插值法

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本章研究对象及目标

设$x_0, x_1, \cdots, x_n \in [a, b]$是$n+1$个相异节点，且有$y_i = f(x_i)$, $i = 0, 1, \ldots, n$, 构造插值函数φ满足

$$\varphi(x_i) = f(x_i),\ i = 0, 1, \ldots, n \tag{1}$$

称

- φ为插值函数
- $x_0, x_1, \cdots, x_n \in [a, b]$为插值节点
- 公式(1)为插值条件
- 在本章中，φ为代数多项式
- $\mathcal{P}_n$为次数不超n的全体多项式的集合。

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Lagrange插值

记

$$\ell_i(x) = \Pi_{j=0,j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)} \Rightarrow l_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} i = 0, 1, \ldots, n$$

Lagrange插值多项式

$$L_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

显然，$L_n(x_j) = f(x_j)$ 且有 $L_n(x) \in \mathcal{P}_n$

定理

设$p \in \mathcal{P}_n$, p的零点数大于n(p的ℓ重零点x^*计为ℓ个零点)，那么$p \equiv 0$

归纳法。设$p \in \mathcal{P}_{n+1}$且假设$p(a) = 0$, 则有

$$p(x) = (x-a)\tilde{p}(x),\ \tilde{p}(x) \in \mathcal{P}_n$$

且$\tilde{p}(x)$的零点个数大于或等于n

定理

给定$n+1$个相异节点$x_0, x_1, \ldots, x_n \in [a, b]$及$n+1$个函数值$f(x_0)$, $f(x_1)$, $\cdots$, $f(x_n)$, 存在唯一的n次多项式$p \in \mathcal{P}_n$, 使得

$$p(x_i) = f(x_i),\ \forall i = 0, 1, \ldots, n$$

插值余项及其估计

余项

$$R_n(x) = f(x) - L_n(x),\ x \in [a,b]$$

定理

在定理2的条件下，对任意的$x \in [a,b]$，存在$\xi = \xi(x) \in (a,b)$，使得

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} w_{n+1}(x)$$

其中$w_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n) \in \mathcal{P}_{n+1}$

设$G(t) = R_n(t) - \frac{w_{n+1}(t)}{w_{n+1}(x)} R_n(x)$，则$G$在$[a,b]$上有$n+2$个零点，重复对$G(t)$使用Rolle中值定理。

推论：若$\max_{a\le x\le b}|f^{(n+1)}(x)|\le M$，则有

$$|R_n(x)|\le \frac{M}{(n+1)!}|w_{n+1}(x)|$$

推论：设$a=x_0<x_1<\cdots<x_n=b$，$h=\max_{1\le j\le n}(x_j-x_{j-1})$，$f\in C^{n+1}[a,b]$，$L_n$为$f$的$n$次插值多项式，那么有

$$\|f-L_n\|_\infty\le \frac{h^{n+1}}{4(n+1)}\|f^{(n+1)}\|_\infty$$

证明：不妨设$x\in[x_k,x_{k+1}]$，则有$|(x-x_k)(x-x_{k+1})|\le \frac{1}{4}h^2$，

$$\begin{matrix}|x-x_{k+2}|\le 2h,\cdots,|x-x_n|\le (n-k)h\\ |x-x_{k-1}|\le 2h,\cdots,|x-x_0|\le (k+1)h\end{matrix}\Rightarrow |w_{n+1}(x)|\le \frac{n!}{4}h^{n+1}$$

例： 设$x_0, x_1, \ldots, x_n$为相异节点，
$\ell_i(i = 0, 1, \cdots, n)$为$n+1$个$n$次Lagrange插值基函数，试证明

$$\sum_{i=0}^{n} \ell_i(x) x_i^k = x^k, k = 0, 1, \cdots, n \tag{2}$$

重要公式： 在公式(2)中，令$k = 0$

$$\sum_{i=0}^{n} \ell_i(x) = 1$$

线性插值与二次插值

线性插值:

$$\ell_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad \ell_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\Rightarrow \quad L_1(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x)$$

$$\Rightarrow \quad R_1(x) = \frac{1}{2} f''(\xi(x))(x - x_0)(x - x_1)$$

二次插值

$$\ell_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \; \ell_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

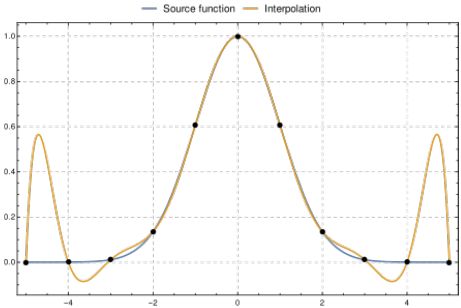
$$\ell_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$\Rightarrow \quad L_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$$

$$\Rightarrow \quad R_2(x) = \frac{1}{6} f^{(3)}(\xi(x))(x - x_0)(x - x_1)(x - x_2)$$

龙格现象

例： 设$f(x) = \frac{1}{1+x^2}$, $x \in [-5, 5]$. 在区间$[-5, 5]$上给出等距节点$x_j - 5 + jh$, $j = 0, 1, \ldots, 10$，作10次Lagrange插值多项式。



当$|x| > 3.63$时，误差很大。

Lagrange插值一致收敛到f的充分条件是任意阶导数一致有界。实际上，该条件往往难于达到。

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均差

$$f[x_k] = f(x_k) \qquad (\text{零阶})$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \qquad (\text{一阶})$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k} \qquad (\text{二阶})$$

$$f[x_k, \ldots, x_{k+j}] = \frac{f[x_{k+1}, \ldots, x_{k+j}] - f[x_k, \ldots, x_{k+j-1}]}{x_{k+j} - x_k} \qquad (\text{j阶})$$

通过画表法可以逐层计算均差。

性质

- k阶均差是$f(x_0), f(x_1), \cdots, f(x_k)$的线性组合且

$$f[x_0, \ldots, x_k] = \sum_{j=1}^{k} \frac{f(x_j)}{\Pi_{i \neq j}(x_j - x_i)}$$

- 对称性。交换均差中的结点位置，均差不变。
- 如果$f[x, x_0, x_1, \ldots, x_k] \in \mathcal{P}_m$，那么$f[x, x_0, \ldots, x_{k+1}] \in \mathcal{P}_{m-1}$
  推论：设$f \in \mathcal{P}_n$，则$f[x, x_0, \ldots, x_n]$恒为零。
- 设$f \in C^n[a, b]$，$x_j \in [a, b]$，$j = 0, 1, \ldots, n$为相异节点，则有

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi), \quad \xi \in (a, b)$$

$$f[x,x_0,x_1]=\frac{f[x_0,x_1]-f[x,x_0]}{x_1-x},f[x,x_0]=\frac{f(x_0)-f(x)}{x_0-x}$$

$$\Rightarrow f(x)=\underbrace{f(x_0)+(x-x_0)f[x_0,x_1]}_{N_1(x)\text{:一次Newton插值多项式}}+(x-x_0)(x-x_1)f[x_0,x_1,x]$$

$N_1(x)$满足：

$$N_1(x_0)=f(x_0),\quad N_1(x_1)=f(x_1)$$

n次Newton插值多项式：

$$N_n(x)=f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$$

且有

$$f(x)=N_n(x)+f[x,x_0,\cdots,x_n](x-x_0)\cdots(x-x_n)$$

可以证明：

$$N_n(x_i) = f(x_i),\ i = 0, 1, \ldots, n$$

从而

$$N_n(x) = L_n(x) \quad \text{(n次Lagrange插值)}$$

则有插值余项：

$$f[x, x_0, \cdots, x_n](x - x_0) \cdots (x - x_n) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} w_{n+1}(x)$$

等距节点

当$x_k = x_0 + kh$时,$x_0, x_1, \cdots, x_n$构成等距节点，可以定义一阶差分

$$\Delta f(x_k) = f(x_{k+1}) - f(x_k) \qquad \text{(向前差分)}$$
$$\nabla f(x_k) = f(x_k) - f(x_{k-1}) \qquad \text{(向后差分)}$$

对于任意m阶差分，有如下定义：

$$\Delta^m f(x_k) = \Delta^{m-1} f(x_{k+1}) - \Delta^{m-1} f(x_k)$$
$$\nabla^m f(x_k) = \nabla^{m-1} f(x_k) - \nabla f^{m-1} f(x_{k-1})$$

则有

$$\begin{aligned} &f[x_0, x_1] = \frac{1}{h}\Delta f(x_0), \quad f[x_0, x_1, x_2] = \frac{1}{2h^2}\Delta^2 f(x_0) \\ &f[x_0, \cdots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0) \quad \text{(可以类似地用向后差分定义)} \end{aligned} \tag{3}$$

利用公式(3),可知对任意$x = x_0 + th$，有

$$
\begin{aligned}
N_n(x) = N_n(x_0+th) =& f(x_0) + t\Delta f(x_0) + \frac{1}{2!}t(t-1)\Delta^2 f(x_0) + \cdots + \\
& \frac{1}{n!}t(t-1)\cdots(t-n+1)\Delta^n f(x_0) \\
=& \sum_{k=0}^{n} \binom{t}{k} \Delta^k f(x_0)
\end{aligned}
$$

且有

$$
R_n(x) = \frac{t(t-1)\cdots(t-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi), \quad \xi \in (x_0, x_n)
$$

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Hermite插值

目标： 插值函数与目标函数在插值节点函数值相同，导数值也相同。

给定$a \le x_0 < x_1 < \cdots < x_n \le b$ 且$y_j = f(x_j)$, $m_j = f'(x_j)$, 选取$H_{2n+1} \in \mathcal{P}_{2n+1}$满足

$$H_{2n+1}(x_j) = y_j, \quad H'_{2n+1}(x_j) = m_j, \ j = 0, 1, \ldots, n$$

构造插值基函数$\alpha_j, \beta_j, j = 0, 1, \ldots, n$, 满足

$$\begin{aligned} \alpha_j(x_k) = \delta_{jk}, \quad \alpha'_j(x_k) = 0 \\ \beta_j(x_k) = 0, \quad \beta'_j(x_k) = \delta_{jk} \end{aligned} \Rightarrow H_{2n+1} = \sum_{j=0}^{n} [y_j \alpha_j(x) + m_j \beta_j(x)]$$

下面考虑如何构造α_j, β_j?

考虑Lagrange插值基函数

$$\ell_j(x)=\prod_{i\neq j}^{n}\left(\frac{x-x_i}{x_j-x_i}\right)\in\mathcal{P}_n\Rightarrow\ell_j(x_k)=\delta_{jk}.$$

设$\alpha_j(x)=(ax+b)\ell_j^2(x)\in\mathcal{P}_{2n+1}$, 则$a,b$应满足

$$ax_j+b=1,\quad a+2\ell_j'(x_j)=0\Rightarrow\alpha_j(x)=[1-2\ell_j'(x_j)(x-x_j)]\ell_j^2(x)$$

设$\beta_j(x)=(ax+b)\ell_j^2(x)$, 则$a,b$应满足

$$\beta_j(x_j)=ax_j+b=0,\beta_j'(x_j)=a\ell_j^2(x_j)+2(ax_j+b)\ell_j(x_j)\ell_j'(x_j)=1$$

$$\Rightarrow\beta_j(x)=(x-x_j)\ell_j^2(x)$$

定理

设$f \in C^1([a,b])$且$x_0, x_1, \ldots, x_n$为相异节点，则存在唯一的多项式$H_{2n+1} \in \mathcal{P}_{2n+1}$ 满足*Hermite*插值条件。

插值余项：

$$R_{2n+1}(x) = f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)(\xi)}}{(2n+2)!} w_{n+1}^2(x)$$

注意到

$$\ell_j(x) = \prod_{i \neq j}^{n} \frac{x - x_i}{x_j - x_i} \Rightarrow \ln \ell_j(x) = \sum_{i \neq j}^{n} \ln \frac{x - x_i}{x_j - x_i}$$

$$\Rightarrow \frac{1}{\ell_j(x)} \ell_j'(x) = \sum_{i \neq j} \frac{x_j - x_i}{x - x_i} \frac{1}{x_j - x_i} \Rightarrow \ell_j'(x_j) = \sum_{i \neq j}^{n} \frac{1}{x_j - x_i}$$

$$\Rightarrow \alpha_j(x) = \left[1 - 2(x - x_j) \sum_{i \neq j}^{n} \frac{1}{x_j - x_i} \right] \ell_j^2(x)$$

重节点均差

定理

设$f \in C^n([a,b])$, $x_0, \ldots, x_n$为$[a,b]$上的相异节点，则有

$$
\begin{aligned}
& f[x_0, x_1, \cdots, x_n] \\
= & \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{n-1}} f^{(n)}(t_n[x_n - x_{n-1}] + \cdots + t_1[x_1 - x_0] + t_0 x_0) dt_n \cdots dt_1
\end{aligned}
$$

推论：

- $f[x_0, x_1, \cdots, x_n]$ 是关于变量$x_0, x_1, \cdots, x_n$的连续函数。
- $f[x_0, x_1, \cdots x_n] = \frac{1}{n!} f^{(n)}(\xi)$, $\xi \in [\alpha, \beta]$, $\alpha = \min(x_0, \cdots, x_n)$, $\beta = \max(x_0, \cdots, x_n)$
- $f[x, \cdots, x] = \frac{1}{n!} f^{(n)}(x)$
- $\frac{d}{dx} f[x_0, \cdots, x_n, x] = f[x_0, x_1, \cdots, x_n, x, x]$

Newton型Hermite插值

设f在相异节点$z_0, z_1, \cdots, z_{2n+1}$上的$2n+1$次Newton插值多项式

$$N_{2n+1}(x) = f(z_0) + f[z_0, z_1](x - z_0) + \cdots + f[z_0, \cdots, z_{2n+1}](x - z_0)\cdots(x - z_{2n})$$

余项为

$$R_{2n+1}(x) = f[z_0, z_1, \cdots, z_{2n+1}, x](x - x_0)\cdots(x - x_{2n+1})$$

由Hermite插值条件可知

$$N_{2n+1}(x_i) = f(x_i), N'(x_i) = f'(x_i)$$

三次Hermite插值

$$\begin{aligned} H_3(x) = & f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 + \\ & f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) \end{aligned}$$

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分段插值

插值函数φ是**分段线性插值，**若其满足

1. $\varphi \in C[a,b]$
2. $\varphi(x_)k = f(x_k)$
3. φ在每一个区间$[x_k, x_{k+1}]$上是线性多项式。

定理

设$f \in C^2[a,b]$，φ是$[a,b]$上的分段线性插值多项式，则有

$$\|f-\varphi\|_\infty \le \frac{h^2}{8}\|f''\|_\infty$$

其中$h = \max_{1\le k\le n}(x_k - x_{k-1})$, $\|f\|_\infty = \max_{a\le x\le b}|f(x)|$

$\varphi \in C[a,b]$

分段三次Hermite插值

插值函数ψ满足

1. $\psi \in C^1[a,b]$
2. $\psi(x_k) = f(x_k)$, $\psi'(x_k) = f'(x_k)$
3. ψ在每个小区间$[x_k, x_{k+1}]$上是一个三次多项式

定理

设$f \in C^2[a,b]$，ψ是$[a,b]$上的分段三次*Hermite*插值多项式，则有

$$\|f - \varphi\|_\infty \le \frac{h^4}{384}\|f^{(4)}\|_\infty$$

其中$h = \max_{1\le k\le n}(x_k - x_{k-1})$, $\|f\|_\infty = \max_{a\le x\le b}|f(x)|$

$\psi \in C^1[a,b]$. 能否构造处更加光滑的插值函数？

三次样条插值

S为$[a,b]$上的**三次样条插值**函数，且满足：

1. $S \in C^2[a,b]$
2. S在每个小区间$[x_k, x_{k+1}]$上是三次多项式
3. $S(x_k) = f(x_k),\ k = 0, 1, \cdots, n$

在区间$[x_j, x_{j+1}]$上为三次多项式，可设为

$$a_j x^3 + b_j x^2 + c_j x + d_j \Rightarrow \text{共计}4n\text{个变量}$$

由于$S \in C^2[a,b]$, 所以有

$$\underbrace{S(x_j) = f(x_j)}_{\text{n+1个方程}}, \underbrace{S(x_j^-) = S(x_j^+)}_{\text{n-1个方程}}, \underbrace{S'(x_j^-) = S'(x_j^+)}_{\text{n-1个方程}}, \underbrace{S''(x_j^-) = S''(x_j^+)}_{\text{n-1个方程}}$$

共$4n-2$个方程，还差2个!

边界条件

1. **I型边界条件：**

$$S'(x_0) = f'(x_0), S'(x_n) = f'(x_n)$$

2. **II型边界条件**

$$S''(x_0) = f''(x_0), S''(x_n) = f''(x_n)$$

若$f''(x_0) = f''(x_n) = 0$，称为自然边界条件。

3. **周期边界条件**

$$S^{(j)}(x_0) = S^{(j)}(x_n), \quad j = 0, 1, 2.$$

计算方法

设$h_j = x_{j+1} - x_j$, $j = 0, 1, \ldots, n-1$; $M_j = S''(x_j)$, $j = 0, 1, \ldots, n$.

S''在$[x_j, x_{j+1}]$上是线性函数，则有

$$S''(x) = M_j \frac{x_{j+1} - x}{h_j} + M_{j+1} \frac{x - x_j}{h_j}, \quad x \in [x_j, x_{j+1}]$$

$$\Rightarrow S'(x) = -\frac{M_j}{2h_j}(x_{j+1} - x)^2 + \frac{M_{j+1}}{2h_j}(x - x_j)^2 + C_1$$

$$\Rightarrow S(x) = \frac{M_j}{6h_j}(x_{j+1} - x)^3 + \frac{M_{j+1}}{6h_j}(x - x_j)^3 + C_1 x + C_2$$

在每个区间上，确定M_j, M_{j+1}, C_1, C_2?

1. 利用$S(x_j) = f(x_j), S(x_{j+1}) = f(x_{j+1})$可知

$$C_1 = \frac{1}{h_j}\left[\left(f(x_{j+1}) - \frac{M_{j+1}}{6}h_j^2\right) - \left(f(x_j) - \frac{M_j}{6}h_j^2\right)\right]$$
$$C_2 = \frac{1}{h_j}\left[x_{j+1}\left(f(x_j) - \frac{M_j}{6}h_j^2\right) - x_j\left(f(x_{j+1}) - \frac{M_{j+1}}{6}h_j^2\right)\right]$$

2. 利用 $S'(x_j^-) = S(x_j^+)$, $j = 1, 2, \cdots, n-1$加上边界条件，可以形成关于$M_0, M_1, \cdots, M_n$的线性方程组，进行求解。（具体算法请参考教材212-214页。）

数值算例

设f为定义在$[0,3]$上的函数，节点剖分为$x_j=0+j, j=0,1,2,3$，并给出$f(0)=0$，$f(1)=0.5$，$f(2)=2$，$f(3)=1.5$，求三次自然样条函数S满足$S(x_j)=f(x_j)$，$j=0,1,2,3$.

$$S(x)=\begin{cases}0.4x^3+0.1x, & x\in[0,1]\\ -(x-1)^3+1.2(x-1)^2+1.3(x-1)+0.5, & x\in[1,2]\\ 0.6(x-2)^3-1.8(x-2)^2+0.7(x-2)+2.0, & x\in[2,3]\end{cases}$$