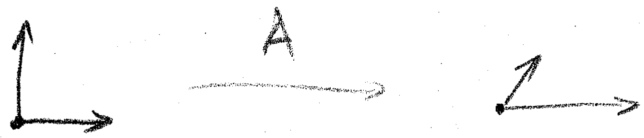
$$A^{ab}\, T_a = T'_b .$$

A^{ab} is a linear transform.

It is eigendecomposible

$$A = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1}$$

If the eigenvalues of A are different, then there exists set of vectors, whose norm will not be scaled uniformly.



Therefore if we want the scaling of norms of all vectors to be the same, then the eigenvalues of A has to be the same.

So for $({}^{(3)}R_{ij})^{kl}$, due to isotropy we want the scaling of norms of all tensors at p to be the same, therefore the eigenvalues of $({}^{(3)}R_{ij})^{kl}$ should be equal.

I.e.,

$$({}^{(3)}R_{ij})^{kl} = Q\,K\,\delta^k_i\,\delta^l_j\,Q^{-1} + Q\,K'\,\delta^k_j\,\delta^l_i\,Q^{-1}$$

$$\Big\downarrow \; ({}^{(3)}R_{ij})^{kl} = -({}^{(3)}R_{ji})^{kl}$$

$$= Q\,K\,\delta^k_{[i}\,\delta^l_{j]}\,Q^{-1}.$$

$$\Big\downarrow \; \text{transformation of coordinates}$$

$$({}^{(3)}R_{ij})^{kl} = K\,\delta^k_{[i}\,\delta^l_{j]}$$

- Lowering the indices by γ_{ij}, we get

$${}^{(3)}R_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

The Ricci tensor becomes

$${}^{(3)}R_{jl} = 2K\gamma_{jl}. \qquad \text{(rm1)}$$

The metric on the hypersurface should also be spherically symmetric ($\Leftarrow$ isotropy + homogeneity). Then we can propose the following form (similar to the derivation of Schwarzschild metric)

$$d\sigma_\Sigma^2 = \gamma_{ij}\,du^i du^j = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta)d\varphi^2$$

I.e., $$[\gamma_{ij}] = \begin{bmatrix} e^{2\beta(r)} & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{bmatrix}$$

The Ricci tensor of this metric is

$$^{(3)}R_{11} = \frac{2}{r}\partial_1\beta$$

$$^{(3)}R_{22} = e^{-2\beta}(r\partial_1\beta - 1) + 1$$

$$^{(3)}R_{33} = \cancel{re^{2}}\ {}^{(3)}R_{22}\sin^2\theta$$

Matching with (rm1), we get

$$\beta(r) = -\frac{1}{2}\ln(1-kr^2), \quad e^{2\beta} = \frac{1}{1-kr^2}$$

Then

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \qquad \text{(m1)}$$

Another change of coordinates

$$r \to \sqrt{|k|}\, r$$

$$a \to \frac{a}{\sqrt{|k|}},$$

$$k \to \frac{k}{|k|}.$$

leaves (m1) invariant. Therefore what matters are the signs of k.

- The Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \qquad \text{(rw)}$$

$k=-1$: constant negative curvature on Σ. The metric/spacetime/universe is called "open".

$k=0$: no curvature on Σ. "flat"

$k=1$: constant positive curvature on Σ. "closed".

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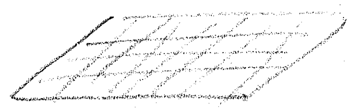
① For $k=0$, the metric on Σ can be converted to

$$d\sigma_\Sigma^2 = dx^2 + dy^2 + dz^2$$

after usual sphereical coordinates to Euclidean coordinates transform.

Therefore, locally the metric describes an flat $\mathbb{R}^3$.



② For $k=1$, we can transform $r = \sin\chi$, then

$$d\sigma_\Sigma^2 = d\chi^2 + \sin^2\chi \, d\Omega^2$$

Globally, the metric describes a 3-sphere.

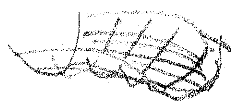
This is not a 2-sphere you can draw, which is embedded in $\mathbb{R}^3$.

It does not have boundary but its size is finite.

③ For $k=-1$, transform $r = \sinh\psi$ brings metric to

$$d\sigma_\Sigma^2 = d\psi^2 + \sinh^2\psi \, d\Omega^2.$$

Space with constant negative curvature. 2-D section e.g.:



Therefore sometimes, the RW metric is also written as

$$ds^2 = -dt^2 + a^2(t) \begin{cases} d\chi^2 + \sin^2\chi \, d\Omega^2 \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi \, d\Omega^2 \end{cases}$$

· Based only on homogeneity and isotropy, we fixed the metric from 10 unknown functions of all spacetime coordinates, to 3 discrete cases with 1 function a(t) of a single coordinates.

$\Longleftarrow$ power of symmetry.

§ classical cosmological models.

1. The E.E

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

★ Work out $G_{\mu\nu}$

From the RW metric (rm), the Ricci tensors

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad \dot{a} = \frac{da}{dt}, \quad \ddot{a} = \frac{d^2a}{dt^2}$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}$$

$$R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$R_{33} = R_{22}\sin^2\theta$$

Ricci scalar $\quad R = \dfrac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

- For the energy-momentum tensor, we assume that the universe is filleded with perfect fluid.

$$T_{\mu\nu} = (p+\rho) U_\mu U_\nu + p g_{\mu\nu}$$

where U_μ is the four-velocity of the fluid.

In a comoving freme, the fluid will be at rest

$$U_\mu = (1, 0, 0, 0)$$

And so $$[T_{\mu\nu}] = \begin{bmatrix} \rho & 0 \; 0 \; 0 \\ 0 & \\ 0 & g_{ij} p \\ 0 & \end{bmatrix}.$$

$$T = T^\mu{}_\mu = -\rho + 3p$$

- The E.E produce

(0.0) component $$-3\frac{\ddot{a}}{a} = 4\pi(\rho + 3p) \quad (00)$$

(1.1)/(2,2)/(3,3) components give only 1 equation

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi(\rho - p) \quad (ii)$$

use (00) in (ii), we finally get two simplified equations

$$\begin{cases} \dfrac{\ddot{a}}{a} = -\dfrac{4\pi}{3}(\rho + 3p) & (f1) \\ \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3}\rho - \dfrac{k}{a^2} & (f2) \end{cases}$$

① They are called "Friedmann equations"

RW metrics that obey these equations describes an "FRW" universe.

② We also define a parameter

$$H \equiv \frac{\dot{a}}{a}$$

Hubble parameter: measures how fast the univese expands at a particular time.

Current measured value $H>0$. $\Rightarrow$ universe expanding.

2. Different fluids.

we finally have 2 Friedmann equations (f1) and (f2).

and three unknowns $a(t)$, $\rho(t)$ and $p(t)$.

We therefore need a equation of state $P = P(\rho)$.

Essentially, the perfect fluid relavant for cosmology obey the simple EOS.

$$P = w\rho \qquad (f3)$$

where w is a constant.

Then we can start to solve them.

$$\frac{d((f2) \times a^2)}{dt} \xrightarrow{\cancel{\text{eliminate } \ddot{a}}} \frac{d(\dot{a}^2)}{dt} = d\left(\frac{8\pi}{3}\rho a^2\right)/dt$$

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}(\dot{\rho} a^2 + 2\rho a\dot{a})$$

using (f1), (f3), $\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$

Solving this we obtain

(ρ) $$\rho = \rho_0 a^{-3(1+w)}$$, ρ_0 is the density when $a(t)=1$.

To further solve $a(t)$ and P, we need to specify particular value for w.

2.1 Dust universe

Collisionless, non relativistic matter. Therefore no pressure.

$$P=0, \quad w=0.$$

- Immediately $\rho = \rho_0 a^{-3}$

This is easy to understand:

the space volume increase as a^3 while the total number ~~number density~~ of the dust ~~do~~ is conserved.

then the number density decrease like a^{-3};

and the energy density $\rho \sim n \cdot m_{each} \cdot c^2 \sim a^{-3}$.

- The (f2) now become

$$\dot{a}^2 - \frac{8\pi}{3}\rho_0 \frac{1}{a} + k = 0 \quad , \quad \frac{8\pi}{3}\rho_0 \equiv C$$

This equation looks quite simple but the result can only be expressed in ~~terms~~ the form of a parameter equation

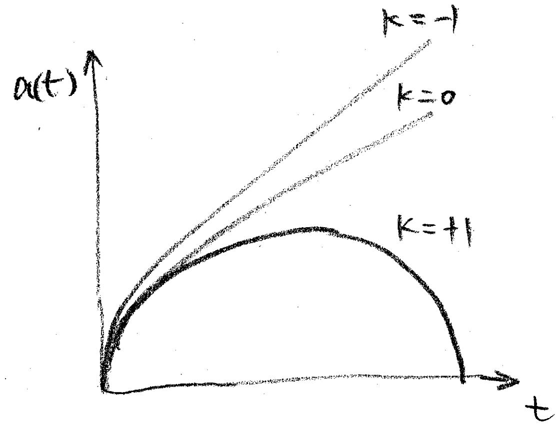
$k=+1$: $a = \frac{1}{2}C(1-\cos\eta)$, where η is related to t by: $t = \frac{1}{2}C(\eta - \sin\eta)$

$k=0$: $a = (\frac{9C}{4})^{\frac{1}{3}} t^{\frac{2}{3}}$

$k=-1$: $a = \frac{1}{2}C(\cosh\eta - 1)$, $t = \frac{1}{2}C(\sinh\eta - \eta)$

Eq. (f1) is automatically satisfied.

•



(A) Since $\dot{a}(\text{now}) > 0$, we are expanding. Left side of the figure.

(B) Depending on k, the universe might

expand forever $\Leftarrow k=-1,\ k=0$

shrink eventually $\Leftarrow k=1$.

(C) In anycase, in the past $a \to 0$.

• The $a \to 0^+$ state dose not mean the size of the universe is zero.

• Rather, the distance between matter is approaching zero.

$\Rightarrow$ Everything is compressed together but the Universe is still expected to be ~~f~~ infinite.

2.2 Radiation Solution

For radiation $p = \frac{1}{3}\rho$, $w = \frac{1}{3}$

- Then immediately from ⓟ

$$\rho = \rho_0 a^{-4}$$

This is not hard to understand either:

The number density goes $n \sim a^{-3}$,

And then the wave length goes like $\lambda \sim a$, $\omega \sim a^{-1}$

Then the energy density become

$$\rho \sim a^{-4}.$$

- The eq. ⓕ2 becomes after the same procedure as in the case of dust

$$\dot{a} - \frac{8\pi}{3}\rho_0 \frac{1}{a^2} + k = 0 , \qquad C \equiv \frac{8\pi}{3}\rho_0 .$$

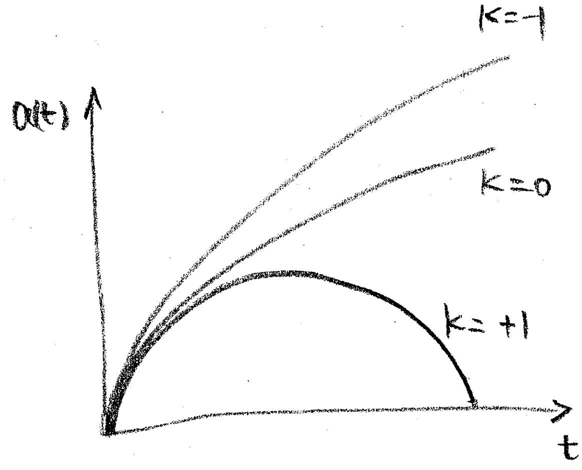
The solution is then given by

$$k = +1 : \quad a = \sqrt{C}\left[1 - \left(1 - \frac{t}{\sqrt{C}}\right)^2\right]^{\frac{1}{2}} ,$$

$$k = 0 : \quad a = (4C)^{\frac{1}{4}} t^{\frac{1}{2}} ,$$

$$k = -1 : \quad a = \sqrt{C}\left[\left(1 + \frac{t}{\sqrt{C}}\right)^2 - 1\right]^{\frac{1}{2}} .$$

Eq. ⓕ1 is automatically satisfied.



Ⓐ Previous statements still hold.

Ⓑ Most importantly, there was also a point of time that $a(t) \to 0^+$!

~~This point turns to be a general feature for~~

2.3 Vacuum energy included

It is also possible that there is an vacuum energy with

$$T^{vac}_{\mu\nu} = -\frac{\Lambda}{8\pi} g_{\mu\nu} ,$$

or $\rho = -P = \frac{\Lambda}{8\pi}$. $w = -1$ in E.os. $P = w\rho$.

The E.E for $T_{\mu\nu}$ including such an vacuum energy should also be modified to

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Here $T_{\mu\nu} = T_{\mu\nu}(\text{other}) + T^{vac}_{\mu\nu}$.

However, we will not further assume other forms of energy-momentum tensor or persuit further solutions. — Beyond ~~the~~ scope of this course.

3. Big bang

The perfect fluids studied in 2 are crude simplified versions.

Reality differs.

- observationally $H = \frac{\dot{a}}{a} \approx 40 - 90$ km/sec/Mpc $\quad$ (Mpc $= 3 \times 10^{24}$ cm)

$$> 0$$

- Without assuming a particular form of ~~$T_{\mu\nu}$~~ E.o.s.,

as long as $\rho > 0$, $P \geq 0$, from (f1), we see that

$$\ddot{a} = -\frac{4\pi}{3}(\rho + 3P)a < 0$$

- then ~~since~~ $\dot{a}(t = \text{past}) > \dot{a}(t = \text{now}) \simeq 40 \sim 90$ km/sec/Mpc $\cdot\, a(\text{now}) > 0$.

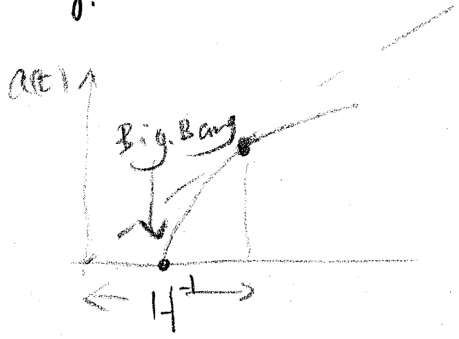
I.e, the universe was accelarating faster.

If it was expanding at ~~the~~ a constant value $\dot{a}_{now}$ then some time

$$T = \frac{a_{now}}{\dot{a}_{now}}$$ ago, $a(t)$ would be zero.

- This means the universe started from a density infinite and Ricci curvature $R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K)$ infinite singularity.

This singularity is called Big Bang.



① Consequently, the age of universe

$$A_{universe} < \frac{a}{\dot{a}} \approx 30 \text{ Billion years}$$

② what is before the big bang?

there are theories on this, but very difficult to check.

③ what is the state/process right after the Big Bang is ~~a~~ intensively studied research area.

④ CMB is a reminant of big bang.

LHC can be used to study some states after Big Bang, QGP.

4 Fate of the universe

The future evolution of the universe is determined by the Friedmann eqs.

From (f2),

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

Using $H = \frac{\dot{a}}{a}$,

$$\frac{k}{a^2H^2} = \frac{8\pi\rho}{3H^2} - 1 .$$

Defining critical density $\rho_{crit} = \frac{3H^2}{8\pi}$, and $\Omega = \frac{\rho}{\rho_{crit}}$, this becomes

$$\frac{k}{a^2H^2} = \Omega - 1 . \qquad \text{(c.d)}$$

Therefore,

$$\rho < \rho_{crit} \Leftrightarrow \Omega < 1 \Leftrightarrow k = -1 \Leftrightarrow \text{"open"},$$

$$\rho = \rho_{crit} \Leftrightarrow \Omega = 1 \Leftrightarrow k = 0 \Leftrightarrow \text{"flat"},$$

$$\rho > \rho_{crit} \Leftrightarrow \Omega > 1 \Leftrightarrow k = +1 \Leftrightarrow \text{"closed"}; \quad \text{(Big crunch).}$$

- Note that eq. (c.d) should hold for all time t, including t_{now}.
  I.e., if we measure H, and then ρ, we can know our future.

- Observationally, we find $\rho \cancel{\frac{3H^2}{8\pi}} \approx \rho_{crit}$ with very small error.
  Therefore the universe is flat.

- Among the entire energy density,

70% → Dark energy   intensive research

25% → Dark matter   intensive research / search

5% → known matter

- 4% interstellar Hydrogen & Helium
- 1%
  - 0.5% stars
  - 0.3% Neutrinos
  - 0.03% Heavy elements including most planet and human beings.
  - ...

§ Effects of G.R. in cosmology.

1. Cosmological redshift

Consider how an comoving observer measure quantities of free falling objects, that is, quantities of geodesic motion.

- For an comoving observer, its four-velocity is

$$U^\mu = (1, 0, 0, 0)$$

Then we can form an tensor

$$K_{\mu\nu} \equiv a^2 (g_{\mu\nu} + U_\mu U_\nu).$$

As one can verify, this satisfies $\nabla_{(\sigma} K_{\mu\nu)} = 0$. (KE)

($K_{\mu\nu}$ is a Killing tensor)

- Now suppose a particle moves along its geodesic. Its four-velocity is

$$V^\mu \equiv \frac{dx^\mu}{d\lambda}.$$

Then we can define

$$K^2 \equiv K_{\mu\nu} V^\mu V^\nu = a^2 [V_\mu V^\mu + (U_\mu V^\mu)^2].$$

- Then I claim that K^2 is a constant of motion along the geodesics.

$$\begin{aligned} \frac{d(K_{\mu\nu} V^\mu V^\nu)}{d\lambda} &= \frac{dx^\alpha}{d\lambda} \nabla_\alpha (K_{\mu\nu} V^\mu V^\nu) \\ &= V^\alpha V^\mu V^\nu \nabla_\alpha K_{\mu\nu} + K_{\mu\nu} V^\alpha (V^\mu \nabla_\alpha V^\nu + V^\nu \nabla_\alpha V^\mu) \\ &= 0 + 0 \end{aligned}$$

where eq. (k-e) and geodesic equation $V^\alpha \nabla_\alpha V^\mu = 0$ are used.

(see also agr.pdf and Wald's book appendix C) for this identity

- Therefore for a time like geodesic $V_\mu V^\mu = -1$,

$$\begin{cases} k^2 = a^2 [-1 + (V^0)^2] \\ (V^0)^2 - g_{ij} V^i V^j = 1 \end{cases}$$

We solve $\vec{V} \equiv \sqrt{g_{ij} V^i V^j} = \dfrac{K}{a}$

That is the particle will slow down w.r.t comoving observers/coordinates as a expands.

This is indeed a true, physically measurable slow down by comoving observers like us.

- For null geodesics, $V_\mu V^\mu = 0$,

we have $-U_\mu V^\mu = +\dfrac{K}{a}$.

(page 149 Sean's book)

(see also notes page 85)

Recall that $-U_\mu V^\mu$ is the energy of the light ray obsewed by the comoving observer, and $E = \hbar\omega$ for light rays, we have

$$\omega = -U_\mu V^\mu = \frac{K}{a}.$$

At two different instants t_1 and t_2, then

$$\frac{\omega_1}{\omega_2} = \frac{a_2}{a_1}$$

The wavelength then ~~follow~~ ~~will~~ follows

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1}{a_2}.$$

There will be a redshift as the universe expands, as obsened by comoving observers.

Usually, we measure the amount of redshift by

$$z \equiv \frac{\lambda_1 - \lambda_2}{\lambda_2}$$

$$= \frac{\lambda_1}{\lambda_2} - 1 \qquad \text{where } \lambda_1 \text{ is in the future of } \lambda_2.$$

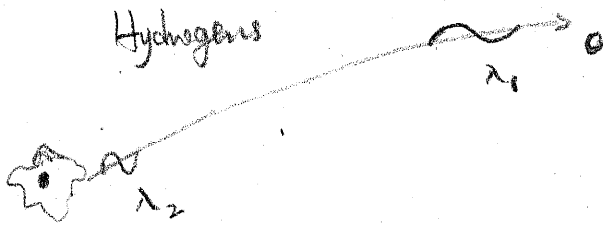
- z is a very useful quantity in observational cosmology.

E.g. we can obsene the hydrogen absorbtion line wavelength redshift.

Say from $n=2 \rightarrow n=3$, $\lambda_2 = 656.3\,nm$, $\lambda_1 > \lambda_2$,

we can identify how much the universe has expended from a partiular event,

such as supernova.



2. cosmological horizon / particle horizon

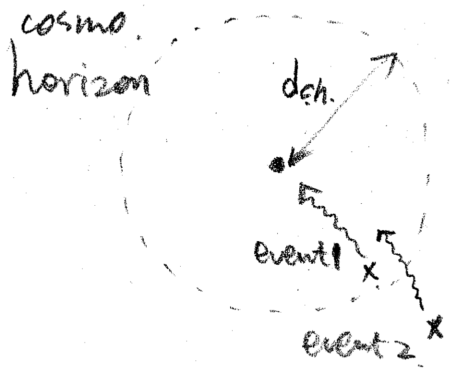
- Since the universe has a finite age, and the light speed is the maximum speed that particles can travel and is also finite,

  there exist a boundary in space beyond which the light signal did not have enough time to travel to us, even if it started right after the big bang.

  ↳ or any comoving observer.

  This boundary is called cosmological horizon,

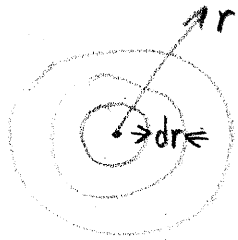
  or particle horizon.



- Let us define a quantity to characterize the size of this horizon.

  we define a "proper distance" to be the distance measured on a slice of constant time. At time t, the proper distance ds between two points with coordinate distance dr, (same θ and φ) is defined through

$$ds^2 = 0 + a^2(t)\frac{dr^2}{1-kr^2} + 0 + 0 \qquad time = t$$



$$ds = \frac{a(t)}{\sqrt{1-kr^2}}\, dr$$

The horizon proper distance ~~is then~~ at time t is then

$$d_{c.h.} \equiv \int ds = a(t) \int_{r=0}^{r=r(\text{horizon})} \frac{1}{\sqrt{1-kr^2}}\, dr$$

Now for the integral part, we know that light travels along null geodesics

$$ds^2 = 0 = -dt^2 + \frac{a^2(t)}{1-kr^2}\,dr^2 + 0 + 0$$

$$\frac{dt}{a(t)} = \frac{1}{\sqrt{1-kr^2}}\,dr$$

This produces

$$d_{c.h.}(t) = a(t)\int_{t'=0^+}^{t'=t} \frac{dt'}{a(t')}$$

- Therefore for different models of cosmology, dust, radiation etc., we can always compute a $d_{c.h.}$.

In particular, for any model we studied, $\boxed{\cancel{a(t') \underset{t'\to 0}{\propto} t'^{\alpha},\quad 0<\alpha<1}}$

we can check that

$$\lim_{t'\to 0} a(t') \to t'^{\alpha}, \quad 0<\alpha<1.$$

therefore $d_{c.h}$ is <u>finite</u>

- observationally, $d_{ch}(\text{now}) \approx 13.7$ giga parsec. (from CMB)

≈ 45 Billion light years.

- There exist other kinds of horizons in some cosmological models.

E.g., cosmological event horizon. Don't mix.

§ Black Holes (B.H)

§ Schwarzschild B.H.

1. Causal structure at $r > 2M$

The S.M. is given by

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2$$

- We have shown that the $r=2M$ surface is not singular:

  Curvatures are finite; and we also found explicit coordinate transforms.

- However, ~~there~~ $r=2M$ still turns out to be an very interesting surface.

Consider a radial null geodesic, satisfying

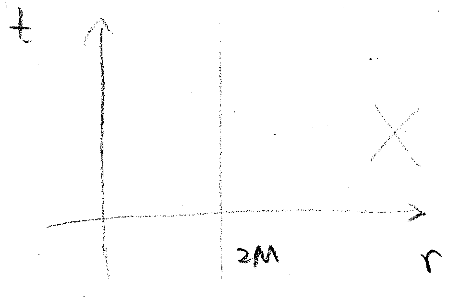
$$ds^2 = 0 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 \cancel{\Omega}$$

That is, $\frac{dt}{dr} = \pm(1-\frac{2M}{r})^{-1}$, or ~~(rg1)~~ $\frac{dr}{dt} = \pm(1-\frac{2M}{r})$. (rg1)

The "−" sign one is called in-going, while "+" is called out-coming.

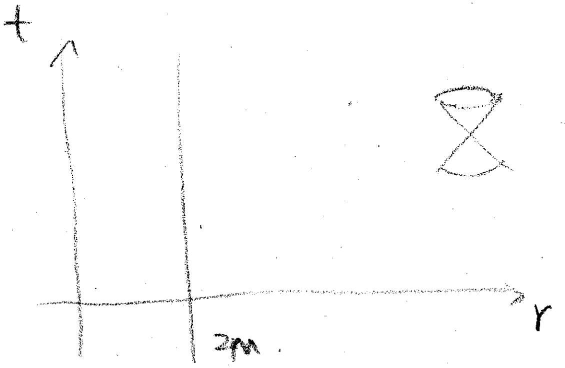
Eq. (rg1) ~~measures~~ gives the boundary of light cone.

At large r, $\left|\frac{dr}{dt}\right| \approx 1$.



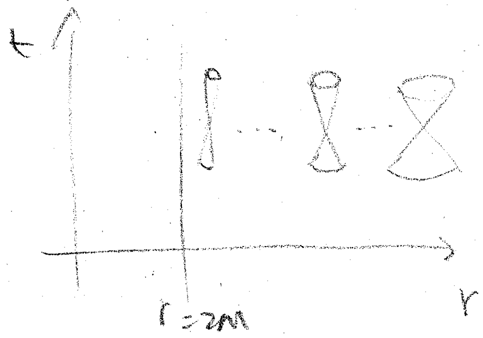
Since timelike particles travels slower, that is $|\frac{dr}{dt}| < \left|\frac{dr}{dt}\Big|_{null}\right|$,

the cone close like



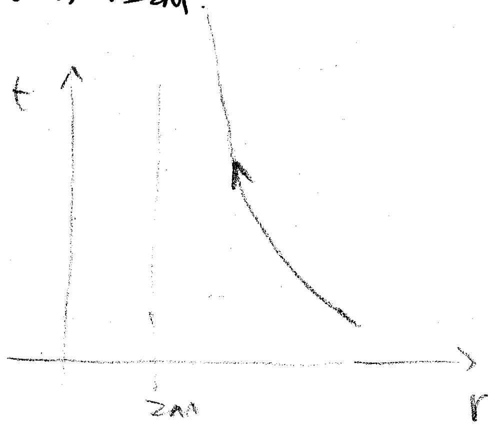
(Draw on the last picture).

With smaller r, the cone closes up, until $r = 2M^+$, where the cone completely closes up.



(Continue drawing on the same picture)

- It might seem that it will take infinite long t for the light ray to reach 2M, and the light ray might never cross r = 2M.

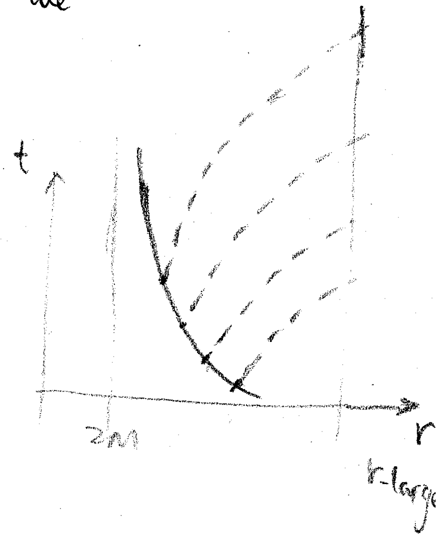


But indeed, infinite t does not mean infinite proper time for the traveler.

~~At Rec~~ t (or δt) is only a good measurement of the propertime at $r = \infty$ where metric is Minkowski.

- Infinite t to reach $r=2M$ only means in the point of view of an ~~ays~~ asymptotic observer at $r=\infty$, the light ray / particle will never reach $r=2M$ in finite time.

- If the traveler sends back a light signal at its own fixed frequency, from our studying of the red-shift, we know that the frequency at larger r is indeed smaller



$$\omega_{r\text{-large}} = \left(\frac{1-2M/r_{small}}{1-2M/r_{large}} \right)^{\frac{1}{2}} \omega_{r\text{-small}} .$$

I.e., when $\omega_{r\text{-small}}$ and r_{large} are fixed,

$\omega_{r\text{-large}}$ approaches infinity as r approaches $2M$.

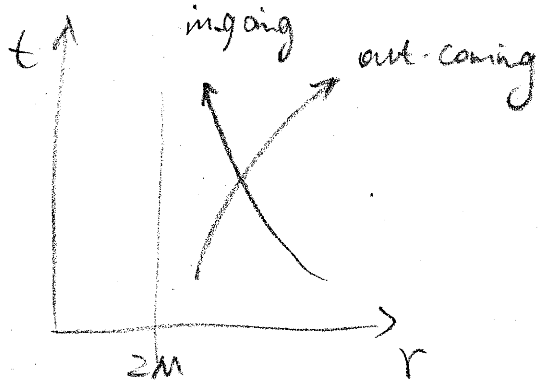
- Indeed, for the proper time of the traveler (or affine parameter of the light ray), it only take finite amount of time to reach and then cross the $r=2M$ surface.

To see this, we need to find a better coordinate system to describe the spacetime around $r=2M$.

2. Coordinate transforms and causal structure around $r \lesssim 2M$

We first notice that the r.g.l

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}$$



is solvable to get

$$t = \pm r^* + \text{constant},$$

where $r^*(r) = \int \frac{1}{1-2M/r} dr = r + 2M \ln\left|\frac{r}{2M} - 1\right|$. (r.s.d.)

I.e., the out-coming light ray move along curve $t - r^* = \text{constant}$

" " in-going " " " " $t + r^* = \text{constant}$.

- This motivate us to try new coordinates defined by

$$u = t + r^*, \quad \text{(u.d)}$$
$$v = t - r^*, \quad \text{(v.d)}$$

where r^* is a function of r.

It is easy to work out relation from $(dt, dr) \to (du, dv)$.

$$\begin{cases} du = dt + \frac{1}{1-2M/r} dr \\ dv = dt - \frac{1}{1-2M/r} dr \end{cases}$$

The S.M. becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega^2 \quad \text{(b.m.)}$$

where r should be thought as a function of u, v, obtainable by inversing $r^*(r) = \frac{1}{2}(u-v)$ using (r.s.d).

The null geodesics are $u = \text{constant}$ or $v = \text{constant}$.

This metric is manifestly non-singular at $r = 2M$. However, the $r=2M$ surface, from (u.d) and (v.d) (r.s.d.) is at ~~$|u| = \infty, |v| = \infty$~~

$u - v = -\infty$.

- we can try to make a half transform too:

$$(t, r) \longrightarrow (u, r)$$

where u is given by (u.d.)

The metric (S.m.) simply becomes

$$ds^2 = -(1-\frac{2M}{r})\,du^2 + (du\,dr + dr\,du) + r^2 d\Omega^2$$

There is no singularity at $r=2M$, which is also finite of the coordinate.

This coordinate system is called Eddington - Finkelstein coordinates.

The null geodesics along radial direction can be solved

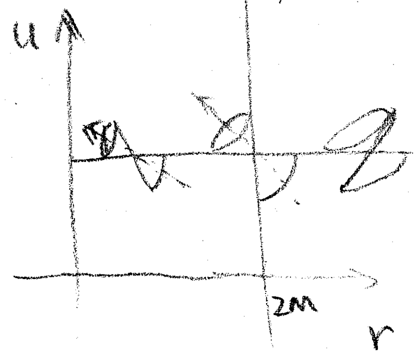
$$ds^2 = 0$$

$$\Rightarrow \frac{du}{dr} = \begin{cases} 0 & \text{"ingoing"} \\ 2(1-\frac{2M}{r})^{-1} & \text{"out coming"} \end{cases}$$

① Now its clear that when $r<2M$,

all future-directed null geodesics, $du>0$, will have $dr<0$,

a dereasing r.



② No problem for both null and timelike geodesics

to pass the $r=2M$ surface.

~~③ A~~

Therefore the $r=2M$ surface is locally perfectly regular,

globally fuction as a surface of no return.

Once a particle pass the $r=2M$ surface, it can never come out.

The $r=2M$ is given the name "event horizon"

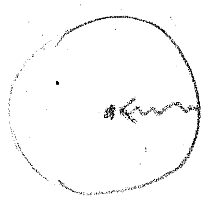
— any event happens inside that horizon can never be observed by an outsider in any long time.

The region bounded by $r=2M$ is then called a "Black hole".

~~Indeed for the S.M, and the when $r<2M$,~~

~~$g_{tt}>0$, $g_{rr}<0$, $g_{\theta\theta}$, $g_{\varphi\varphi}>0$.~~

~~Therefore by our convention~~



• Indeed from the definition of proper time,

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\Omega^2 ,$$

the maximum amount of proper time it takes for a timelike geodesic to travel from $r=2M$ to $r=0$ is when $dt^2=0=d\Omega^2$.

$$\Delta\tau_{max} = \int d\tau = \int_{2M}^{0} -\frac{1}{\sqrt{\frac{2M}{r}-1}}\,dr = \pi M$$

This is the proper time that you will take when you energy is 0^+ right at the surface $r=2M$ after you enter the horizon.

3. Kuskal coordinates

- We can try to scale the $u-v=-\infty$ surface ($r=2M$ surface) to somewhere finite.

To do this, study the behavior of the metric near $r=2M$.

From (r.s.d),

$$r^* \simeq 2M \ln |r/2M - 1|$$

$$\Rightarrow \quad r/2M \simeq 1 \pm e^{r^*/2M} = 1 \pm e^{(u-v)/4M}$$

$$\Rightarrow \quad 1 - \frac{2M}{r} \simeq \pm e^{(u-v)/4M}$$

where "+" sign corresponds to $r > 2M$.

"−" " " " $r < 2M$.

Then the metric (b.m.) around $r=2M$ becomes

$$ds^2 \simeq \mp (e^{u/4M} du)(e^{-v/4M} dv) + r^2 d\Omega^2 \qquad \begin{cases} \text{"−"} & r \gtrsim 2M^+ \\ \text{"+"} & r \lesssim 2M^- \end{cases}$$

This motivates us to introduce another transform

$$V = \mp e^{-v/4M}, \quad U = e^{u/4M} \qquad \begin{cases} \text{"−"} & r > 2M \\ \text{"+"} & r < 2M \end{cases}$$

then $dV = \pm \frac{1}{4M} V dv$, $\quad dU = \frac{1}{4M} U du$

$$UV = \mp e^{(u-v)/4M} = \mp e^{r^*/2M} = -e^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right), \qquad \text{(UVp)}$$

The metric (b.m.) for all r becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\left(\frac{4M}{U}\right)\cdot\left(\pm\frac{4M}{V}\right) dU dV + r^2 d\Omega$$

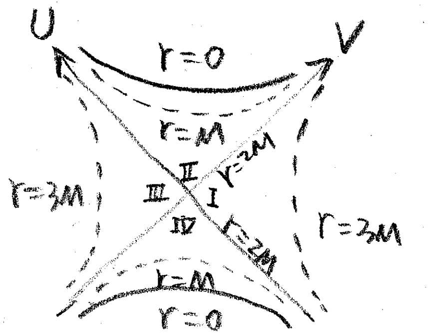
$$= -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega$$

The coordinates U and V are called <u>null Kruskal coordinates</u>.

- Kruskal diagram:

  A diagram obtained by drawing constant r and t lines on a U, V grid.

  We tilt the grid s.t. null geodesics are 45°.



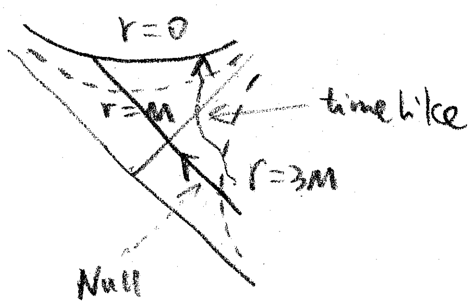
① From (UVP), constant r curves are constant UV curves.

② $r=2M$ are $U=0$, $V=0$ axis.

③ Now there are two copies for each $r=$ constant.
Therefore the Kruskal coordinates reveals a large portion of manifold than covered by the original Schwarzschild coordinates.

④ The original Schwarzschild coord. covers region I & II.
It is seen the hiting to $r=0$ after $r<2M$ is unavoidable.

- One can also do a rotation



$$\tilde{V} = (U+V)/2,$$
$$\tilde{U} = (U-V)/2.$$

The metric is then

$$ds^2 = \frac{32M^2}{r} e^{-r/2M} (d\tilde{U}^2 - d\tilde{V}^2) + r^2 d\Omega^2.$$

The $\tilde{U}, \tilde{V}$ are called Kruskal-Szekeres coordinates. $r = r(\tilde{U}, \tilde{V})$.

In this system, the relation (t, r) and $(\tilde{U}, \tilde{V})$ are worked out:

$$\left.\begin{matrix}\tilde{U}\\ \tilde{V}\end{matrix}\right\} = \pm \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \left\{\begin{matrix}\cosh(\frac{t}{4M})\\ \sinh(\frac{t}{4M})\end{matrix}\right. \qquad r > 2M$$

$$\left.\begin{matrix}\tilde{U}\\ \tilde{V}\end{matrix}\right\} = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \left\{\begin{matrix}\sinh(\frac{t}{4M})\\ \cosh(\frac{t}{4M})\end{matrix}\right. \qquad r < 2M.$$

§ Reissner - Nordström BH.

Reissner 1916

Nordstrom 1918.

1. The metric.

- Consider a sphereically symmetric metric

$$ds^2 = -e^{2\psi(t,r)} f dt^2 + f^{-1} dr^2 + r^2 d\Omega$$

where $f = f(t,r)$, $\psi = \psi(t,r)$. This is the most general form of a sphereically symmetric spacetime.

Consider a electro-magnetic field in this spacetime. Its field strength $F^{\alpha\beta}$ has no θ or φ direction components. This ensures it's purely electric when measured by stationary observers.

$$[F^{\alpha\beta}] = \begin{bmatrix} 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Maxwell equation in vacuum

$$\nabla_\beta F^{\alpha\beta} = 0 = |g|^{-\frac{1}{2}} \partial_\beta (|g|^{\frac{1}{2}} F^{\alpha\beta})$$

That is, $\partial_r(e^{\psi} r^2 F^{tr}) = 0$, $\partial_t(e^{\psi} r^2 F^{rt}) = 0$,

Solution then $F^{tr} = e^{-\psi} \frac{Q}{r^2}$, Q being an integral constant.

The energy momentum ~~tensor~~ ~~becomes~~

$$T^\alpha{}_\beta \equiv \frac{1}{4\pi} (F^{\alpha\mu} F_{\beta\mu} - \frac{1}{4} \delta^\alpha_\beta F^{\mu\nu} F_{\mu\nu})$$

becomes $[T^\alpha{}_\beta] = \frac{Q^2}{4\pi r^4} \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

- The E.E from (g.m.) becomes

$$\frac{\partial m(r,t)}{\partial r} = 4\pi r^2(-T^t_{\ t}), \qquad \text{where} \quad f(r,t) = 1 - \frac{2m(r,t)}{r}.$$

$$\frac{\partial m(r,t)}{\partial t} = -4\pi r^2(-T^r_{\ t}),$$

$$\frac{\partial \psi(r,t)}{\partial r} = 4\pi r f^{-1}(-T^t_{\ t} + T^r_{\ t}).$$

Substituting $T^\alpha_{\ \beta}$,
$$\begin{cases} \dfrac{\partial m}{\partial r} = \dfrac{Q^2}{2r^2} \\ \dfrac{\partial m}{\partial t} = 0 \end{cases}$$

we get $m(r) = M - \dfrac{Q^2}{2r}$, M being the integrational constant.

- The metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2. \quad \text{(RN)}$$

One can show that Q is the total electric charge in the spacetime ~~true~~ (we will not do this though).

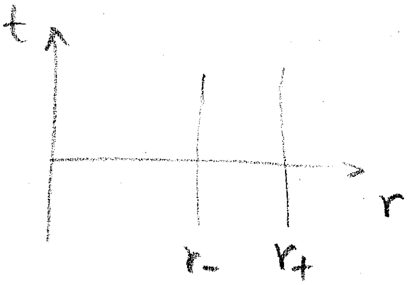
2. B.H.

The metric components have coordinate singularities at
$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$$

or
$$\begin{cases} r_\pm = M \pm \sqrt{M^2 - Q^2}. & |M| \geq |Q| \\ r = 0 & |M| < |Q|. \end{cases}$$

- There are two surfaces, $r = r_+$ and $r = r_-$, $r_+ > r_-$. when $|M| > |Q|$.

  when $|M| = |Q|$, $r_+ \Leftrightarrow r_-$.



  We can show that the r_+ is an event horizon and therefore metric (rnm) contains a B.H, called Reissner-Nordström BH.

  when $Q = M$, this BH is called an extreme R.N B.H.

- The $r = r_-$ surface is not an event horizon, but an apparent horizon. (which we ~~do not hav~~ have not teach the proper background to understand this).

- The $r = 0$ point is a time-like singularity.

  That is, travelers do not have to hit this point. They are free to avoid encounter it if they choose so.

§ Kerr B.H. and Kerr-Newnan B.H. (G.12)

* For completeness, we also give the last two of the four ~~of three~~ most well known BHs.

1. The first ~~this~~ is called a Kerr B.H. (discovered by Roy Kerr in 1963).

Its metric is

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dt\,d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

(km)

where $\rho^2 = r^2 + a^2\cos^2\theta$,

$\Delta = r^2 - 2Mr + a^2$,

$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$.

This metric describes an stationary and axially symmetric spacetime.

M is the total mass and $L = aM$ is the angular momentum.

So a is the ratio of angular momentum and mass.

The metric describe ~~an rotat~~ the spacetime of an rotating body.

- One can show that it also has an event horizon when $M \geq a$.

This is obtained from g_{rr} component, when $\Delta = 0$, we get

$$r = r_\pm = M \pm \sqrt{M^2 - a^2}.$$

Therefore (K.m.) also describes a black hole when $M \geq a$.

- The other coordinate singularity is from ~~dt dφ~~ $g_{t\phi}$,

when $\rho = 0$, we get

$$\begin{cases} r = 0 \\ \text{and } \theta = \frac{\pi}{2} \end{cases}$$

These points can be shown to be true curvature singularities.

2. Kerr-Newman B.H. (Newman 1965)

The Kerr-Newman metric

$$ds^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + (dt - a\sin^2\theta\, d\phi)^2 \frac{\Delta}{\rho^2} - ((r^2+a^2)d\phi - a\,dt)^2 \frac{\sin^2\theta}{\rho^2}$$

where $a = \frac{J}{M}$, (knm)

$\rho^2 = r^2 + a^2\cos^2\theta$,

$\Delta = r^2 - 2Mr + a^2 + Q^2$.

Here M is the total mass, J is angular momentum while Q is charge.

when $M^2 > a^2 + Q^2$, the $\Delta = 0$ gives coordinate singularities

$$r = r_\pm = M \pm \sqrt{M^2 - (a^2+Q^2)}.$$

One can show that $r = r_+$ is an event horizon.

Therefore ~~that~~ the (knm) also can describe a B.H. called Kerr-Newman B.H.

§ The Lagrangian formalism of general relativity.

This part will only be considered when a 54-class course is taught.