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Quiz 1

(10 points, 20 minutes)

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(4 points=2+2)

1.3—X-Ray photons are produced in a cloud of radius R at the uniform rate Γ (photons per unit volume per unit time). The cloud is a distance d away. Neglect absorption of these photons (optically thin medium). A detector at earth has an angular acceptance beam of half-angle $\Delta\theta$ and it has an effective area of ΔA.

a. **Assume that the source is completely resolved. What is the observed intensity (photons per unit time per unit area per steradian) toward the center of the cloud.**

b. **Assume that the source is completely unresolved. What is the observed average intensity when the source is in the beam of the detector?**

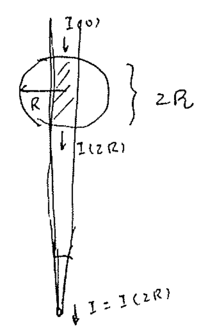
Solution:

a. For that is well resolved,

$$j = \frac{\Gamma}{4\pi} = \frac{dE}{dt\, dV\, d\Omega}$$

So the intensity satisfied $\frac{dI_N}{ds} = j$, $I_N(0) = 0$

Integral it we have (in the cloud) $I = j \cdot 2R = \frac{\Gamma}{4\pi} \cdot 2R = \frac{\Gamma R}{2\pi}$ ✓



and keep constant when propagation until into our telescope.

b. For that is not resolved, total emitted energy/number

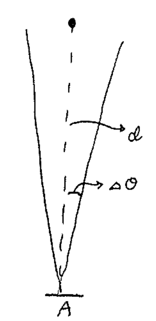
$$\frac{dN}{dt} = \Gamma \cdot \frac{4}{3}\pi R^3$$

So the flux at receiver is $F_N = \frac{1}{4\pi d^2}\frac{dN}{dt}$, and

$$\frac{dN_{received}}{dt} = \frac{\Delta A}{4\pi d^2}\frac{dN}{dt}$$

So the intensity is

$$I_N = \frac{dN_{received}}{dt \cdot \Delta A \cdot \pi \Delta\Theta^2} = \frac{R^3 \Gamma}{3d^2 \pi \Delta\Theta^2}$$ ✓



$$\frac{4}{3}\frac{\pi\,12^3 \cdot 5}{d^2 \cdot \pi\,00^2}$$

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(6 points = 2+3+1)

3.2—A particle of mass m and charge e moves at constant, nonrelativistic speed v_1 in a circle of radius a.

2 **a. What is the power emitted per unit solid angle in a direction at angle θ to the axis of the circle?**

2.5 **b. Describe qualitatively and quantitatively the polarization of the radiation as a function of the angle θ.**

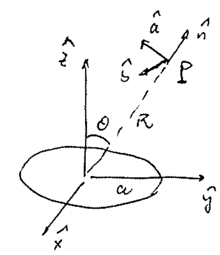
1 **c. What is the spectrum of the emitted radiation?**

Note: for (a): i.e. the time-averaged power.

for (b): i.e. what is the components of the E-field? The Stocks parameters? The polarization in the general case and in some special cases?

Solution:

a. we can choose the frame as shown in this figure



So the power $\frac{dE}{dt\,d\Omega} = \frac{q^2 \dot{u}^2 \sin^2\theta'}{4\pi c^3}$ where the θ' $= \langle \hat{n}, \vec{v} \rangle$ should depend on time $\sin\theta' = \frac{|\hat{n}\times\vec{v}|}{|\vec{v}||\hat{n}|} = \sqrt{\sin^2\omega t + \cos^2\omega t \cos^2\theta}$

$$\left\langle \frac{dE}{dt\,d\Omega} \right\rangle = \left\langle \frac{q^2 \dot{u}^2 \sin^2\theta'}{4\pi c^3} \right\rangle = \cdots = \frac{q^2(\omega^2 a)^2}{4\pi c^3} \langle \sin^2\theta' \rangle$$

$$= \frac{q^2(\omega^2 a)^2}{4\pi c^3}\left(\frac{1}{2} + \frac{1}{2}\cos^2\theta\right)$$

$\vec{x} = a\cos\omega t\ \hat{x} + b\sin\omega t\ \hat{y}$

$\vec{v} = \dot{\vec{x}} = \omega a(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$\dot{\vec{v}} = \ddot{\vec{x}} = -\omega^2 a(\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$\hat{n} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}$

$\langle \sin^2\omega t \rangle = \frac{1}{2}$

b. We can calculate the Electric field at P, it's

$$\vec{E}_{rad} = \frac{q}{Rc^2}\hat{n}\times(\hat{n}\times\dot{\vec{u}}) = -\frac{q}{Rc^2}\omega^2 a\left(\cos\theta\cos\omega t\,\hat{a} - \sin\omega t\,\hat{b}\right) = E_a\,\hat{a} + E_b\,\hat{b}$$

and expand it in the $\{\hat{a}, \hat{b}, \hat{n}\}$ basis. we should see it is a elliptially polarized field.

where

$$\begin{cases} E_a = -\frac{q}{Rc^2}\omega^2 a\cos\theta\cos\omega t \\ E_b = \frac{q}{Rc^2}\omega^2 a\sin\omega t \end{cases} \Rightarrow \left(\frac{E_a}{\left(\frac{q\omega^2 a}{Rc^2}\cos\theta\right)}\right)^2 + \left(\frac{E_b}{\left(\frac{q}{Rc^2}\omega^2 a\right)}\right)^2 = 1 \Rightarrow$$

in general elliptically polarized.

The Stocks parameters:

$$\begin{cases} I = \varepsilon_1^2 + \varepsilon_2^2 = \left(\frac{q}{Rc^2}\omega^2 a\right)^2(1+\cos^2\theta) \\ Q = \varepsilon_1^2 - \varepsilon_2^2 = \left(\frac{q}{Rc^2}\omega^2 a\right)^2(1-\cos^2\theta) \\ U = 2\varepsilon_1\varepsilon_2\cos(\phi_1-\phi_2) = -\left(\frac{q}{Rc^2}\omega^2 a\right)^2 \cdot 2\cos\theta \\ V = 2\varepsilon_1\varepsilon_2\sin(\phi_1-\phi_2) = 0 \end{cases}$$

For $\theta = 0$, it is circular polarized.

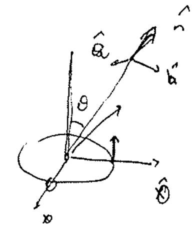
$\theta = \frac{\pi}{2}$. linear polarized

c. The Spectrum

$$\frac{dW}{d\omega\,d\Omega} = cR|\tilde{E}(\omega)|^2$$

But in this case, the $\vec{E}_{rad}$ and also $\vec{B}_{rad}$ is single-frequency, with circular frequency $\omega_{field} = \omega$.

So its spectrum is located at single point.



$$\frac{1}{T}\int_0^{\frac{2\pi}{\omega}} \sin^2\omega t \, dt$$

$$= \frac{1}{2}$$

$$\hat{n} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{a} = -\cos\theta\,\hat{y} + \sin\theta\,\hat{z}$$

$$\hat{b} = \hat{x}$$

$$x = a\cos\omega t\,\hat{x} + a\sin\omega t\,\hat{y}$$

$$\vec{v} = \dot{x} = -a\omega\sin\omega t\,\hat{x} + \omega a\cos\omega t\,\hat{y}$$

$$= \omega a(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y}).$$

$$\cancel{\hat{n} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}}$$

$$\ddot{u} = -\omega^2 a\Big(\sin\omega t\,(\hat{b})$$

$$+\cos\omega t\,(-\cos\theta\,\hat{a} + \sin\theta\,\hat{n})\Big).$$

$$= -\omega^2 a\,(-\cos\theta\cos\omega t\,\hat{a}$$

$$\cancel{\hat{n}\times\ddot{u}} = \quad +\sin\theta\cos\omega t\,\hat{n} + \sin\omega t\,\hat{b})$$

$$\hat{n}\times$$

$$\hat{n}\times\ddot{u} = -\omega^2 a\,(-\cos\theta\cos\omega t\,\hat{b} - \sin\omega t\,\hat{a})$$

$$\sin\theta = \frac{|\vec{v}\times\hat{n}|}{|\vec{v}|\,|\hat{n}|}$$

$$= \Big|\cancel{\omega a}\,(-\sin\theta\sin\omega t\,\hat{z} + \sin\omega t\cos\theta\,\hat{y} + \cos\omega t\cos\theta\,\hat{x})\Big|$$

$$\cancel{\omega a}$$

$$\hat{n}\times\hat{n}\times\ddot{u} = -\omega^2 a\,(+\cos\theta\cos\omega t\,\hat{a} - \sin\omega t\,\hat{b})$$

$$\cancel{\sin\theta} = \sin^2\omega t\,\cancel{\sin^2\theta} + \cos^2\omega t\cos^2\theta$$

$$\vec{a} = -\omega^2 a\,(\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$$

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Quiz 2
(10 points, 60 minutes)

Student name:

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3 Problem 1: Bremsstrahlung. (4 points = 1+2+1)

5.2 – Suppose X-rays are received from a source of known distance L from a flux F (erg cm^{-2} s^{-1}). The X-ray spectrum has the form of Fig. 5.5. It is conjectured that these X-rays are due to Bremsstrahlung from an optically thin, hot, plasma cloud, which is in hydrostatic equilibrium around a central mass M. Assume that the cloud thickness ΔR is roughly its radius R, $\Delta R \sim R$.
(a) Estimate the temperature T of the cloud from the X-ray spectrum..
(b) Find R and the density of the cloud, ρ, in terms of F, L, T, and conjectured mass M.
(c) If F, L, and T are given, what are the constraints on M such that the source would indeed be effectively thin (for self-consistency)? Does electron scattering play any role?

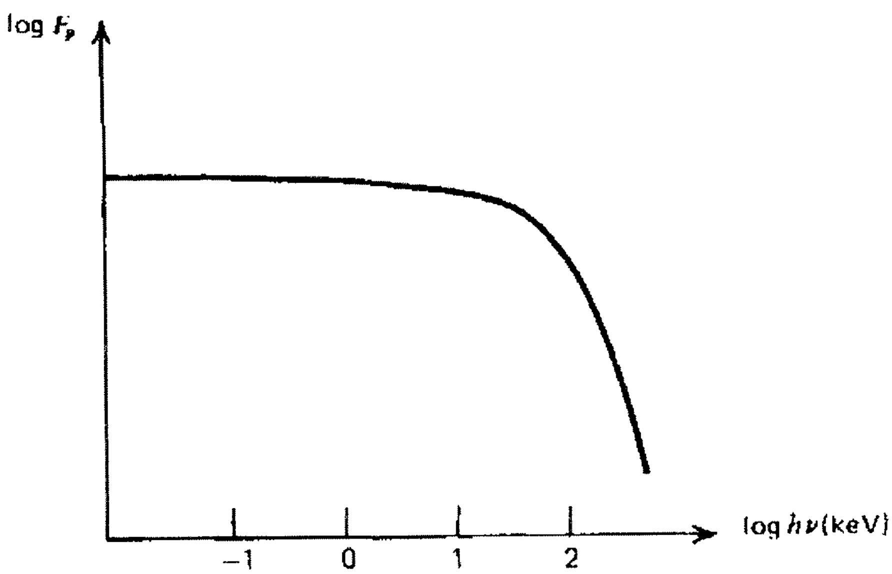

Figure 5.5 Detected spectrum from an X-ray source.

Solution

(a) The exponential cut-off of F_ν is $\frac{dw}{dt\,dV\,d\nu} \propto e^{-h\nu/kT}$, so

$$h\nu_{\text{cut-off}} = kT = 10^2\,\text{keV}$$

$$\Rightarrow \quad T \simeq 10^9\,K$$

(b) From observed flux F. we can have

$$\begin{cases} \frac{dw}{dt\,dV} \cdot \frac{4}{3}\pi R^3 = F \cdot 4\pi L^2 \\ \frac{dw}{dt\,dV} \simeq 1.4\times10^{-27}\, T^{1/2} n_e n_i Z^2 \bar{g}_B \end{cases}$$

$(\bar{g}_B \simeq 1)$

Combine this two, we obtain

$$R = \left[\frac{F \cdot 4\pi L^2}{\frac{4}{3}\pi \cdot 1.4\times10^{-27}\, T^{1/2} n_e n_i Z^2} \right]^{1/3}$$

From hydro-static equlibrium, we know $\frac{3}{2}kT\times 2 \simeq \frac{GMm_e}{R}$. The density of the cloud is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}, \quad \text{where } R \text{ is given by } R = \frac{GMm_e}{3kT} \text{ or by } R = \left(\frac{F\cdot 4\pi L^2}{\frac{4}{3}\pi \times 1.4\times10^{-27}\, T^{1/2} n_e n_i Z^2} \right)^{1/3}$$

Finally if we let $Z \simeq 1$, $n_e n_i \simeq n_i^2 = \frac{1}{m_H^2}\rho^2$, then $\rho = \frac{F\cdot 4\pi L^2}{1.4\times10^{-27} \frac{T^{1/2}}{m_H^2} M}$, $R = \frac{GMm_e}{3kT}$

(c) The opacity of the medium is $\kappa_* = \sqrt{\kappa_R^{\cancel{ff}}\kappa_{es}}$, where $\kappa_R^{ff} = \frac{\varepsilon^{ff}}{\rho B_\nu}$, $\frac{\kappa_R^{ff}}{\kappa_{es}} = \frac{0.7\times10^{23}\rho T^{-7/2}}{0.4}$

Then the optical depth

$$\tau_* = \rho\kappa_*\Delta R \overset{\Delta R \simeq R}{\simeq} \rho\kappa_* \frac{GMm_e}{3kT}$$

The optical thin require $\tau_* \ll 1$. Thus $M \ll \frac{3kT}{Gm_e\rho\kappa_*}$.

The electron scattering comes to this equation by $\underline{\underline{\kappa_{es}}}$, so it does play some role. But from $\frac{\kappa_R^{ff}}{\kappa_{es}}$ we know $\kappa_R^{ff} \gg \kappa_{es}$. So the free-free process plays major role.

6 Problem 2. Synchrotron Losses. (6 points = 2+1+2+1)

(a) Obtain an analytic expression for the energy of a single relativistic electron as a function of time, $E(t)$, taking into account its energy loss by synchrotron radiation. Your expression should contain only the variables $E(0)$ (the initial energy of the electron), B (the magnetic field, here held fixed with time, following the rest of the world, though one should worry in general about the field changing with time just as the electron energy spectrum changes with time), time t, and fundamental constants. Assume $\sin\alpha = 1$ (the electron pitch angle is 90 degrees) for simplicity.

For (synchrotron) problems of interest to us, the electron always remains relativistic. It merely evolves from a large $\gamma \gg 1$ to a smaller $\gamma \gg 1$.

(b) How can you reconcile the loss of energy of the electron with the bald statement of Rybicki & Lightman on page 168 that "γ is constant"?

(c) We have made arguments in class that power-law distributions of electrons in astrophysical sources are maintained against synchrotron losses by continuous energization by central engines (a.k.a. *injection*). The injection (input) spectrum of electrons is modified by synchrotron losses to produce a steady-state (output) distribution.

Call $\eta(E,t) = dN/dE$ the differential energy spectrum of electrons as discussed repeatedly in class. Continuity of electrons in energy space reads

$$\frac{\partial \eta(E,t)}{\partial t} + \frac{\partial}{\partial E}[\dot{E}\eta(E,t)] = I \tag{1}$$

where I is the rate of injection of electrons with some input distribution and $\dot{E}$ is the rate of energy loss of a single electron by synchrotron radiation. This equation should not mystify you; it merely describes how the number of electrons in a given energy bin changes with time, taking into account a flux divergence (the second term on the left-hand-side) and a source term (the right-hand-side).

We have assumed in class a *steady-state* distribution of electron energies for which $\eta \propto E^p$. Given p, how must I scale with E? Give only the scaling and forget about the numerical coefficients.

As with most scaling problems, you don't have to solve anything in detail. Ruthlessly work to order of magnitude.

(d) Electrons having a given energy must wait a characteristic time before synchrotron losses become important. Before this time elapses for all such electrons, how does η scale with E?

(a) For synchrotron radiation, the Power $P = \frac{2}{3} r_0^2 c \gamma^2 B^2 \beta^2$, so the energy loss is

$$\frac{dE}{dt} = \frac{d}{dt}\left(\gamma m c^2\right) = -\frac{2}{3} r_0^2 c \gamma^2 B^2 \beta^2$$

Using $\gamma^2\beta^2 = \gamma^2 - 1 \simeq \gamma^2$, we have

$$\frac{d\gamma}{\gamma^2} = -\frac{2}{3} r_0^2 \frac{cB^2}{mc^2} dt$$

By integral of it we obtain

$$\gamma = \frac{1}{\frac{1}{\gamma_0} + \frac{2}{3} r_0^2 \frac{cB^2}{mc^2} t}$$

Or use $E = \gamma mc^2$:

$$E = \frac{1}{\frac{1}{E(0)} + \frac{2}{3} r_0^2 \frac{cB^2}{(mc^2)^2} t}$$

(b)

Actually the loss of energy may come from the electro-magnetic field around the particle, so γ remain nearly constant doesn't mean there isn't radiation.

Consider both field change would reconcile the problem.

(c)

The energy loss rate $\dot{E} = P = \frac{2}{3} r_0^2 c \gamma^2 B^2 \beta^2 \propto E^2$

and for final stable state $\eta \propto E^p$.

From the equation $\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial E}\left(\dot{E}\eta(E,t)\right) = I$ we know

$$I = \frac{\partial}{\partial E}\left(\dot{E}\eta\right) \propto \frac{\partial}{\partial E}\left(E^2 E^p\right) \propto E^{p+1}$$

(d) Before the characteristic time $(t < t_c)$, the energy loss $\dot{E} \simeq 0$. so

$$\frac{\partial \eta}{\partial t} = I$$

$$\Rightarrow \eta = It \propto E^{p+1} t \propto E^{p+1}$$