研究生课程

天体物理中的辐射机制　茅奕 ymao@tsinghua.edu.cn

- 电动力学 → 物质的辐射(机制)（经典）
  的基本理论　动力…
  ↓ 量子：谱线 → 非经典：Compton (Inverse)
  谱线. f. Af. 多电子原子. 双原子分子的谱线

- 教材：Rybicki & Lightman（有答案）follow, 跳过不重要部分
  Frank H. Shu
  盗版书网站 http://gen.lib.rus.ec/
  参考：尤峻汉（二卷，非常详实清楚）

- Grades:
  30%, 书上的题, 有solution 但不许抄（一周后, 按时交）
  30%. Quiz, 3-in-class
  40%, Final Exam } open, one page of formulae
  周末. 或有回家自己考

Lecture 1

Fundamentals of radiation and radiation transfer

作业：
R&L 1.1, 1.2, 1.3, 1.4
下周一交

天体辐射的类型
- 电磁辐射(光子) ~ 本课程
- 引力波辐射(引力子)
- 中微子辐射(中微子) 不成熟, 引力也不强
- 宇宙线辐射(高能粒子) 不是严格的辐射 但也是传递信息的媒介

Electromagnetic Spectrum
电磁辐射谱

prism / grating / slit / … 分光 ~ $c=\lambda\nu$, $E=h\nu$, $T=\frac{E}{k}$

e.g.
硬X与γ有交叠
软X与紫外交叠

分光 → 中国的望远镜：FAST, HXMT, LAMOST

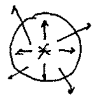
→ 大气层对波段的吸收：
地面天文(光学, 射电)
空间天文(紫外, X, γ, 红外, 亚毫)

e.g. 不同波段的太阳 / MW 图像
(天体, 不同波段的能谱 不一定是黑体谱, 有结构, 如（太阳）quasar)

1. Elementary properties of radiation

- radiation flux

假设光是直线传播（忽略衍射等…）
Assume light travels in a straight line. Flux $F=\frac{dE}{dt\,dA}$



各向同性的发光体：$F\cdot4\pi r^2=\frac{dE}{dt}$ = power

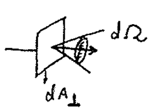
对power不变时，$F\propto\frac{1}{r^2}$（能量守恒的结果）

我们希望描述光的物理量是恒定的（与传播距离无关），F 不是很好

- Specific intensity (brightness), for individual rays
  比强度

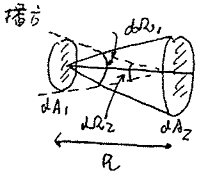
$$I_\nu(t,\vec{x},\hat{n},\nu)=\frac{dE}{dt\,dA\,d\Omega\,d\nu}$$

位置　传播方向 与 $d\Omega$ 有关



比强度的"比" $d\nu$，没有 $d\nu$ 时称 intensity

prove: constancy of I_ν along rays in free space, $\frac{dI_\nu}{ds}=0$（沿传播方向）— 无吸收, 发射, 散射



经过 dA_1 的光也必经过 dA_2

$$dE=I_{\nu_1}dt_1dA_1d\Omega_1d\nu_1$$
$$=I_{\nu_2}dt_2dA_2d\Omega_2d\nu_2$$

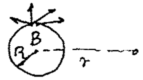
$dt_1=dt_2$ 不考虑相对论效应
$d\nu_1=d\nu_2$ 不考虑红移

$d\Omega_1=\frac{dA_2}{R^2}$
$d\Omega_2=\frac{dA_1}{R^2}$

$$\frac{dA_2}{d\Omega_1}=\frac{dA_1}{d\Omega_2}\Rightarrow dA_1d\Omega_1=dA_2d\Omega_2$$
$$\Rightarrow I_{\nu_1}=I_{\nu_2}\text{，即 }I_\nu=\text{Constant}$$

说明：不与平方反比律矛盾，可以导出平方反比律（说明它们的一致性）
The inverse square law from the constancy of I_ν

$$\frac{dF}{d\nu}=\int I\cos\theta\,d\Omega=B\int_0^{2\pi}d\phi\int_0^{\theta_c}\cos\theta\sin\theta\,d\theta$$
$$=2\pi B\sin^2\theta_c\cdot\frac{1}{2}$$
$$=\pi B\left(\frac{R}{r}\right)^2\propto\frac{1}{r^2}$$

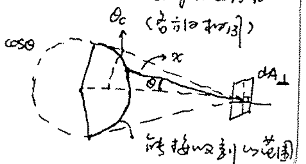


ball, uniform brightness B（各方向相同）

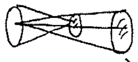
说明：与前面的差别是，这里是有限大的球体，不是点状发光体，但仍是 $\frac{1}{r^2}$

而 $I=\begin{cases}B, & \text{if ray intersects with sphere}\\0. & \text{without}\end{cases}$　$\delta\left(\frac{dF}{d\nu}\right)=\frac{dE}{dt\,dA\,d\nu}=I\cos\theta\,d\Omega$

对x光束 $dA_\perp=dA\cdot\cos\theta$



Note: (1) at large r，可以简单地理解 Constancy I_ν 与平方反比的关系



$$I = \frac{dE}{dt\,dA\,d\Omega\,d\nu}$$

同一束光，在不同位置 $dA \propto r^2$

$$d\Omega \approx \pi\left(\frac{R}{r}\right)^2 \propto \frac{1}{r^2}$$

所以 $I = \text{Const}$

(2) at the Surface $r=R$

$$\frac{dF}{d\nu} = \pi B \neq 2\pi B$$
$$\neq 2\pi B \times 2$$

和光源的直觉不同

因为这不止一束的光被接收到

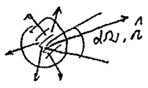
2. Radiative transfer

Along the ray, $\frac{dI_\nu}{ds} = 0$ in free transfer

ds = differential element of length along the ray

有哪些因素会改变辐射的强度？ Emission, Absorption, Scattering …

- Emission



不一定各向同性

$$dE = j_\nu\, dV\, d\Omega\, dt\, d\nu$$

$\uparrow$ 辐射系数

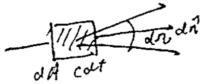
spontaneous emission coefficient

(自发，与后面的受激辐射对应，

只与自发辐射率有关

定义 $j = \int j_\nu\, d\nu$ 也叫自发辐射系数

For a ray traveling through the volume



体积 $dV = c\,dt\,dA = ds\,dA$

经辐射后，比强度增量 $dI_\nu = \frac{dE}{dt\,dA\,d\Omega\,d\nu} = \frac{j_\nu\, ds\, dA\, dt \cdot d\Omega\, d\nu}{dt\, dA\, d\Omega\, d\nu} = j_\nu \cdot ds$

或 $\frac{dI_\nu}{ds} = j_\nu$

另外 emissivity (按质量来 specify)

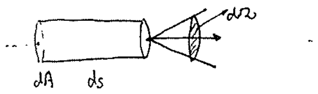
$$dE = \epsilon_\nu \underbrace{\rho\, dV}_{dm}\, dt\, \underbrace{\frac{d\Omega}{4\pi}}_{\text{the fraction of energy radiated into } d\Omega}\, d\nu$$

而 j_ν 与 ϵ_ν 的关系 $j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$

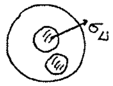
- Absorption

(自发)发射只与发光率有关

吸收与入射光有关，也与吸收率有关



toy 玩具 model: consider a beam of I_ν travels through a cloud of gas with number density n



某个粒子的截面，与 ν 有关

粒子的截面足够小(间距足够大)，σ_ν 不重叠

gas particle presents an effective absorbing area σ_ν (cross section)

number of absorber $= n\, dA\, ds$

Total absorbing area $= n\, \sigma_\nu\, dA\, ds$

Total energy absorbed

$$dE = -I_\nu \underbrace{(n\, \sigma_\nu\, dA\, ds)}_{\text{粒子的总照射面积}}\, d\Omega\, dt\, d\nu$$

而 $dE = dI_\nu \cdot dA \cdot dt\, d\Omega\, d\nu$

combine of two $\frac{dI_\nu}{ds} = -I_\nu \cdot \underbrace{n\sigma_\nu}$

* independent of incoming ray
* only dep of gas
α_ν

In general, we define α_ν as absorption coefficient ($[L]^{-1}$)

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

只与吸收率有关.
在 toy model 里，与 $n \cdot \sigma_\nu$ 有关
一般地，与 $n \cdot \nu$，材料有关

Note: (1) assumption: (在 toy model 里 粒子间有足够大的距离)

- linear scale of $\sigma_\nu \ll$ mean interparticle distance

$$\sqrt{\sigma_\nu} \ll d \sim n^{-1/3}$$

即 $\left(\frac{\alpha_\nu}{n}\right)^{1/2} \cdot n^{1/3} \ll 1 \Rightarrow \frac{\alpha_\nu}{n} \cdot n^{2/3} = \alpha_\nu \cdot n^{-1/3} \sim \alpha_\nu \cdot d \ll 1$

- random distribution

(吸收的粒子必须随机分布，对热平衡的系统一般是满足的)

(2) 这里的 "Absorption" = true absorption (纯粹吸收)
$+$
stimulated emission

(受激发射，也有同样的关系，也被包含在内)

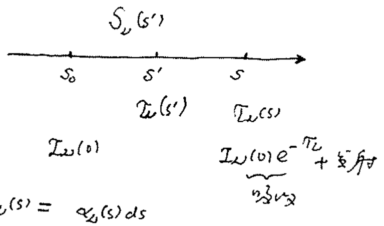
$\uparrow$ 有时，可以有 $\alpha_\nu < 0$

(3) 这里的 "Absorption" does NOT include scattering

（"吸收"是减少入射的流量
"散射"只改变 ν, angle 的分布, 没有减少流量，不能用直线传播来描述）

- Radiation transfer equation (No Scattering)
  辐射转移方程

$$\frac{dI_\nu(s)}{ds} = \underset{\substack{\uparrow \\ \text{emission}}}{j_\nu(s)} - \underset{\substack{\uparrow \\ \text{absorption}}}{\alpha_\nu(s) I_\nu(s)}$$



可以有形式解, 但无显解, define optical depth 光深 $d\tau_\nu(s) = \alpha_\nu(s)ds$

(when $\alpha_\nu(s)>0$, τ_ν monotonically increasing along s.
$\tau_\nu(s) \leftrightarrow s$)

$\tau_\nu(s) = \int_{s_0}^{s}\alpha_\nu(s')ds'$ （无量纲函数），光深是一个积分量，依赖的是积分区段，不只与某点的结果有关

求解 $I_\nu(s)$ 等价于求解 $I_\nu(\tau_\nu)$

将等式两边除以 α_ν，有

$$\frac{dI_\nu}{d\tau_\nu} = \frac{dI_\nu}{\alpha_\nu ds} = \frac{j_\nu(s)}{\alpha_\nu} - I_\nu(s) \qquad S_\nu = \frac{j_\nu}{\alpha_\nu} \text{ source function}$$

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu(s) + S_\nu$$

常数变易法和 Green 函数法？

求解 $\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = S_\nu e^{\tau_\nu}$ 得 $I_\nu(s)e^{\tau_\nu(s)} = \underbrace{I_\nu(s_0)e^{\tau_\nu(s_0)}}_{1} + \int_0^{\tau_\nu} \underbrace{e}_{记为\ e^{\tau_\nu'(s')}}{}^{-(\tau_\nu-\tau_\nu')} S_\nu(\tau_\nu')d\tau_\nu'$

也就是

$$I_\nu(s) = \underset{\substack{\uparrow \\ 入射光强 \\ \text{brightness at } \infty}}{I_\nu(0)e^{-\tau_\nu}} + \underbrace{\int_0^{\tau_\nu} e^{-(\tau_\nu-\tau_\nu')} S_\nu(\tau_\nu')d\tau_\nu'}_{发射}$$

$I_\nu(0)e^{-\tau_\nu}$ 为入射的吸收后剩量

$S_\nu(s')e^{-(\tau_\nu-\tau_\nu')}$ 为重发射的光被吸收后的量（新发光的吸收后的结果）

[formal solution
并非 $I_\nu \sim s$ 关系,
而是 $I_\nu \sim \tau_\nu$ 关系, 只是形式上解决了, $\tau_\nu \sim s$ 关系仍未知] → 往往需要数值模拟, 沿光线方向 tracing, 积分

Note : (1) for case $\tau_\nu > 1$, optically thick or opaque
$\tau_\nu < 1$, optically thin or transparent

(2) for only emission, $\alpha_\nu = 0$ （没有光深）

$$\frac{dI_\nu}{ds} = j_\nu$$

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'$$

(3) for only absorption $j_\nu=0$, $S_0=0$

$$I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu}$$

τ_ν 是积分量.

(4) S_ν is a constant

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1-e^{-\tau_\nu}\right)$$

also $I_\nu - I_\nu(0) = (S_\nu - I_\nu(0))(1-e^{-\tau_\nu})$

有时, 观测到的光有背景光 $I_\nu(0)$ background, 例如 21 cm, 需要的是 $I_\nu - I_\nu(0)$

(5) 极特殊的情况

when $S_\nu \neq 0$ only at a point （忽略热展宽、量子展宽的理想化情况, 如 21 cm 谱中）

可证明也有 $I_\nu(s) - I_\nu(s_0) = (S_\nu(s) - I_\nu(s_0))(1-e^{-\tau_\nu})$

往往只在光学薄的情况 $\tau_\nu << 1$，或者否则后面径迹全吸收发射，此时

$$I_\nu(s) - I_\nu(s_0) = [S_\nu(s) - I_\nu(s_0)] \cdot \tau_\nu$$

- Mean free path (MFP)

有吸收时, 光不可能无限传播; 在能传播的有限距离, 称平均自由程;

Mean free path is defined as the average distance a photon can travel thru an absorbing material w/o (without) being absorbed.

$$I_\nu = I_\nu(0)e^{-\tau_\nu}$$

mean $\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu \underbrace{e^{-\tau_\nu}}_{\substack{光子能传播\ \tau_\nu\ 的几率 \\ \text{probability of a photon travelling } \tau_\nu}} d\tau_\nu = 1$

homogeneous medium

$$\langle\tau_\nu\rangle \equiv \alpha_\nu l_\nu = 1$$

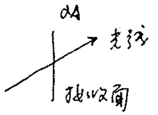
$$\Rightarrow \text{MFP } l_\nu = \frac{1}{\alpha_\nu}$$

在非均匀介质中, 可定义局地量 local MFP $l_\nu = \frac{1}{\alpha_\nu}$ at the point in an inhomogeneous medium

注: 经常用来估计光子的传播尺度, 作不同尺度的近似（如再电离光子）

Some more definitions

(与ν,φ,θ有关)

- 

$$dF_\nu = \frac{dE}{dt\,dA\,d\nu} = I_\nu\cos\theta\,d\Omega$$

如果只在乎经过截面的总的光子数，不在乎方向，则定义 net flux

$$F_\nu = \int_{\text{all direction}} I_\nu\cos\theta\,d\Omega$$

- Mean intensity （不在乎 intensity 的方向）

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$$

- Radiation energy density

$$dE = I_\nu\,dt\cdot dA\cdot d\Omega\cdot d\nu$$

$$= u_\nu^{(\hat{n})}\cdot \underbrace{c\,dt\cdot dA}_{dV}\,d\Omega\,d\nu$$

则有 $u_\nu^{(\hat{n})} = \frac{I_\nu}{c}$

同样所对方向积分 $u_\nu = \int u_\nu(\hat{n})d\Omega = \frac{4\pi}{c}J_\nu$，称为 Radiation energy density

在读文献时，有的叫法可能不一样，要仔细 check

Lecture 2　03/05/2018　六教6B211

Review:

Fundamentals of radiations　$I_\nu = \frac{dE}{dt\,dA_\perp\,d\Omega\,d\nu}$ （光线传播过程中不变的 I_ν）

$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$ — 入射　（转移方程）

j_ν：发光率　α_ν：发光率/吸收率

$\tau_\nu = \int_{s_0}^{s}\alpha_\nu ds$　（光深）　$\Rightarrow$　$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$

形式解　$I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu-\tau_\nu')}d\tau_\nu'$

→ 辐射是平衡的

Blackbody radiation is a radiation which is itself in thermal equilibrium.　$I_\nu = B_\nu(T)$

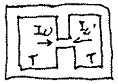
Thermal radiation is a radiation from some material which is at thermal equilibrium with itself.

发光体是平衡的　(but not necessarily with the radiation)　$S_\nu = B_\nu(T)$ (only at $\tau >> 1$)

Blackbody Spectrum　I_ν " $I_\nu(t,\vec{x},\nu,\hat{n})$ $= B_\nu(T)$　Planck function，$I_\nu = I_\nu(t,\vec{x},\nu,\hat{n},T) = I_\nu(\nu,T)$

由于是热平衡，与 t，$\vec{x}$，$\hat{n}$ 无关

Proof for $I_\nu = I_\nu(T,\nu)$



if $I_\nu \neq I_\nu'$, energy flows

but T is the same, no flows (2nd law)

$\Rightarrow$ So that I_ν is indep of t, $\vec{x}$, $\hat{n}$

$\hat{n}$：腔可开在任意位置，任意方向

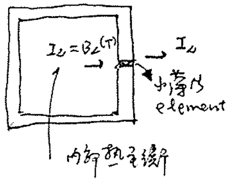
Thermal Radiation

热辐射不一定是平衡的，所以 $I_\nu \neq B_\nu(T)$

但是 source function $S_\nu = \frac{j_\nu}{\alpha_\nu} = B_\nu(T)$　　(Kirchhoff's law)

S_ν：描述物体的发射/吸收性质

Proof for $S_\nu = B_\nu(T)$



（足够黑/大，完全吸收）

Assume $\tau \to \infty$, at the element

$I_\nu = S_\nu$

If $S_\nu \neq B_\nu \Rightarrow I_\nu \neq B_\nu$

which is impossible, because for the element absorption must be equal to radiation.

由于薄片可以任取，所以 $S_\nu = B_\nu(T)$ 是整个黑体的性质

Planck Spectrum　$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1}$　(Planck's law)

Properties

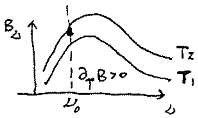
(1) $h\nu << kT$，Rayleigh - Jeans law (typically radio radiation)

$I_\nu(T) = \frac{2\nu^2}{c^2}kT$ ~ 线性近似

(2) $h\nu >> kT$，Wien's law

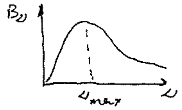
$I_\nu(T) = \frac{2h\nu^3}{c^2}e^{-h\nu/kT}$

(3) $\frac{\partial B_\nu(T)}{\partial T} > 0$



两条曲线一定不相交

(4) 黑体谱是 ν 的单峰函数　$\frac{\partial B_\nu}{\partial \nu}(\nu_{max}) = 0$



$$\frac{h\nu_{max}}{kT} = 2.82$$

$$\nu_{max}/T = 5.88\times10^{10}\ Hz\cdot K^{-1}$$

所以 ν_{max} 的位置可用来判断温度.

(5) $F = \int F_\nu d\nu = \sigma T^4$，$\sigma = \frac{2\pi^5 k^4}{15c^2h^3}$

亮温度

Brightness temperature 对于射电来说重要

For any value of I_ν, we can fit $I_\nu = B_\nu(T_b)$

Radio Astronomy (for $h\nu \ll kT$) $T_b = \frac{c^2}{2\nu^2 k} I_\nu$ →也是ν的函数

所以，例如CMB，总是可以用温度表示强度

The transfer function equation:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \Rightarrow \frac{dT_b}{d\tau_\nu} = -T_b + T$$

→ temperature of the material

有时，不能测到全波段，只能测到 ν_{max} 附近

由 ν_{max} 确定的温度 T_c，称为 color temperature

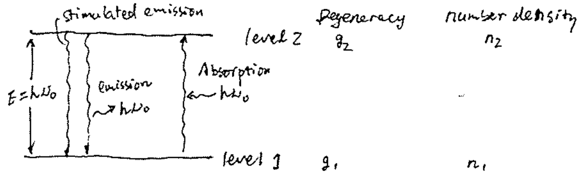
有时，只能得到辐射总量 F，利用 $F = \sigma T_{eff}^4$，称为 effective temperature.

T_b, T_{eff} depend only on the magnitude of the intensity, but T_c depend on the shape.

Einstein Coefficient

$$\frac{j_\nu}{\alpha_\nu} = S_\nu = B_\nu(T) \rightarrow j_\nu = \alpha_\nu B_\nu(T)$$

, Einstein Consider



Spontaneous emission: occurs even in the absence of radiation

(也可是自发的)

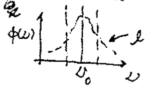
A_{21}: transition probability per unit time for spontaneous emission

$[A_{21}] = sec^{-1}$

Absorption:

probability ∝ photon density or mean intensity J_ν at ν_0 (对于密度相当)

但事实上，能级间的吸收/发射有展宽



← line profile function $\phi(\nu)$

描述多大的范围内能吸收的

Assume, if J_ν changes slow over $\Delta\nu$

$\int_0^\infty \phi(\nu)d\nu = 1$. 类似于δ-function

$\phi(\nu) \sim \delta(\nu - \nu_0)$

$B_{12}\bar{J}$ = transition probabilities per unit time for absorption

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu \xrightarrow{\phi(\nu) \approx \delta(\nu-\nu_0)} \bar{J} = J_{\nu_0}$$

Stimulated emission:

$B_{21}\bar{J}$ = transition probability per unit time for stimulated emission

由于这组系数描述原子在相关能级之间跃迁的性质，所以这组系数之间应该有相互关系

（应可以从原子物理导出，实际上 Einstein 从热力学导出）

Relations between Einstein A, B Coefficients

假如系统处于热平衡，由 Detailed balance relation:

$$\underset{\uparrow \text{ absorption rate}}{n_1 B_{12}\bar{J}} = \underset{\uparrow \text{ spontaneous emission}}{n_2 A_{21}} + \underset{\uparrow \text{ stimulated emission}}{n_2 B_{21}\bar{J}}$$

$$\Rightarrow \bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

In thermal equilibrium

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E/kT}}{g_2 e^{-(E+h\nu_0)/kT}} = \frac{g_1}{g_2} e^{h\nu_0/kT}$$

代入 $\bar{J}$ 的表达式后

$$\bar{J} = \frac{A_{21}/B_{21}}{\frac{g_1 B_{12}}{g_2 B_{21}} e^{h\nu_0/kT} - 1}$$

（从系统的热平衡得）

In thermal equilibrium

$I_\nu = B_\nu$, $J_\nu = B_\nu$, 所以 $\bar{J} = B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu_0/kT} - 1}$ (即 Black-body law)

在任意温度，任意温度都成立，因而

$$\left.\begin{aligned} g_1 B_{12} &= g_2 B_{21} \\ A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \end{aligned}\right\} \text{Einstein relations}$$

① irrelevant with T

② irrelevant / holds whether or NOT in thermal equilibrium

虽是从热平衡热力学导出，但由于是原子本身的性质，在非热平衡时也应成立.

③ 所以，为了给出 Planck function，必须有受激发射的过程

Stimulated emission is required to get Planck function.

在短波，低温时，$n_1 \gg n_2$，可不考虑受激发射，应能得到 Wien law.

Absorption and emission Coefficients:

spontaneous emission : amount of energy

$$d\mathcal{E} = j_\nu dV dt d\Omega d\nu = n_2 \cdot dV \cdot A_{21}\, dt \cdot \phi(\nu) d\nu \cdot \frac{d\Omega}{4\pi} \cdot h\nu_0$$

自发辐射是 isotropic 的

每个光子的能量

所以自发辐射系数 $j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)$

Absorption: total energy absorbed

$$h\nu_0 \cdot n_1 B_{12} \cdot \bar{J}\, dt \cdot dV = h\nu_0\, n_1 B_{12} \int \frac{d\Omega}{4\pi} \int d\nu\, \phi(\nu) I_\nu \times dt\, dV$$

考虑受激辐射 修正为 $n_1 B_{12} - n_2 B_{21}$

Stimulated emission

$n_1 B_{12} - n_2 B_{21}$

$c\, dt\, dA = ds\, dA$

$$\alpha_\nu I_\nu dt\, dA\, ds\, d\Omega\, d\nu = \text{上面的}\ dV dt \cdot d\Omega \cdot d\nu \cdot \frac{h\nu_0}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu) I_\nu$$

最终可以得到 $\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) (n_1 B_{12} - n_2 B_{21})$

由于 A、B 系数与热平衡无关. α_ν, j_ν 与 A、B 的关系是一般的. 所以这里的结果总是正确的

Source function $S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$

代入AB 关系 $= \frac{2h\nu^3/c^2}{\frac{g_2}{g_1}\frac{n_1}{n_2} - 1}$

这与 Planck 函数差一些

在 thermal equilibrium, $\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_0/kT}$

代入 S_ν 的表达式 即有 $S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu_0/kT} - 1}$ (Planck function)

可将此关系写为

Generalized Birchhoff's law

在热平衡下. 应当有 $S_\nu = B_\nu$

Thermal emission (LTE)

if $\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_0/kT}$, then $S_\nu = B_\nu(T)$

No thermal emission, $n_1/n_2 \neq \frac{g_1}{g_2} e^{h\nu_0/kT}$

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{n_1 g_2}{n_2 g_1} - 1} \neq B_\nu(T)$$

所以. 是否热平衡决定了 Birchhoff's law 是否成立

Normally, $\frac{n_1}{g_1} > \frac{n_2}{g_2}$, if existing inverted population $\frac{n_1}{g_1} < \frac{n_2}{g_2}$. 此时 $\alpha_\nu \propto (n_1 B_{12} - n_2 B_{21}) < 0$

粒子数反转

Absorption Stimulated emission

So we have $\alpha_\nu \begin{cases} > 0 & \text{normal} \\ < 0 & \text{inverted population (masers)} \end{cases}$

Classical Theory of Radiative Processes

HW2
R&L 1.7 1.9
2.1 2.2 2.4

- Maxwell's equations (Gaussian units)

$\nabla\cdot\vec{D} = 4\pi\rho$ $\quad \nabla\cdot\vec{B}=0$ ↙无磁单极子

$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\quad \nabla\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ (*4)

↑ 法拉第定律

偏微分
(线性、耦合、一阶方程组)
why?

$\vec{D}=\varepsilon\vec{E}$ $\quad \vec{B}=\mu\vec{H}$

介电常数 dielectric constant $\quad$ 磁导率 magnetic permeability

微观描述 $\rho = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i$

$\vec{j} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i\cdot\vec{v}_i$

Lorentz force $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})$

$\vec{f} = \rho\vec{E} + \frac{\vec{J}}{c}\times\vec{B}$

↑ force density

energy Conservation

$$\vec{j}\cdot\vec{E} + \frac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S}$$

EM momentum per unit volume

$$\vec{P}_{em} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{\vec{S}}{c^2}$$

为什么 $\vec{P}_{em}$ 和 $\vec{S}$ 是同向的呢？因为光是直线传播的？

energy-momentum tensor

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & \text{energy flux} \\ \hline T_{0i} & \text{momentum flux} \\ \parallel & \\ T_{i0} & \end{array}\right)$$

momentum Conservation

$$\frac{\partial\vec{P}_{em}}{\partial t} - \nabla\cdot\overleftrightarrow{\sigma} + \rho\vec{E} + \vec{j}\times\vec{B} = 0$$

用四维方程可以写得简单一些 $\partial_\mu T^{\mu\nu} = 0$

charge conservation $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$

(可从 *4 求散度导出)

力学功率密度 $\frac{du_{mech}}{dt} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i \vec{F}_i\cdot\vec{v}_i$

$= \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i(\vec{E} + \frac{\vec{v}_i}{c}\times\vec{B})\cdot\vec{v}_i$

$= \vec{j}\cdot\vec{E}$

u_{mech} = mechanical energy per unit volume

由 $\vec{j}\cdot\vec{E} + \frac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S}$

($\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ Poynting Vector
~electromagnetic flux vector)

由于 $\vec{j}\cdot\vec{E} = \frac{1}{4\pi}\left(c\left(\nabla\times\vec{H}\right)\cdot\vec{E} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)$

↓

$\nabla\cdot(\vec{E}\times\vec{H})$
$= -\vec{E}\cdot(\nabla\times\vec{H})$
$+ \vec{H}\cdot(\nabla\times\vec{E})$
↑ 代入 Faraday law

$$-\nabla\cdot\left(\frac{c}{4\pi}\vec{E}\times\vec{H}\right) = \vec{j}\cdot\vec{E} + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$$

↓ if ϵ and μ indep of time
use $E^2 = \vec{E}\cdot\vec{E}$, $\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} = \frac{1}{2}\frac{\partial(E^2)}{\partial t}$

$$\underbrace{\vec{j}\cdot\vec{E}}_{\frac{\partial u_{mech}}{\partial t}} + \frac{1}{8\pi}\frac{\partial}{\partial t}\underbrace{\left(\epsilon E^2 + \frac{B^2}{\mu}\right)}_{u_{em}} = -\nabla\cdot\left(\frac{c}{4\pi}\vec{E}\times\vec{H}\right)$$

EM energy density $\quad$ EM energy flux $\vec{S}$

$$u_{em} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right)$$

03/12/2018 Week3

- Plane waves

In vacuum, $\rho=0$, $\vec{j}=0$, $\epsilon=\mu=1$

$$\begin{cases}\nabla\cdot\vec{E}=0,\ \nabla\times\vec{B}=0 \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\ \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\end{cases}$$

(linear, coupled, first-order partial differential equations)

$\Big\downarrow$ generally

linear 2nd-order ordinary differential equations

$\Big\downarrow$

wave solutions (e.g. G.R.)

it is invariant under $\vec{E}\to\vec{B}$, $\vec{B}\to-\vec{E}$
(有什么含义)

平面波解:

$$\nabla\times\nabla\times\vec{E} = -\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \rightarrow \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t^2} = 0$$

$\parallel$
$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$
$\parallel$
0

同样 $\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial\vec{B}}{\partial t^2} = 0$

Plane wave solution: $\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{B} = \hat{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

(E_0, B_0 are complex → 实际物理量的实部
monochronic, linearly polarized)
单色平面波

传播方向: $\vec{k} = k\hat{n}$, $\hat{n}$ = direction of propagation

ω = circular frequency
圆频率

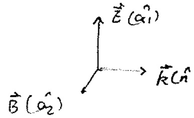
实际上: E is the real part of $\vec{E}$!

计算: $\nabla\to i\vec{k}$, $\frac{\partial}{\partial t}\to -i\omega$

将平面波解代入 Maxwell eq.

① $i\vec{k}\cdot\hat{a}_1 E_0 = 0$, ② $i\vec{k}\cdot\hat{a}_2 B_0 = 0$

③ $i\vec{k}\times\hat{a}_1 E_0 = \frac{i}{c}\omega\hat{a}_2 B_0$, ④ $i\vec{k}\times\hat{a}_2 B_0 = -\frac{i}{c}\omega\hat{a}_1 E_0$



①② → $\vec{k}\perp\hat{a}_1$, $\vec{k}\perp\hat{a}_2$
+
③④ → $\hat{k}(\hat{n})$, $\hat{a}_1$, $\hat{a}_2$ 构成正交 set.

还可以得 $kE_0 = \frac{\omega}{c}B_0$ $\quad kB_0 = \frac{\omega}{c}E_0$ ⇒ $E_0 = \left(\frac{\omega}{kc}\right)^2 E_0$

所以对应的频率 $\omega = ck$, and $E_0 = B_0$ (B, E 同频, 同速)

Phase velocity $v_{ph} = \frac{\omega}{k} = c$,
Group velocity $v_g = \frac{\partial\omega}{\partial k} = c$ } identical

↳ 信息传播速度. 在介质中也 < c.

能量密度 $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$E_0 = B_0$

平均值才有意义 $\langle\vec{S}\rangle = \frac{c}{4\pi}\left\langle Re\vec{E}\times Re\vec{B}\right\rangle = \frac{c}{8\pi}|E_0|^2$

$\langle u\rangle = \frac{1}{8\pi}\left[\left\langle Re E\cdot Re E\right\rangle + \left\langle Re B\cdot Re B\right\rangle\right]$

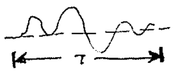
$= \frac{1}{16\pi}\left[|E_0|^2 + |B_0|^2\right] = \frac{1}{8\pi}|E_0|^2 = \frac{\langle S\rangle}{c}$

定理: $A(t) = Ae^{i\omega t}$
$B(t) = Be^{i\omega t}$ (同频)
$\Rightarrow \langle Re A(t) Re B(t)\rangle$
$= \frac{1}{2}Re(AB^*)$
$= \frac{1}{2}Re(A^*B)$

上面都是在真空中而言; 若在介质中, ϵ, μ 都可以是 ω 的函数.
平面波

- 实际的波不可能是单色平面波，或是有空间宽度
  或是有时间宽度（对应光子的测量有不确定性）

$\Delta\omega\Delta\tau > 1$, for any wave theory of light.



谱的分解由 Maxwell 方程的线性性决定.

$\vec{E}(t)$ can only have two $_{(\text{independent})}$ components $\perp \vec{k}$

联系着 $m_\gamma = 0$ （光子的质量为 0） 如果质量是非零，则一定可以变换到静止参考系

且 spin $= 1$ 在此坐标系下没有波动，可以有三个分量.

联系着 (helicity)

parity conservation for EM interaction.（左右对称性）

既然光有两个独立分量，可以讨论一个，然后叠加

光场的傅利叶分解

$$\tilde{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)\, e^{i\omega t} dt$$

$$\underset{\text{real}}{\underset{\uparrow}{E(t)}} = \int_{-\infty}^{\infty} \underset{\text{complex}}{\underset{\uparrow}{\tilde{E}(\omega)}}\, e^{-i\omega t} dt$$

$$\Downarrow$$

$$\tilde{E}(-\omega) = \overset{*}{E}(\omega)$$

i.e. negative frequencies are NOT indep modes

等价. 可以只讨论正频率，然后叠加

（一般不讨论光的负频率）

能流 $S = \frac{dW}{dt\,dA} = \frac{c}{4\pi}E^2(t)$

那么，在频率空间 $\frac{dW}{d\omega\, dA} = ?$ energy per unit area per unit frequency. for the entire pulse

对整个波列有意义

利用 Parseval's thm. $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\overset{*}{g}(t)\, dt = \int_{-\infty}^{\infty}\tilde{f}(\omega)\,\overset{*}{\tilde{g}}(\omega)\, d\omega$

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\, dt = \frac{c}{4\pi}2\pi\int_{-\infty}^{\infty}|\tilde{E}(\omega)|^2 d\omega = c\int_0^{\infty}|\tilde{E}(\omega)|^2 d\omega$$

$$\frac{dW}{dA\, d\omega} = c\left|\tilde{E}(\omega)\right|^2 \quad \text{(for the entire pulse)}$$

"power spectrum" 称为功率谱

相比于宇宙学的功率谱 $P(k) = |\tilde{\delta}(k)|^2$，这才是真正的功率谱

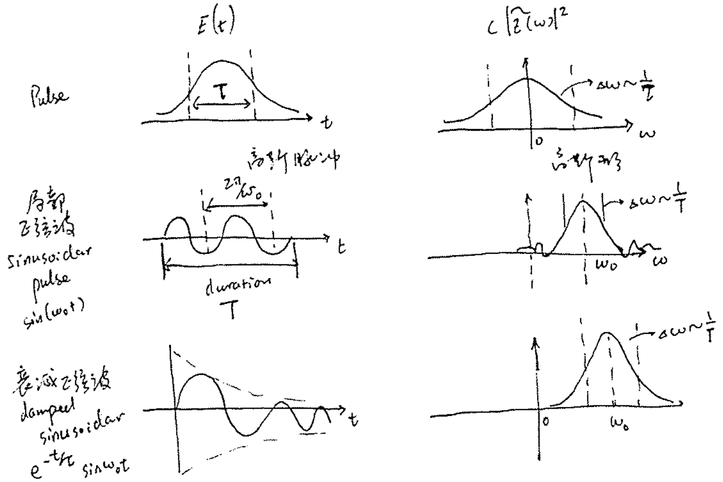
能否定义 $\frac{dW}{dt\,dA\,d\omega}$? 不能，因为与 $\Delta\omega\Delta\tau > 1$ 违背.

BUT, formally,

对某种形式的波 在某段时间内有稳定 特征 $\frac{dW}{dt\,dA\,d\omega} := \frac{1}{T}\frac{dW}{dA\,d\omega} = \frac{c}{T}\left|\tilde{E}(\omega)\right|^2$, if a long signal has same period over its entire length.

此时有 $\frac{dW}{dt\,dA\,d\omega} = c\lim_{T\to\infty}\frac{1}{T}\left|\tilde{E}_T(\omega)\right|^2$ indep of T for large T.

典型的光场的 Fourier 变换



generally, for a pulse with time extent T

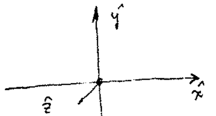
→ the width $\Delta\omega \doteq \frac{1}{T}$ of the first feature

(sinusoidal) time dependence

→ spectrum concentrated at ω_0

- Polarization 极化

$\vec{E} = \hat{e}_1 E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ linearly polarized i.e. $\vec{E}(t) \parallel \hat{e}_1$



光沿 $\hat{z}$ 方向传播

由于光场有①两个独立分量，还可以有另一个分量，At $\vec{r}=0$，看对时间的依赖

$$\hat{x}E_1 e^{i(\omega t)} + \hat{y}E_2 e^{i(-\omega t)} = \vec{E}_0 e^{-i\omega t}$$

E_x, E_y 是 complex，且

可以有②相对的相位差，e.g. $E_1 = \mathcal{E}_1 e^{i\phi_1}$ $E_2 = \mathcal{E}_2 e^{i\phi_2}$

$$E_x(t) = \mathrm{Re}\left(\mathcal{E}_1 e^{i\phi_1} e^{-i\omega t}\right) = \mathcal{E}_1\cos(\omega t - \phi_1)$$

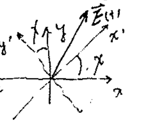
$$E_y(t) = \mathrm{Re}\left(\mathcal{E}_2 e^{i\phi_2} e^{-i\omega t}\right) = \mathcal{E}_2\cos(\omega t - \phi_2)$$

实的电场 $\vec{E}(t) = \mathcal{E}_1\cos(\omega t - \phi_1)\hat{x} + \mathcal{E}_2\cos(\omega t - \phi_2)\hat{y}$

$$= \cos\omega t\left(\mathcal{E}_1\cos\phi_1\hat{x} + \mathcal{E}_2\cos\phi_2\hat{y}\right) + \sin\omega t\left(\mathcal{E}_1\sin\phi_1\hat{x} + \mathcal{E}_2\sin\phi_2\hat{y}\right)$$

可以定义 $\mathcal{E}_0$ 和 β, χ s.t. $\mathcal{E}_0\cos\beta\cos\chi = \mathcal{E}_1\cos\phi_1$ $\mathcal{E}_0\sin\beta\sin\chi = \mathcal{E}_1\sin\phi_1$

$\mathcal{E}_0\cos\beta\sin\chi = \mathcal{E}_2\cos\phi_2$ $-\mathcal{E}_0\sin\beta\cos\chi = \mathcal{E}_2\sin\phi_2$

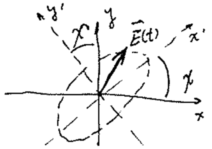


$\vec{E}(t) = \cos\omega t\ \mathcal{E}_0\cos\beta\ \hat{x}' - \sin\omega t * \mathcal{E}_0\sin\beta\ \hat{y}'$（沿着新的坐标轴分解）

立即就有
$$\left(\frac{E_{x'}}{\cos\beta\cdot\mathcal{E}_0}\right)^2+\left(\frac{E_{y'}}{\mathcal{E}_0\sin\beta}\right)^2=1$$

这是一个主轴为 $\mathcal{E}_0\cos\beta$, $\mathcal{E}_0\sin\beta$ 的椭圆 (ellipse w/ principal axes)

所以，当 E_1, E_2 存在相位差时，一般来讲 E 的端迹为椭圆.



- linear polarized $\beta=0,\ \pm\frac{\pi}{2}$
- circular polarized $\beta=\pm\frac{\pi}{4}$
- general case, elliptically polarized

$0<\beta\le\frac{\pi}{2}$ clock-wise = right-handed polarization = negative helicity

$-\frac{\pi}{2}\le\beta<0$ counter clockwise = left-handed polarization = positive helicity

如果相位完全随机 (ϕ_1,ϕ_2无关联), 则是无偏振的.

之前是用四参数 $\left(\mathcal{E}_1,\mathcal{E}_2,\underbrace{\phi_1,\phi_2}_{\text{相差才有意义}}\right)\leftrightarrow$ 现在用三参数 $(\mathcal{E}_0,\beta,\chi)$

与物理量关联不紧密，且计算不方便
引入Stock's parameters for monochromatic waves.

$I\equiv\mathcal{E}_1^2+\mathcal{E}_2^2=\mathcal{E}_0^2$ ∝ total energy flux

$$\begin{cases}Q\equiv\mathcal{E}_1^2-\mathcal{E}_2^2=\mathcal{E}_0^2\cos2\beta\cos2\chi\\ U\equiv2\mathcal{E}_1\mathcal{E}_2\cos(\phi_1-\phi_2)=\mathcal{E}_0^2\cos2\beta\sin2\chi\end{cases}$$
orientation of the ellipse

$V\equiv2\mathcal{E}_1\mathcal{E}_2\sin(\phi_1-\phi_2)=\mathcal{E}_0^2\sin2\beta$

$I^2=Q^2+U^2+V^2$ (只有3个量是独立的)

β. V 描述椭圆的圆度
circularity parameter

$$\begin{cases}\mathcal{E}_0=\sqrt{I}\\ \sin2\beta=\frac{V}{I}\\ \tan2\chi=\frac{U}{Q}\end{cases}$$

- Electromagnetic potentials

vector potential $\vec{A}=\nabla\times\vec{B}$, s.t. $\nabla\cdot\vec{B}=0$

scalar potential ϕ. s.t. $\nabla\times\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{A}=0$

$\nabla\times\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)\overset{\downarrow}{=}0\nearrow\vec{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

现在，将泊松方程写为 $\nabla\cdot\vec{E}=4\pi\rho\Rightarrow\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A})=-4\pi\rho$

将位移电流方程写为 $\nabla\times\vec{H}=\nabla\times(\nabla\times\vec{A})=-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})\Rightarrow\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\left(\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)=-\frac{4\pi}{c}\vec{j}$

Gauge invariance

$\vec{A}\to\vec{A}+\nabla\psi$ 保证 $\vec{B}\to\vec{B}$

$\phi\to\phi-\frac{1}{c}\frac{\partial\psi}{\partial t}$ 保证 $\vec{E}\to\vec{E}$

所以. $\vec{A},\phi$ 的选择有一定的自由度

So. free to chose the gauge

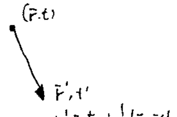
choose Lorentz gauge $\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0\longrightarrow\begin{cases}\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho\\ \nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi}{c}\vec{j}\end{cases}$

(是个Lorentz 不变的条件.
而其它 gauge condition 不是 Lorentz 不变的)

上述方程的解
$$\phi(\vec{r},t)=\int\frac{d^3r'[\rho]}{|\vec{r}-\vec{r}'|}$$
$$\vec{A}(\vec{r},t)=\int\frac{d^3r'[\vec{j}]}{|\vec{r}-\vec{r}'|}$$
称为 retarded potential
(推迟势)

其中 $[\rho]=\rho\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)$

$[\vec{j}]=\vec{j}\left(\vec{r}',t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$



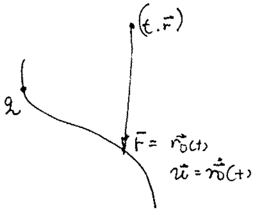
$t'=t+\frac{1}{c}|\vec{r}-\vec{r}'|$

考虑Green函数解法?

例3: Radiation from moving charges

$\rho=q\delta(\vec{r}-\vec{r}_0(t))$

$\vec{j}=q\vec{u}(t)\delta(\vec{r}-\vec{r}_0(t))$



因为 $\phi(\vec{r},t)=\int d^3r'\rho\left(\vec{r}',t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)/|\vec{r}-\vec{r}'|$

$=\int d^3r'\underbrace{\int dt'\,\delta\left(t'-t+\frac{1}{c}|\vec{r}-\vec{r}'|\right)}_{\text{积掉一项，结果为1}}\frac{\rho\left(\vec{r}',t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|}\quad\nearrow q\delta\left(\vec{r}'-\vec{r}_0(\overset{\uparrow}{t'})\right)$ $t-\frac{1}{c}|\vec{r}-\vec{r}'|$

$=q\int dt'\frac{\delta\left(t'-t+\frac{1}{c}|\vec{r}-\vec{r}_0(t')|\right)}{|\vec{r}-\vec{r}_0(t')|}=q\int dt'\,R^{-1}(t')\,\delta\left(t'-t+\frac{R(t')}{c}\right)$

这里 $\vec{R}(t')=\vec{r}-\vec{r}_0(t')$

类似的 $\vec{A}(\vec{r},t)=\frac{q}{c}\int dt'\,\vec{u}(t')R^{-1}(t')\delta\left(t'-t+\frac{R(t')}{c}\right)$

定义 $t''=t'-t+\frac{R(t')}{c}$, $dt''=dt'+\frac{\dot{R}(t')}{c}dt'=\left(1+\frac{\dot{R}(t')}{c}\right)dt'$

利用关系 $R^2(t')=\vec{R}(t')\cdot\vec{R}(t')$

$2R(t')\dot{R}(t')=2\vec{R}(t')\cdot\dot{\vec{R}}(t')=-2\vec{R}(t')\dot{\vec{r}}_0(t')$

$=-2\vec{R}(t')\vec{u}(t')$

$=-2R(t')\hat{n}\cdot\vec{u}(t')$, 这里 $\hat{n}(t)=\frac{\vec{R}(t)}{R(t)}$

最后 $\dot{R}(t')=-\hat{n}(t')\vec{u}(t')$

所以 $dt''=\kappa(t')dt'$, 其中 $\kappa(t')\equiv1-\frac{1}{c}\hat{n}(t')\cdot\vec{u}(t')$

所以. Scalar potential

$\phi(\vec{r},t)=q\int dt''\kappa^{-1}(t')R^{-1}(t')\delta(t'')$ when $t''=0$, $t'=t_{retard}$

s.t. $c|(t_{ret}-t)|=R(t_{ret})$

$=\frac{q}{\kappa(t_{ret})R(t_{ret})}=\left[\frac{q}{\kappa R}\right]$ (Liénard-Wiechart potential)

同样的 $\vec{A}(\vec{r},t)=\frac{q\vec{u}(t_{ret})}{c\kappa(t_{ret})R(t_{ret})}=\left[\frac{q\vec{u}}{\kappa R}\right]$

$[\cdots]$ 指的是取 t_{ret} 时的值

κ 是一个修正项. 来源于 dt'' 和 dt' 的关系,
代表时间的

Note 1. $\phi\sim\frac{1}{R}$ static case $E\sim\frac{1}{R^2}$

Now $\phi\sim1/R$ at t_{ret}. implicitly depend on R

E falls slower than $1/R^2$.

i.e. $E\neq0$ at $r=\pm\infty$ (EM waves cases)

2. $\kappa=1-\frac{\vec{n}\cdot\vec{u}}{c}$ $\phi\sim1/\kappa$ (极端相对论的情况) (就是同步辐射产生 beam, 称"beaming effect")

when $\beta=u/c\to1$ $\kappa\to0$ along $\vec{u}$. $\kappa\to1\perp\vec{u}$, ϕ is concentrated into a narrow cone along $\vec{u}$.

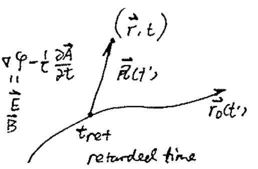
0319/2018 Week 4

HW
R.L.
3.1 3.2 3.3

Summary: define $\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')$

$c(t - t_{ret}) = R(t_{ret})$

Solution $\phi(\vec{r},t) = \left[\frac{q}{\kappa R}\right]$ $\quad \vec{A}(\vec{r},t) = \left[\frac{q\vec{u}}{\kappa R}\right] \Rightarrow$ to get $\overset{-\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}{\vec{E}}$, $\vec{B}$



其中 κ 是所谓 Doppler 因子, $\kappa(t') = 1 - \frac{\hat{n}(t')}{c}\cdot\vec{u}(t') = 1 - \hat{n}\cdot\vec{\beta}$, $\quad \vec{\beta} = \vec{u}/c$

[···] denotes the evaluation at tret

$\hat{n}(t') = \vec{R}(t') / |\vec{R}(t')|$

电场强度不一定是波. 所以
不能直接称为 Doppler 因子

$$\vec{E}(\vec{r},t) = q\underbrace{\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]}_{\text{velocity field (dept. on } \vec{\beta})} + \frac{q}{c}\underbrace{\left[\frac{\hat{n}}{\kappa^3 R}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\right]}_{\text{acceleration field (dept. on } \vec{\beta}, \dot{\vec{\beta}})}$$

acceleration field → $E_a \propto \frac{1}{R}$, $S \propto \frac{1}{R^2}$, $P \propto \int \vec{S}\cdot d\vec{A} \propto$ const

⇒ RADIATION FIELD

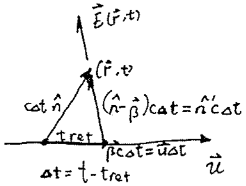
$\vec{E}(\vec{r},t) \perp \hat{n}(t_{ret})$

$\vec{B}(\vec{r},t) = \hat{n} \times \vec{E}(\vec{r},t)$

$\kappa(t') = 1 - \frac{1}{c}\hat{n}(t')\cdot\vec{u}(t')$

Case 1. particles move with constant velocity

$$\vec{E}(\vec{r},t) = \frac{q(1-\beta^2)\hat{n}'}{\kappa^3 R^2}, \quad \hat{n}' = \hat{n} - \vec{\beta} \ (\text{Not a unit vector})$$

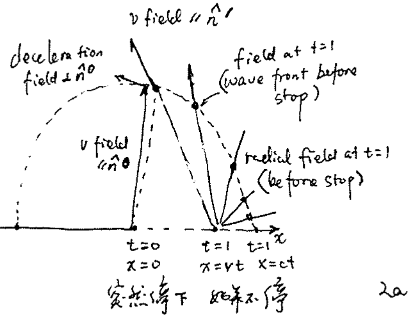


- $\vec{E}(\vec{r},t) \parallel \hat{n}'$, i.e. 'radial' to $\hat{n}'$, along the line toward the current position of the charge.
- When $\beta \ll 1$, $\vec{E} \to$ Coulomb's law
- $E \propto \frac{1}{R^2}$, $S \propto |E^2| \propto \frac{1}{R^4}$, $P \propto \int \vec{S}\cdot d\vec{A} \propto \frac{1}{R^2} \xrightarrow[R\to\infty]{} 0$

  i.e. No radiation at this case. (away to infinity)

  即, 有电场, 但没有辐射

Example. Charged particle moving at uniform velocity (in +x direction) is stopped at $x=0$ and $t=0$.



$\nabla\cdot\vec{E} = 0$, No Source, $\vec{E}$ preserve flux

power spectrum $\frac{dW}{d\omega dA} = c|\tilde{E}(\omega)|^2$

$dA = R^2 d\Omega$

$$\Rightarrow \frac{dW}{d\omega d\Omega} = c\,|R\hat{E}(\omega)|^2, \quad \hat{E}(\omega) = \frac{1}{2\pi}\int \vec{E}(t)e^{i\omega t}dt$$

$$= \frac{c}{4\pi^2}\left|\int R\vec{E}(t)e^{i\omega t}dt\right|^2 = \frac{q^2}{4\pi^2 c}\left|\int [\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\kappa^{-3}]_{t'}\, e^{i\omega t}dt\right|^2$$

对初始时间积分

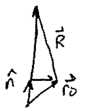
because $t' = t - \frac{R(t')}{c}$, $t = t' + \frac{R(t')}{c}$, $\quad dt = dt'\left(1 + \frac{\dot{R}(t')}{c}\right) = \kappa(t')dt'$

$R^2 = \vec{R}\cdot\vec{R} \to R\dot{R} = \vec{R}\cdot\dot{\vec{R}} = -\vec{R}\cdot\vec{u} \to \dot{R} = -\hat{n}\cdot\vec{u}$

$e^{i\omega t}$ can be written to $e^{i\omega t} = e^{i\omega(t' + \frac{R(t')}{c})}$

For large distance $|\vec{r}_0| \ll |\vec{r}|$ $\quad R(t') \cong |\vec{r}| - \hat{n}\cdot\vec{r}_0(t')$

range of motion of charge



So $e^{i\omega t} = e^{i\omega(t' + \frac{R(t')}{c})} = e^{i\omega(t' + \frac{|\vec{r}|}{c} - \frac{\hat{n}\cdot\vec{r}_0(t')}{c})}$

$= e^{i\omega\frac{|\vec{r}|}{c}} \times e^{i\omega(t' - \frac{\hat{n}}{c}\cdot\vec{r}_0(t'))}$

Substiture $e^{i\omega t}$, dt into $\frac{dW}{d\omega d\Omega}$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\kappa^{-2}e^{i\omega(t' - \frac{\hat{n}}{c}\cdot\vec{r}_0(t'))}dt'\right|^2$$

where $e^{i\omega\frac{|\vec{r}|}{c}}$ is canceled after $|\,|^2$

just a phase!

Use identity $\frac{d}{dt'}[\kappa^{-1}\hat{n}\times(\hat{n}\times\vec{\beta})] = \hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\kappa^{-2}$ ... (*1)

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\text{surface term} - \int \kappa^{-1}\hat{n}\times(\hat{n}\times\vec{\beta})\, i\omega\kappa\, e^{i\omega(t' - \frac{\hat{n}}{c}\vec{r}_0(t'))}dt'\right|^2$$

surface term: 0 at $t \to \pm\infty$
分部积分之后也有

$\kappa = \frac{d}{dt'}\left(t' - \frac{\hat{n}\cdot\vec{r}_0(t')}{c}\right)$

$$= \frac{q^2\omega^2}{4\pi^2 c}\left|\int \hat{n}\times(\hat{n}\times\vec{\beta})\exp[i\omega(t' - \hat{n}\cdot\vec{r}_0(t')/c)]dt'\right|^2$$

Proof for (*1)

$\hat{n}$ fixed, $\kappa(t')$, $\vec{\beta}(t')$

$$\text{LHS} = \kappa^{-2}\left[(\hat{n}\cdot\dot{\vec{\beta}})\hat{n}\times(\hat{n}\times\vec{\beta}) + \kappa\,\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right] = \kappa^{-2}\{(\hat{n}\cdot\dot{\vec{\beta}})[(\hat{n}\cdot\vec{\beta})\hat{n} - \vec{\beta}] + (1-\hat{n}\cdot\vec{\beta})[(\hat{n}\cdot\dot{\vec{\beta}})\hat{n} - \dot{\vec{\beta}}]\}$$

$$= \kappa^{-2}\{(\hat{n}\cdot\dot{\vec{\beta}})[(\hat{n}\cdot\vec{\beta})\hat{n} - \vec{\beta}] + \cdots\} = \kappa^{-2}\{(\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}\cdot\vec{\beta}) - \dot{\vec{\beta}}(1-\hat{n}\cdot\vec{\beta})\} = \text{RHS}$$

$\cdots + (\hat{n}\cdot\dot{\vec{\beta}})\hat{n} - (\vec{n}\cdot\vec{\beta})(\hat{n}\cdot\dot{\vec{\beta}})\hat{n} + \cdots$

$\cdots - \dot{\vec{\beta}} + (\hat{n}\cdot\vec{\beta})\dot{\vec{\beta}}$

$t' \simeq t_{ret}$

讲3: Radiation from non-relativistic system of particles.

$$\frac{E_{rad}}{E_{vel}} \cong \frac{R\dot{u}}{c^2} \cong \frac{Ru\nu}{c^2} = \beta\frac{R}{\lambda}, \quad \beta = \frac{u}{c}$$

If there exists a characteristic frequency of oscillation ν, $\dot{u} \sim u\nu$

"Near Zone" $\quad R \lesssim \lambda$, $E_{vel} \gg E_{rad}$, $\frac{E_{vel}}{E_{rad}} \propto \beta^{-1}$

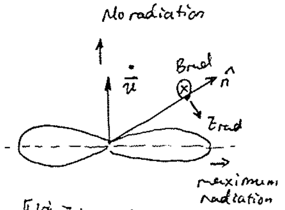
"Far Zone" $\quad R \gg \lambda\beta^{-1}$, $E_{rad} \gg E_{vel}$ $\quad \frac{E_{rad}}{E_{vel}} \propto R$

Larmor's Formula: for a single accelerated charge

$$\vec{E}_{rad} = \frac{q}{Rc^2}\underbrace{\hat{n}\times(\hat{n}\times\dot{\vec{u}})}_{\dot{u}\sin\theta}$$

for non-relativistic case
$\kappa \to 1$, $|\vec{\beta}| \ll 1$, $\vec{\beta}\times\dot{\vec{\beta}} \to 0$

$\vec{B}_{rad} = \hat{n}\times\vec{E}_{rad}$



辐射电场 → linear polarized
因为 $\hat{n}$, $\dot{\vec{u}}$ 决定了 $\vec{E}_{rad}$ 的方向

$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$, $\quad S = \frac{c}{4\pi}\left(\frac{q}{Rc^2}\right)^2[\dot{u}\sin\theta]^2 = \frac{dW}{dt\cdot dA} = \frac{dW}{dt\cdot d\Omega\cdot R^2}$

note: $\int_0^{\frac{\pi}{2}}\sin^3\theta\, d\theta = \frac{2}{3}$

$\frac{dW}{dt\cdot d\Omega} = \frac{c}{4\pi}\left(\frac{q}{c^2}\dot{u}\sin\theta\right)^2 = \frac{q^2\dot{u}^2\sin^2\theta}{4\pi c^3}$

$\frac{dW}{dt} = \int\frac{dP}{d\Omega}d\Omega = \frac{2q^2\dot{u}^2}{3c^3}$ (Larmor's Formula) $\quad \frac{dP}{d\Omega} \propto q^2, \dot{u}^2, \sin^2\theta$, $\quad P \propto (\underset{\sim}{q\dot{u}})^2$

辐射功率

偶极矩的导数

Dipole approximation

忽略各粒子的 t_{ret} 的差别，以某个中心的 t_{ret} 代替

引入 dipole moment $\vec{d} = \sum_i q_i \vec{r}_i$

The radiation field

$$\vec{E}_{rad} = \sum_i \frac{q_i}{c^2} \frac{\hat{n}\times\hat{n}\times\dot{\vec{u}}_i}{R_i} \quad (\text{at } t_{ret})$$

for 'far zone' radiation $|R_i - R_0|/R_0 \ll 1$, $R_i \to R_0$

$$\vec{E}^{(t)}_{rad} = \frac{\hat{n}\times(\hat{n}\times\ddot{\vec{d}})}{c^2 R_0}$$ (like single particle $qr \to d$)

The radiation power

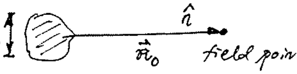
$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\theta$$

$$P = \frac{2|\ddot{\vec{d}}|^2}{3c^3}$$ (dipole approximation)

给定 $(t, \vec{r})$，不同的粒子的 $t_{retarded}$ 有差别

But for the case $\tau \gg \frac{L}{c}$ → L = typical size of system

τ: typical time scale for changes in $\vec{E}_{rad}$



We may neglect the difference in t_{ret}.

① $\tau \sim \frac{1}{\nu} \sim \lambda/c$

The condition can be written $\lambda \gg L$

② $\tau \sim \frac{l}{u}$. (l → orbit, u → velocity) $\quad \frac{l}{u} \gg \frac{L}{c} \to \frac{u}{c} \ll \frac{l}{L} < 1$

The condition can be written $u \ll c$ (non-relativistic)

The dipole can be decomposed to Fourier Component

$$d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\tilde{d}(\omega)\, d\omega$$

$$\ddot{d}(t) = \int_{-\infty}^{\infty} -\omega^2 \tilde{d}(\omega) e^{-i\omega t}\, d\omega \leftarrow \mathcal{F}[\ddot{d}(t)]$$

So as the field $\tilde{E}_{rad}(\omega) = \frac{\sin\theta^{2-1}}{c^2R_0}\mathcal{F}[\ddot{d}(t)] = -\frac{\omega^2}{c^2R_0}\tilde{d}(\omega)\sin\theta^{2-1}$

$$\frac{dW}{d\omega d\Omega} = c^{\overset{-1}{2}} R_0^2 |\tilde{E}_{rad}(\omega)|^2 = \frac{1}{c^3}\omega^4 |\tilde{d}(\omega)|^2 \sin^2\theta$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 \qquad \frac{8}{3}\pi$$

偶极辐射是可以看成来自辐射的最低阶近似，因而简单而有用.

例如：如果有相隔不远的两个偶极，他们之间的 t_{ret} 是存在差别的（相位差），要叠加先场，然后求 P.

Thomson Scattering (Electron Scattering)

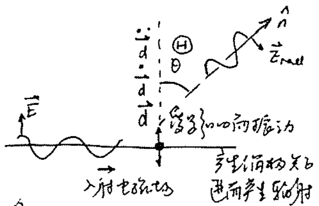
A free charge radiates in response to an incident EM wave in a Non-relativistic limit.

Suppose $\vec{E} = \hat{\epsilon} E_0 e^{i(\vec{k}\cdot\vec{r} - \omega_0 t)}$ is the incident field, the equation of motion

$$m\ddot{\vec{r}} = \vec{F} = e\hat{\epsilon} E_0 \sin\omega_0 t \quad (\text{at } \vec{r} = \vec{0})$$

$$\vec{d} = e\vec{r},\ \ddot{\vec{d}} = e\ddot{\vec{r}} = \frac{e^2E_0}{m}\hat{\epsilon}\sin\omega_0 t$$

$$\vec{d}(t) = -\left(\frac{e^2E_0}{m\omega_0^2}\right)\sin\omega_0 t\,\hat{\epsilon} = -d_0\sin\omega_0 t$$ where $\vec{d}_0 = \frac{e^2E_0}{m\omega_0^2}\hat{\epsilon}$



最简单的 Thomson Scattering (electron scattering)

The incident flux $\langle S \rangle = \frac{c}{8\pi}E_0^2$

The Scattering EM field

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\frac{e^2E_0^{-1}}{m}\right)^2 \sin^2\omega_0 t \sin^2\Theta$$

or $$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^4E_0^2}{8\pi m^2c^3}\sin^2\Theta$$

or total Power $$\langle P \rangle = \frac{e^4E_0^2}{3m^2c^3}$$

Define the differential cross section （微分）散射截面

$$\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{\langle S\rangle} = \frac{e^4\sin^2\Theta}{m^2c^4} = r_0^2\sin^2\Theta,$$

↑ the probability of the incident EM wave being scattered by the point source charge.

where $r_0 = \frac{e^2}{mc^2}$ denote the 'size' of the pointer charge ($\frac{e^2}{r_0} = mc^2$)

The total cross section $\sigma = \frac{8\pi}{3}r_0^2$. If r_0^2 is larger, so is the σ.

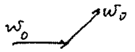
So for electron and proton, $r_0/r_0' \sim 10^3$, $\sigma/\sigma' \sim 10^6$

在CMB的辐射中，电子的散射远比质子强，后者往往被忽略.

$r_0 \sim 2.82\times10^{-13}$ cm

$\sigma_T = 0.665\times10^{-24}$ cm^2

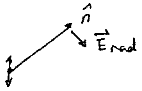
Note: ① 散射功率和散射截面都与入射频率无关.
($\frac{dP}{d\Omega}$ — 散射功率, σ — 散射截面, frequency — 频率)



This is elastic scattering.

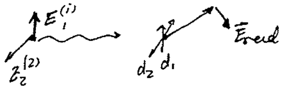
- This is valid for sufficient low ω, such that $h\nu < mc^2 = 0.511$ MeV for electron
  (→ non relativistic)
- Valid for sufficiently low intensity of radiation field.
  Otherwise the charge may move(s) relativisticly, the dipole approx fails.

②



散射波是 linearly polarized (if incident wave is linearly polarized)

如果入射波是非极化的



$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{polarized}(\theta) + \left(\frac{d\sigma}{d\Omega}\right)_{polarized}\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}r_0^2(1+\sin^2\theta) = \frac{1}{2}r_0^2(1+\cos^2\alpha)$$

(α → 散射角)

- $\sigma_{unpol} = \frac{8}{3}\pi r_0^2 = \sigma_{pol}$
- But here are 2 polarized direction, degree of polarization $\Pi = \frac{1-\cos^2\alpha}{1+\cos^2\alpha} = \begin{cases} 0\%, & \alpha = 0 \text{ along } \hat{k} \\ 100\%, & \alpha = \frac{\pi}{2}\ \perp \hat{k} \end{cases}$
  That means unpolarized incident wave produces a partially polarized outgoing wave.

对于CMB来说，入射是非极化的，Thomson 散射后，变为部分极化的.

入射是 T. $\underset{0}{\underset{\shortparallel}{E}}$. $\underset{0}{\underset{\shortparallel}{B}}$ 型极化 → T的四极，$E \neq 0$, $B = 0$ → B型极化不行
↓ 产生出E型极化

0326/2018 week 5 6B211 狭义相对论复习
free-free emission

HW
R & L
4.3; 4.7; 4.14

Relativistic Covariance and Kinematics

Contra-variant Vector $x^\mu = (ct, x, y, z)$, $\mu = 0,1,2,3$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \; 1 \end{pmatrix}$$

covariant vector $x_\mu = \eta_{\mu\nu} x^\nu$

Invariant 不变量 $S^2 = x_\mu x^\mu$ under K → K'

Lorentz transformation along X-axis

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-1/2}$$

contra-variant under Λ $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Invariant $S^2 = \eta_{\mu\nu} x^\mu x^\nu = \eta_{\sigma\tau} x'^\sigma x'^\tau = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu$

S^2 is invariant for arbitrary x^μ, only if $\eta_{\mu\nu} = \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \eta_{\sigma\tau}$

or $\eta = \Lambda^T \eta \Lambda$ $= (\Lambda^T)_\mu{}^\sigma \eta_{\sigma\tau} (\Lambda)^\tau{}_\nu$

or $\det \eta = (\det\Lambda)^2 \det\eta$

$$\Rightarrow \det\Lambda = \pm 1 \begin{cases} +1, & \text{proper Lorentz transformation} \quad \cdots ①' \\ -1, & \text{reflection } \vec{x} \rightarrow -\vec{x} \end{cases}$$

也说明4元等关系 ⋯②'
另外，时序不可颠倒的 $\Lambda^0{}_0 \geq 1$ isochronous (i.e. the flow of time is the same in K and K')
NOT CONSIDERED HERE

Covariant Vector $x_\mu \rightarrow x'_\mu = \tilde{\Lambda}_\mu{}^\nu x_\nu$

it can also put Constraint on $\eta_{\mu\nu}$:

$$S^2 = x'_\mu x'^\mu = \tilde{\Lambda}_\mu{}^\sigma \Lambda^\mu{}_\tau x_\sigma x^\tau = \delta^\sigma_\tau x_\sigma x^\tau = x_\sigma x^\sigma$$

$$\Rightarrow \tilde{\Lambda}_\mu{}^\sigma \Lambda^\mu{}_\tau = \delta^\sigma_\tau$$

$$\Rightarrow \tilde{\Lambda} = \Lambda^{-1}$$

we conclude the transformation of 4-vector satisfies

$$\begin{cases} \text{Contra-variant} & A^\mu \rightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu & A_\mu = \eta_{\mu\nu} A^\nu \\ \text{Covariant} & A_\mu \rightarrow A'_\mu = \tilde{\Lambda}_\mu{}^\nu A_\nu & A^\mu = \eta^{\mu\nu} A_\nu \end{cases}$$

for any A^μ, B_μ, their inner product $A^\mu B_\mu$ is invariant. (4-scalar).

four velocity $\underline{U}^\mu \equiv \frac{dx^\mu}{d\tau}$, $\underline{U}^0 = \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\gamma_\mu$
$= \gamma_u(c, \vec{u})$

$\gamma_\mu = \left(1 - u^2/c^2\right)^{-1/2}$, $\vec{u} = \frac{d\vec{x}}{dt}$ ⤳ 和坐标系的变换没有关系 $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$$\underline{U}^i = \frac{dx^i}{d\tau} = \gamma_\mu \frac{dx^i}{dt} = \gamma_u u^i$$

EM wave phase $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$
↳ $\vec{k}\cdot\vec{x} - \omega t$ should be invariant

$$\vec{k}\cdot\vec{x} - \omega t = \eta_{\mu\nu} k^\mu x^\nu$$

where $k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$
$x^\nu = (ct, \vec{x})$

Thus k^μ should be a 4-vector, and

$$k^2 = -\left(\frac{\omega}{c}\right)^2 + \vec{k}^2 = 0$$

↳ $\frac{\omega}{c} = |\vec{k}|$ for plane-wave

i.e. k^μ is a null-vector.

Tensor analysis

zeroth-rank tensor = Lorentz scalar (invariant)

1st-rank tensor = 4-vector

2nd-rank tensor $T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}$

$$T_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\tau} T^{\sigma\tau}$$

$$T_{\mu\nu} \rightarrow T'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\sigma \tilde{\Lambda}_\nu{}^\tau T_{\sigma\tau}$$

Tensor Satisfies:

1. Addition: $A^{\mu\nu} \pm B^{\mu\nu}$ is a tensor of the same rank with $A^{\mu\nu}$, $B^{\mu\nu}$
2. Multiplication: $A^\mu B^\nu G_{\sigma\tau}$ is a 4th-rank tensor
3. Raising and lowering indices using $\eta^{\mu\nu}$, $\eta_{\mu\nu}$. respectively
4. Contraction e.g. $A^\mu A_\mu$, $A^\mu B_\mu$ is a scalar $T^{\mu\nu}{}_\mu$ is a 4-vector
5. Gradients: if λ is a 4-scalar:

$$\partial_\mu \lambda \equiv \frac{\partial\lambda}{\partial x^\mu} = \lambda_{,\mu}$$ is a covariant vector.

similarly for tensor $T^{\mu\nu}$:

$$T^{\mu\nu}{}_{,\sigma} \equiv \frac{\partial T^{\mu\nu}}{\partial x^\sigma}$$ is a 3rd order tensor.

tensor 的好处是，可以方便地构建不变量；
所有的物理方程都应是 tensor equation. ⟵ if it is true in frame K then it is true in all frames K'
That means the LHS, RHS should has same rank.

Maxwell equations in covariant form

charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0 \quad \Rightarrow \quad j^\mu = (\rho c, \vec{j})$$

$$\partial_\mu j^\mu = 0$$

Gauge potential form of Maxwell equation

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j} \qquad A^\mu = (\phi, \vec{A})$$

$$\Rightarrow \quad \partial^2 A^\mu = \eta^{\alpha\beta}\partial_\alpha\partial_\beta A^\mu = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)A^\mu = -\frac{4\pi}{c}j^\mu$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$$

under Lorentz gauge $\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \Rightarrow \partial_\mu A^\mu = 0$

What about $\vec{E}$ and $\vec{B}$?

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a 2nd-rank tensor

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell equations

$$\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

e.g. Use $\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$

$$\underbrace{\partial^\mu\partial^\nu}_{\substack{\text{symmetric} \\ \mu\leftrightarrow\nu}} \underbrace{F_{\mu\nu}}_{\substack{\text{anti} \\ \mu\leftrightarrow\nu}} = \frac{4\pi}{c}\partial^\mu j_\mu \Rightarrow \partial^\mu j_\mu = 0$$

Transformation under Λ

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta F_{\alpha\beta}$$

e.g. $\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0$ automatically satisfied for antisymmetric $F_{\mu\nu}$

$$\Leftrightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

For a boost along $\vec{v} = c\vec{\beta}$

$$\vec{E}_\parallel{}' = \vec{E}_\parallel, \quad \vec{B}_\parallel{}' = \vec{B}_\parallel$$

$$\vec{E}_\perp{}' = \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}), \quad \vec{B}_\perp{}' = \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E})$$

- 在一个坐标系下. 纯存在E场. 则 Boost之后可能会产生B场

例如. A pure E and/or B field is NOT Lorentz invariant.

Useful invariant

① $F^{\mu\nu}F_{\mu\nu} = 2\underbrace{(|\vec{B}|^2 - |\vec{E}|^2)}_{\substack{\text{Lorentz invariant} \\ \text{不变要记得}}}$

② For any 2nd-rank tensor $A_{\mu\nu}$, its determinant $\det A$ is a scalar

$$\det A'_{\alpha\beta} = \det\left(A_{\mu\nu}\tilde{\Lambda}_\alpha{}^\mu\tilde{\Lambda}_\beta{}^\nu\right) = \det\left(\tilde{\Lambda}_\alpha{}^\mu A_{\mu\nu}(\tilde{\Lambda}^T)^\nu{}_\beta\right) = \det(\tilde{\Lambda}A\tilde{\Lambda}^T)$$

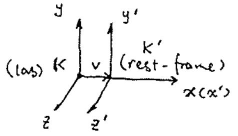
$$= (\det\tilde{\Lambda})^2 \cdot \det A = \det A$$

For EM field

$\det F_{\mu\nu} = (\vec{E}\cdot\vec{B})^2$ is invariant

$\vec{E}\cdot\vec{B}$ is also invariant.

Example : fields of a uniformly moving charge



In the Rest frame $\vec{B} = 0$

$$\begin{cases} E_x' = q x'/r'^3 \\ E_y' = q y'/r'^3 \\ E_z' = q z'/r'^3 \end{cases} \qquad r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

In the lab frame

$$E_x = q x'/r'^3, \quad B_x = 0$$

$$\vec{E}_\perp = \gamma\vec{E}_\perp', \quad \vec{B}_\perp = \gamma\vec{\beta}\times\vec{E}_\perp' \qquad \left.\begin{cases} B_y = -q\gamma\beta z/r^3 \\ B_z = q\gamma\beta y/r^3 \end{cases}\right\} \vec{B} = \frac{q\gamma}{r^3}\vec{\beta}\times\vec{r}$$

$$\begin{cases} E_x = \frac{q\gamma}{r^3}(x - vt) \\ E_y = q\gamma y/r^3 \\ E_z = q\gamma z/r^3 \end{cases} \qquad r = \sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}$$

- 与静电场公式 对比. 在坐标系变换

Relativistic Mechanics

4-momentum $p^\mu = m_0 \underline{u}^\mu = (\frac{E}{c}, \vec{p})$

$$p^0 = c m_0 \gamma_\mu, \quad p^i = m_0\gamma_\mu\vec{u}$$

$$p^2 = p^\mu p_\mu = -m_0^2\gamma_\mu^2(-c^2 + u^2) = -m_0^2c^2 \quad \Rightarrow \quad |\vec{p}|^2c^2 + m_0^2c^4 = E^2$$

$$\parallel$$

$$-\frac{E^2}{c^2} + |\vec{p}|^2$$

For photons. $p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right) = \hbar k^\mu$

$$\omega = c|\vec{k}|$$

4-acceleration

$$a^\mu = \frac{d\underline{u}^\mu}{d\tau}$$

4-force $F^\mu \equiv m_0 a^\mu = \frac{d\underline{p}^\mu}{d\tau}$

Lorentz force $\vec{F} = e(\vec{E} + \frac{\vec{v}\times\vec{B}}{c})$

$$\underbrace{F^\mu}_{\text{Lorentz force}} = \frac{e}{c}\underbrace{F^\mu{}_\nu}_{\text{anti-symmetric tensor}}\underline{u}^\nu \quad \text{or} \quad a^\mu = \frac{e}{m_0 c}F^\mu{}_\nu\underline{u}^\nu$$

For $\mu = 0$ $\underbrace{\frac{dW}{dt}}_{\text{power}} = e\vec{E}\cdot\vec{v}$

$\mu = i$ $\frac{dp_x}{dt} = e\left(E_x + \left(\frac{\vec{v}\times\vec{B}}{c}\right)_x\right)$

$$F^\mu u_\mu = \frac{e}{c}F^\mu{}_\nu\underline{u}^\nu\underline{u}_\mu$$

$$= \frac{e}{c}\underbrace{F_{\mu\nu}}_{\substack{\text{anti} \\ \mu\leftrightarrow\nu}}\underbrace{\underline{u}^\mu\underline{u}^\nu}_{\substack{\text{sym.} \\ \mu\leftrightarrow\nu}} = 0$$

例如. 在回到静止坐标系下 F^μ, $\underline{u}^\mu$ 是正交向量.

Emission From Relativistic Particles

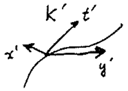
非相对论性的一束辐射，可以用 Larmor 公式求得

相对论性的辐射，求得复杂些（思路：在静止系下用 Larmor 公式，再变回来）

- 总的辐射功率

Instantaneous rest frame K' : In K', particle has zero velocity at a certain time.

For infinitesimally neighboring time the particle moves NON-relativistic



$\Rightarrow$ we can use Larmor's (or dipole) Formula in K'

In K', total amount of energy dW'

K (lab frame) moves $-\vec{v}$ w.r.t K', $dW = \gamma dW'$ (from the transformation of p^μ)

$$dt = \gamma dt'$$

这里的"dw'"是什么意义？以什么速度运动

$$P = \frac{dW}{dt} = P' \quad (\text{两者是Lorentz 不变的})$$

From Larmor's formula $P' = \frac{2q^2}{3c^3}\left|\vec{a}'\right|^2$

并非Lorentz不变的，须注意

Easy to show $a_\mu u^\mu = \frac{du_\mu}{d\tau} u^\mu = \frac{1}{2}\frac{d}{d\tau}\left(u_\mu u^\mu\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^2\right) = 0$

$\frac{dx^\mu}{d\tau} = \gamma_\mu(c, \vec{u})$

In K', $\underline{u}'^\mu = (c, \vec{0})$

$$a'_\mu u'^\mu = a_0' u'^0 = 0 \Rightarrow a_0' = 0$$

$$a'_\mu a'^\mu = a_0' a'^0 + |\vec{a}'|^2 = |\vec{a}'|^2$$

So the Power $P = \frac{2q^2}{3c^3} a^\mu a_\mu$ (Covariant form)

In K', $a'_\parallel = \gamma^3 a_\parallel$ $\Rightarrow$ $P = \frac{2q^2}{3c^3}\vec{a}'\cdot\vec{a}' = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$

$a'_\perp = \gamma^2 a_\perp$

(textbook P140)

- 辐射的角分布

Angular distribution of emitted and received power.

In K', $d\Omega' = \sin\theta' d\theta' d\phi'$

$\mu' = \cos\theta'$

$$\gamma dW'\left(1 + \beta c \underbrace{\frac{dp_x'}{dp'}}_{\mu'} \underbrace{\frac{dp'}{dW'}}_{\frac{1}{c}}\right)$$

Since 4-momentum p^μ is a 4-vector, $dW = \gamma\left(dW' + v dp_x'\right) = \gamma dW'\left(1 + \beta\mu'\right)$

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \quad (\text{see R.L. eq 4.8b}) \quad \Rightarrow \sin^2\theta' = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2}$$

and $d\mu = \frac{d\mu'}{\gamma^2(1+\beta\mu')^2}$

$$d\phi = d\phi'$$

$$\Rightarrow d\Omega = \frac{d\Omega'}{\gamma^2(1+\beta\mu')^2}$$

$$\frac{dW}{d\Omega} = \gamma^3(1+\beta\mu')^3\frac{dW'}{d\Omega'}$$

To divide dt

$\rightarrow dt = \gamma dt'$ emitted power

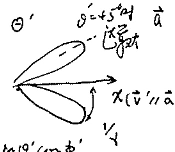
$\rightarrow dt_A = \gamma(1-\beta\mu)dt'$ the time interval of the radiation as received by stationary observer in K.

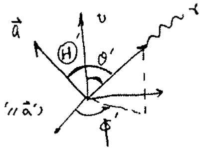
received power 用这个 P_r

而 $P = \frac{dW}{dt}$, so $\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{(\gamma^2 a_\parallel{}^2 + a_\perp^2)}{(1-\beta\mu)^4}\sin^2\Theta'$.

Case 1. acceleration $\vec{a}' \parallel \vec{v}$ elocity, $\Theta' = \theta'$

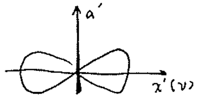
$$\frac{dP_\parallel}{d\Omega} = a_\parallel^2 \frac{\sin^2\theta'}{(1-\beta\mu')^4} \times \text{const.}$$

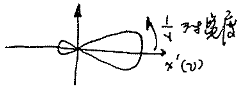




Case 2. acceleration ⊥ velocity, $\cos\Theta' = \sin\theta'\cos\phi'$

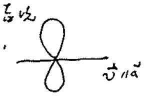
$$\frac{dP}{d\Omega} = \frac{1}{(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right] \times \text{Const.}$$





这是在静止系下的情况

in K'



$\beta\mu \sim 1$ 时分布极大值

$\Delta\mu = \frac{1}{\gamma}$

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \cong \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4$$

width $\theta \sim \frac{1}{\gamma}$, peaked at 0.

0402/2018 week6 6B211

小测试
第一章~第三章
可携带A4纸双面
网络学堂有作业答案

作业
R&L 5.2

自由的离子，在经过的场中加速（减速）的辐射 → Free ~ Free emission
电子

经典的用偶极辐射描述；

（带电粒子与电磁波的作用 ~
散射）

Bremsstrahlung (free-free) 韧致辐射

经典的偶极辐射 $\vec{d} = \sum_i q_i \vec{r}_i \propto \sum_i m_i \vec{r}_i \propto \vec{r}_c$

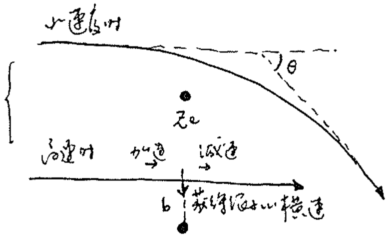
if like-ions

质心是没有加速度的

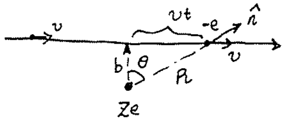
No bremsstrahlung for like particle (e.g. $e e^-$ $p p^-$)

e ~ ion Bremsstrahlung (e.g. p-e, p的加速度很小，电子可视作 as moving in the fixed Coulomb field of p) $m_e \ll m_p$

- Emission from single-speed electrons



讨论高速情况



$$\vec{d} = -e\vec{R}$$

$$\ddot{d} = -e\dot{\vec{v}}$$

Fourier transformation $\frac{d}{dt} \to -i\omega$

$$-\omega^2 \tilde{d}(\omega) = \frac{-e}{2\pi} \int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t} dt$$

Collision time (碰撞时间，受到较大力作用的特征时间) $\sim \tau = \frac{b}{v}$

for $\omega\tau \gg 1$, $e^{i\omega t}$ oscillates rapidly for $|t| \lesssim \tau$
$(b \gg \frac{v}{\omega})$

$$\int \dot{\vec{v}} e^{i\omega t} dt \simeq 0$$

for $\omega\tau \ll 1$, $e^{i\omega t} \simeq 1$

$$\int \dot{\vec{v}} e^{i\omega t} dt \simeq \int \dot{\vec{v}}\, dt = \Delta\vec{v}$$

Combine these

$$\tilde{d}(\omega) = \begin{cases} \frac{e}{2\pi\omega^2} \Delta\vec{V}, & b \ll \frac{v}{\omega} \ (\lesssim) \\ 0, & \text{otherwise} \end{cases}$$

In dipole approximation, power spectrum

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\tilde{d}(\omega)|^2 = \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta\vec{v}|^2, & b \lesssim \frac{v}{\omega} \\ 0, & \text{otherwise} \end{cases}$$

From Coulomb's law $\ddot{\vec{v}} = -\frac{Ze^2}{mR^2}\hat{n}$

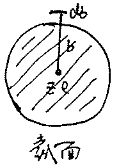
$$\ddot{\vec{v}}_\perp = -\frac{Ze^2}{mR^3}\vec{b}$$

$$|\Delta\vec{v}| = \int_{-\infty}^{\infty} \frac{Ze^2 b}{m(b^2+v^2t^2)^{3/2}} dt = \frac{Ze^2 b}{mvb^2} \underbrace{\int_{-\infty}^{\infty} \frac{d\left(\frac{vt}{b}\right)}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}}}_{=2} = \frac{2Ze^2}{mvb}$$

Small-Scale
small-angle scattering regime

$$\frac{dW(b)}{d\omega} = \begin{cases} 8Z^2e^6 / 3\pi c^3 m^2 v^2 b^2, & b \lesssim \frac{v}{\omega} \\ 0, & \text{otherwise} \end{cases}$$

- For a flux of electrons $\frac{dN_e}{dA\,dt} = n_e v$, incident on 1 ion



$$\frac{dW}{d\omega\, dV\, dt} = \int_{b_{min}}^{\infty} \frac{dW}{d\omega} \underbrace{n_e \cdot v \cdot 2\pi b\, db}_{\frac{dN_e}{dt}} \times \underbrace{n_i}_{\frac{dN_i}{dV}} = \int_{b_{min}}^{\infty} n_e n_i \cdot 2\pi v \frac{dW(b)}{d\omega} b\, db$$

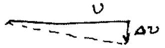
～单位体积的离子数

$$= n_e n_i \cdot 2\pi v \int_{b_{min}}^{b_{max}} \frac{8Z^2e^6}{3\pi c^3 m^2 v^2} \frac{db}{b} = \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 \ln\frac{b_{max}}{b_{min}}$$

b_{max} 的估算十分容易 $b_{max} \simeq \frac{v}{\omega}$

b_{min} 的决定因素有：

① straight-line approximation $\Delta v \ll v$ $(\lesssim)$



$\Leftrightarrow b \gtrsim \frac{2Ze^2}{mv^2}$, $b_{min} = \frac{2Ze^2}{mv^2}$

书里面有3-4 $(\frac{2}{\pi})^2$ 因子，$b^{(1)}_{min} = \frac{2Ze^2}{mv^2}\left(\frac{2}{\pi}\right)$?

② Uncertainty principle $\Delta x \Delta p \gtrsim \hbar/2$
$\Delta x \sim b$, $\Delta p \sim mv$

If $b^{(1)}_{min} \gg b^{(2)}_{min}$ $\Leftrightarrow b^{(2)}_{min} = \frac{\hbar}{mv \cdot 2}$

Classical treatment is valid, $b_{min} = b^{(1)}_{min}$

$\frac{2Ze^2}{mv^2} \gg \frac{\hbar/2}{mv}$ $\Leftrightarrow$ $v \ll \frac{2Ze^2}{\hbar/2}$

$\Leftrightarrow$ $\frac{1}{2}mv^2 \ll Z^2 \times 4 \frac{me^4}{2\hbar^2/4} = 4Z^2 R_y$

where $R_y = \frac{me^4}{2\hbar^2} \approx 13.6eV$ 是电离的能量

是氢原子的电离能
Rydberg energy for Hydrogen

书里面是3，4 4倍．即 $4Z^2R_y$

Alternatively, $\frac{1}{2}mv^2 \gg Z^2 R_y$

$b_{min} = b^{(2)}_{min}$

Quantum limit

In general

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3m^2v} n_e n_i Z^2 g_{eff}(v.\omega)$$

where Gaunt factor $g_{eff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\frac{b_{max}}{b_{min}}$

- Thermal Bremsstrahlung emission

Average the single-speed expression over a thermal distribution of speeds.

粒子的速度应是有分布的
如热平衡下是 Maxwell 分布

$\begin{cases}\text{Thermal : Maxwell distribution} \\ \text{Non-thermal: some distribution of speeds.}\end{cases}$

Maxwell probability $dp \propto e^{-E/kT} d^3v \propto v^2 e^{-\frac{mv^2}{2kT}} dv$

$$\frac{dW(T.\omega)}{d\omega dV dt} = \frac{\int_{v_{min}}^{\infty}\frac{dW(v.\omega)}{d\omega dV dt} v^2 e^{-\frac{mv^2}{2kT}} dv}{\underset{\uparrow \text{归一化系数}}{\int_0^{\infty}} v^2 e^{-mv^2/2kT} dv}$$

辐射产生的光子的能量应
$h\nu \le \frac{1}{2}mv^2$
$\Rightarrow v_{min} = \sqrt{\frac{2h\nu}{m}}$ 最低速度

引入无量纲参数 $x = \frac{v}{v_{min}}$
$\frac{mv^2}{2kT} = \frac{mv_{min}^2}{2kT}x^2$

$$= \frac{\frac{16\pi e^6 n_e n_i Z^2}{3\sqrt{3}c^3m^2}}{\sqrt{\pi}/4\left(\frac{m}{2kT}\right)^{3/2}} \int_{v_{min}}^{\infty} g_{eff}(v.\omega)\, v e^{-\frac{mv^2}{2kT}} dv$$

$$= \frac{2h\nu}{m} e^{-h\nu/kT} (\quad) \int_1^{\infty} g_{eff}(v.\omega)\, e^{-\frac{h\nu}{kT}(x-1)} x\, dx$$

$$= \frac{kT}{m} e^{-h\nu/kT} (\quad)\ \bar{g}_{ff}$$

$$= \frac{32\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-\frac{h\nu}{kT}} \bar{g}_{eff}$$

where $\bar{g}_{ff} = \frac{2h\nu}{kT}\int_1^{\infty} g_{ff}(v.\omega)\, e^{-\frac{h\nu}{kT}(x-1)} x\, dx$ is velocity averaged Gaunt factor.

typically of order 1 $\sim \mathcal{O}(1)$ for $\frac{h\nu}{kT} \sim 1$

define $\varepsilon_\nu^{eff} = \frac{dW}{d\nu dV dt} = 2\pi\frac{dW}{d\omega dV dt} = 6.8\times10^{-38} Z^2 n_e n_i T^{-1/2} e^{-\frac{h\nu}{kT}} \bar{g}_{eff}$

$(\omega = 2\pi\nu)$
$d\omega = 2\pi d\nu$

$\text{erg}\cdot\text{s}^{-1}\text{cm}^3\text{Hz}^{-1}$

Gaunt factor $\tilde{g}_{eff} \sim \begin{cases}\mathcal{O}(1) & \text{for } \frac{h\nu}{kT}\sim 1 \\ 1\sim5 & \text{for } 10^{-4} < \frac{h\nu}{kT} < 1\end{cases}$

Total power per volume

$$\frac{dW}{dt dV} = \int \frac{dW}{dt dV d\nu} d\nu = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{32\pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B$$

$$= 1.4\times10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B$$

相当于一个没有角度的辐射系数

where $\bar{g}_B = \int \bar{g}_{ff} e^{-\frac{h\nu}{kT}} d\left(\frac{h\nu}{kT}\right)$

frequency averaged of the velocity averaged Gaunt factor $\approx 1.1-1.5$

- Thermal bremsstrahlung

Free-Free Absorption （是 bremsstrahlung 的反过程）

free-free emission $4\pi j_\nu^{ff} = \frac{dW}{dt dV d\nu}$
这是没有角分布的.

free-free absorption $\alpha_\nu^{ff} = \frac{j_\nu^{ff}}{B_\nu(T)}$，所以利用 $\overset{\text{Planck func}}{B_\nu(T)}$ 易得 $\alpha_\nu^{ff} \approx 3.7\times10^8 T^{-1/2} Z^2 n_e n_i \nu^{-3} (1-e^{-h\nu/kT})\bar{g}_{ff}$

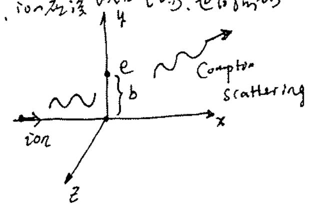
for $h\nu \gg kT$, $\alpha_\nu^{ff} \propto \nu^{-3}$
absorption 的贡献，发射为主

for $h\nu \ll kT$, $\alpha_\nu^{ff} = 0.018\, T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff} \propto \nu^{-2}$
低频的吸收更强，发射也存在

Relativistic Bremsstrahlung (free-free) = Compton Scattering of the virtual quanta of the ion's electrostatic field as seen in electron's frame.

思路：通过 Lorentz 变换，要到静止参考系.
"method of virtual quanta" 虚光子方法

设在电子坐标系下，ion应该只有电场、也有磁场



electro static field → radiation
被Compton散射（速度高, $v_{ion} \sim c$ 不是Thomson散射）
（电磁，不是平面波）

欲求得辐射功率谱

$$\frac{dW'}{dA'd\omega'} = c\left|\tilde{E}(\omega')\right|^2 = \frac{\overset{Z^2e^2}{q^2}}{\pi^2b'^2c}\left(\frac{b'\omega'}{\gamma c}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma c}\right)$$

这里 $\tilde{E}(\omega) = \frac{1}{2\pi}\int E_2(t) e^{i\omega t}dt = \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}(\gamma^2v^2t^2+b^2)^{-3/2} e^{i\omega t}dt$, $\gamma = (1-\beta^2)^{-1/2}$

所有的 prime 都表示在电子坐标系下

$$= \frac{q}{\pi b v} K_1\left(\frac{b\omega}{\gamma v}\right)\frac{b\omega}{\gamma v}.$$

$K_1(x)$ modified Bessel function of the order 1

In the long-wavelength limit $\hbar\omega' \lesssim m_e c^2$，这里是 Thomson 散射

$$\frac{dW'}{d\omega'} = \frac{dW'}{dA'd\omega'}\int d\sigma_T = \frac{dW'}{dA'd\omega'}\underset{\uparrow \text{Thomson 散射截面}}{\sigma_T}$$

$\underset{\uparrow \text{功率谱，不变}}{\frac{dW'}{d\omega'} = \frac{dW}{d\omega}}$　$\underset{\uparrow \text{垂直速度方向的长度，不变}}{b' = b}$, $\omega = \gamma\omega'(1+\beta\cos\theta')$ → in average about 0. $\omega = \gamma\omega'$

$$\sigma_T = \frac{8}{3}\pi\left(\frac{e^2}{mc^2}\right)^2$$

In lab frame

$$\frac{dW}{d\omega} = \frac{8 z^2 e^6}{3\pi b^2 c^5 m^2}\left(\frac{b\omega}{\gamma^2 c}\right)^2 K_1^2\left(\frac{b\omega}{\gamma^2 c}\right)$$

书里面公式(5.23)的typo. 少了平方

实验室坐标系下的功率谱

If $v \to c$, and add γ, from $\underline{NR}$ to Relativistic. formuler

Non-Relativistic.

for a flux of electrons

$$\frac{dW}{d\omega dt dV} = \int_{b_{min}}^{\infty} \frac{dW(b)}{d\omega} \cdot 2\pi b\, db\, n_e v \cdot n_i \sim \frac{16 z^2 e^6 n_e n_i}{3 c^4 m^2} \ln\left(\frac{0.68\gamma^2 c}{\omega b_{min}}\right)$$

其中 ln 的参数应该是 $\int_{x=\frac{b_{min}\omega}{\gamma^2 c}}^{\infty} K_1^2(x)\cdot x\,dx$

$$= x K_0(x) K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right) \xrightarrow{\hbar\omega \ll \gamma m c^2} \ln\left(\frac{0.68\gamma^2 c}{\omega b_{min}}\right)$$

In the high-frequency, $\hbar\omega' \gtrsim mc^2$ (高能极限)

$$\frac{dW'}{d\omega'} = \frac{dW'}{d\omega' dA'}\,\sigma$$

σ → Compton Scattering cross-section

Klein-Nishina cross-section

$$\frac{dW}{dt dV} \approx 1.4\times10^{-27} T^{1/2} z^2 n_e n_i \bar{g}_B \underbrace{(1+4.4\times10^{-10} T)}_{\text{relativistic Correction}}$$

对ω积分

total power per volume

relativistic Correction (Novikov + Thorne 1973)

0409/2018 week 7 GB211

Synchrotron Radiation 同步辐射

(低速, 使用 Larmor 公式, 谱是单色的

高速(相对论性)
- 谱不是单色的 (beam effect) *1
- 单粒子的辐射功率 $\propto F(\frac{\omega}{\omega_c})$ *2
- 总的辐射功率 $\propto \omega^{-s}$ *3

Particles accelerated by B-field can radiate

- Non-relativistic particles : $\omega = \omega_g$, cyclotron radiation

  称回旋加速辐射 $\Delta t \to \infty$ $\Delta\omega \to 0$

- Ultra-relativistic particles : spectrum of ω, synchrotron radiation

  称同步加速辐射 $\Delta t_A \sim \frac{1}{\omega_c}$, $\omega_c \cong \frac{3}{2}\gamma^3 \omega_B \sin\alpha$

  arrival

  critical frequency

  pitch angle α: $\vec{v}$ 与 $\vec{B}$ 的夹角

介于两者之间的情况不好讨论. 略去

前者只低速, 这里讨论后者

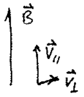
Consider a particle, m, q, in a B-field

$$a^{\mu} = \frac{q}{m_0 c} F^{\mu}{}_{\nu} u^{\nu}$$

$$\begin{cases} \frac{dW}{dt} = q\vec{E}\cdot\vec{v} = 0 & ① \\ \frac{d\vec{p}}{dt} = q(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) = \frac{q}{c}\vec{v}\times\vec{B} & ② \end{cases}$$

From ① $\frac{d}{dt}(\gamma mc^2) = 0$, γ is constant $\Rightarrow$ v is constant

From ② $LHS = \frac{d}{dt}(\gamma m\vec{v}) = \gamma m\frac{d\vec{v}}{dt}$ $\Rightarrow$ $\frac{d\vec{v}}{dt} = \frac{q}{\gamma mc}\vec{v}\times\vec{B}$



将速度投影 ∥$\vec{B}$ 和 ⊥$\vec{B}$ 方向 : $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$

w.r.t $\vec{B}$

所以 $\frac{d\vec{v}_{\parallel}}{dt} = 0 \Rightarrow \vec{v}_{\parallel}$ is constant

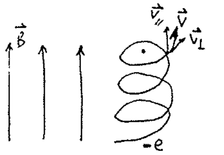
$$\frac{d\vec{v}_{\perp}}{dt} = \frac{q}{\gamma mc}\vec{v}_{\perp}\times\vec{B} \Rightarrow \dot{\vec{v}}_{\perp} \perp \vec{v}_{\perp}$$

$|\vec{v}_{\perp}| = $ constant $|\vec{a}_{\perp}| = $ constant

所以 运动是 helical orbit

回旋频率 $\omega_B = \frac{qB}{\gamma mc}$ ✓ (因为 $\underset{a}{\overset{\text{w.r.t.}\vec{B}}{a_{\perp}}} = \left|\frac{d\vec{v}_{\perp}}{dt}\right| = v_{\perp}\omega_B$ ✓)

gyration frequency



Total Emission 用上一节的公式

$$P = \frac{2q^2\gamma^4}{3c^3}\left(\underset{\text{加速}}{a_{\perp}^2} + \gamma^2 \underset{\substack{0\\ \text{w.r.t}\ \vec{v}}}{a_{\parallel}^2}\right) = \frac{2q^2\gamma^4}{3c^3}\left(\frac{qB}{\gamma mc}\right)^2 \beta_{\perp}^2 c^2 . \quad \beta_{\perp} = \frac{v_{\perp}}{c}$$

$\beta_{\perp}$: 与磁场垂直的速度

$$= \frac{2}{3} r_0^2 c\gamma^2 B^2 \beta_{\perp}^2 , \quad r_0 = \frac{q^2}{mc^2} \text{ is classical radius}$$ ✓

(见 Thomson 散射)

对于多粒子系统, 可以看成均匀的, 需求平均

速度方向的方向均匀分布

但速度大小确定 $\langle\beta_{\perp}^2\rangle = \int(\beta\sin\theta)^2 \frac{d\Omega}{4\pi} = \frac{2}{3}\beta^2$

$d\Omega$: $\sin\theta d\theta d\phi$

$$\langle P\rangle = \frac{2}{3}r_0^2 c\gamma^2 B^2 \frac{2}{3}\beta^2 = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B$$

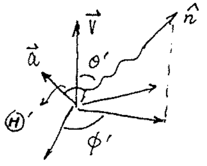
σ_T 是 Thomson scattering cross section $\frac{8}{3}\pi r_0^2$

$U_B = \frac{1}{8\pi}B^2$ is magnetic field energy density.

辐射的角分布

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{(\gamma^2 a_{\parallel}^2 + a_{\perp}^2)}{(1-\beta\mu)^4}\sin^2\Theta'$$

w.r.t. $\vec{v}$



" ' " 指在 rest frame

Θ' : angle between acceleration and the direction of emission

在 $\vec{a}\perp\vec{v}$ 的情况下, $\cos\Theta' = \frac{\vec{a}'\cdot\hat{n}'}{|\vec{a}'|} = \hat{e}_x\cdot\hat{n}'$

将 $\vec{a}, \vec{v}$ 分别放在 $\hat{x}, \hat{z}$ 方向

$$= \sin\theta'\cos\phi'$$

而 $\sin^2\Theta' = 1 - \sin^2\theta'\cos^2\phi'$

而另外 $\phi = \phi'$

利用 $\underset{\cos\theta}{\mu} = \frac{\mu'+\beta}{1+\beta\mu'} \xrightarrow{\beta\to-\beta} \mu' = \frac{\mu-\beta}{1-\beta\mu}$

所以 $\sin^2\theta' = 1-\mu'^2 = 1-\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2}$

$$\sin^2\Theta' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}$$

最后结果是 $\frac{dP}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{(1-\beta\mu)^4}\left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right)$

在速度非常大时. $\mu = \cos\theta \doteq 1 - \frac{\theta^2}{2}$

$\beta = (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2\gamma^2}$

$1-\beta\mu \simeq \frac{1+\gamma^2\theta^2}{2\gamma^2}$

当 $\theta = 0$ 时. $1-\beta\mu$ 最小

当 $\theta = \frac{1}{\gamma}$ 时. $1-\beta\mu$ 已经很大

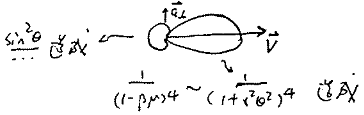
所以辐射集中在 $\frac{1}{\gamma}$ 范围的立体角内 (相对于速度方向)

此时 $\frac{dP_\perp}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{16\gamma^8}{(1+\gamma^2\theta^2)^4}\left(1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right) = \frac{4q^2a_\perp^2\gamma^8}{\pi c^3}\cdot\frac{1-2\gamma^2\theta^2\cos2\phi+\gamma^4\theta^4}{(1+\gamma^2\theta^2)^6}$

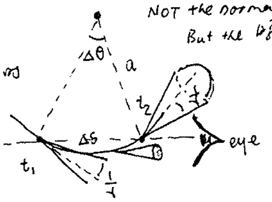
θ 很大时 $\propto \frac{\gamma^4\theta^4}{(\gamma^2\theta^2)^6} \simeq \frac{1}{(\gamma\theta)^8}$

$\phi = 90°$ 时 $\propto \frac{(1+\gamma^2\theta^2)^2}{(1+\gamma^2\theta^2)^6} = \frac{1}{(1+\gamma^2\theta^2)^4}$

The factor is sharply peaked at $\theta = 0$
with an angular scale $\frac{1}{\gamma}$
This is the beam effect along $\vec{v}$



观察的频谱: 在观察者的



NOT the normal plane to B
But the 瞬时曲率圆平面

a 是 radius of the Curvature of the path

a' 是 ⊥ $\vec{B}$ 方向投影运动的半径

$a' = \frac{v\sin\alpha}{\omega_B}$

$\Delta\theta \doteq 2\frac{1}{\gamma}$ 可以观察到辐射的角度

$\Delta S = a\Delta\theta = \frac{2a}{\gamma}$ 路径长

$a = \frac{\Delta S}{\Delta\theta} = \frac{v\Delta t}{|\Delta V|/V} = \frac{v^2}{|\Delta V|} = \frac{v^2}{\frac{q}{\gamma mc}vB\sin\alpha} = \frac{v}{\omega_B\sin\alpha}$

$\gamma m\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$ $\qquad \omega_B = \frac{qB}{\gamma mc}$

α 是螺旋角, 即 $\vec{v}$ 与 $\vec{B}$ 的夹角

可观测时间 $\underset{\text{arrival time}}{\Delta t^A} = \Delta t^A_2 - \Delta t^A_1$. 由测不准关系 $\Delta t^A \Delta\omega \gtrsim 1$ 即 $\Delta\omega \gtrsim \frac{1}{\Delta t^A}$

$\Delta t = \frac{\Delta S}{v} = \frac{2}{\gamma\omega_B\sin\alpha} \simeq \Delta t^A$. 所以 $\Delta\omega \gtrsim \frac{\gamma\omega_B\sin\alpha}{2} \gg \frac{\omega_B}{2\pi}$

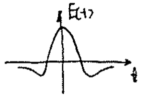
在经典的非相对论情况时. 是接近单色的; 但由于 beam effect, 观测 Δt^A 有限. $\Delta\omega$ 有下限

更精确的估计 $\Delta t^A = \Delta t - \frac{\Delta S}{c} = \Delta t(1-\frac{v}{c}) = \frac{2}{\gamma\omega_B\sin\alpha}(1-\frac{v}{c})$

利用 $1-(\frac{v}{c})^2 = \gamma^{-2}$. $\frac{v}{c} \simeq (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2\gamma^2}$, $1-\frac{v}{c} \simeq \frac{1}{2\gamma^2}$ for $\gamma \gg 1$

则 $\Delta t^A \simeq \frac{1}{\gamma^3\omega_B\sin\alpha}$

所以有 $\Delta t^A = \Delta t(1-\frac{v}{c}) = \frac{1}{\gamma^3\omega_B\sin\alpha}$



$\frac{\Delta t^A}{\underset{\text{回旋时间}}{\Delta t_{gyro}}} = \frac{1/\gamma^3\omega_B\sin\alpha}{2\pi/\omega_B} \sim \frac{1}{\gamma^3} \ll 1$ 所以接收到的是一个个脉冲

定义 Critical frequency $\omega_c = \frac{3}{2}\gamma^3\omega_B\sin\alpha$, 则 $\Delta t^A = \frac{3}{2}\omega_c^{-1}$

根据测不准关系 $\Delta t^A\Delta\omega \sim 1$, 得辐射展宽 $\Delta\omega \sim \frac{2}{3}\omega_c$
extend of the spectrum

在非相对论情况时
$\Delta t^A \to \infty$
得单色谱

总的辐射功率

由于 $\Delta\theta \sim \frac{1}{\gamma}$, 可以假设 $E(t) = F(\gamma\theta)$
即 $\gamma\theta$ 唯一决定了 $E(t)$

由几何关系

$\theta \simeq \frac{s}{a}$, $t = \frac{s}{v}(1-\frac{v}{c})$

$\gamma\theta = \frac{\gamma s}{a} = \frac{\gamma}{a}vt/(1-\frac{v}{c}) \simeq \frac{\gamma vt}{\frac{3}{2}\frac{\gamma^3 v}{\omega_c}\times\frac{1}{2\gamma^2}} = \frac{4}{3}\omega_c t$

所以 $E(t) \propto g(\omega_c t)$, 写出其 Fourier Transform

$\tilde{E}(\omega) \propto \int_{-\infty}^{\infty} g(\omega_c t)e^{i\omega t}dt$. 若令 $\xi = \omega_c t$ 则 $\tilde{E}(\omega) = \int_{-\infty}^{\infty} g(\xi)e^{i\frac{\omega}{\omega_c}\xi}d\xi$

最后 $\tilde{E}(\omega)$ 应是 $\tilde{g}(\frac{\omega}{\omega_c})$, 即关于 $\frac{\omega}{\omega_c}$ 的函数; 功率谱定义为 $\tilde{P}_\Omega(\omega) = \frac{dW}{d\omega d\Omega} \propto |\tilde{E}(\omega)|^2$.

$$\tilde{P}(\omega) = \frac{dW}{d\omega dt} = \underset{\text{independent of }\omega}{\frac{1}{T}}\int\frac{dW}{d\omega d\Omega}d\Omega \propto |\tilde{E}(\omega)|^2 = \underset{\text{some constant}}{C_1}F(\frac{\omega}{\omega_c}) \rightsquigarrow \text{dimensionless}$$

同步辐射总功率 $P = \int_0^\infty \tilde{P}(\omega)d\omega = C_1\omega_c\int_0^\infty F(x)dx$, $x = \frac{\omega}{\omega_c}$

另一方面由之前的辐射功率 应是 $P = 2q^4B^2\gamma^2\beta^2\sin\alpha/3m^2c^3$. 所以 $C_1\int_0^\infty F(x)dx = \frac{P}{\omega_c} = \frac{4q^3B\beta^2\sin\alpha}{9mc^2}$ → $\beta \simeq 1$

$\simeq \frac{4q^3B\sin\alpha}{9mc^2}$

最后 $\tilde{P}(\omega) = \frac{\sqrt{3}}{2\pi}\frac{q^3B\sin\alpha}{mc^2}F(\frac{\omega}{\omega_c})$
↳ ω_c only dependent on γ

Special Cases

$\tilde{P}(\omega) \propto \omega^{-s}$
power law. 经常见到
这里的 s 称为 spectral index

对于多粒子系统，若速度服从 distribution of velocity

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma \quad , \quad \gamma_1<\gamma<\gamma_2$$

总的辐射功率 $P_{tot}(\omega) = C\int_{\gamma_1}^{\gamma_2}\tilde{P}(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2}F(\frac{\omega}{\omega_c})\gamma^{-p}d\gamma \propto -\int_{x_1}^{x_2}F(x)\omega_c^{-\frac{p}{2}+\frac{1}{2}}\frac{dx}{x} \propto \omega^{-\frac{(p-1)}{2}}\int_{x_2}^{x_1}F(x)x^{\frac{p-3}{2}}dx$

$x=\frac{\omega}{\omega_c}$, $\omega_c \propto \gamma^2$

$= \frac{3\gamma^2 qB\sin\alpha}{2mc} \propto \gamma^2$

$x \propto \gamma^{-2}$

$\frac{dx}{x} \simeq -2\frac{d\gamma}{\gamma}$

若 $\omega_c(\gamma_1) << \omega << \omega_c(\gamma_2)$，可令 $x_2\to 0$，$x_1\to\infty$，则

$$P_{tot}(\omega) \propto \omega^{-\frac{(p-1)}{2}}$$

所以，在极端相对论情况下，若 velocity 分布以 p 为 power law index，则 $P_{tot}(\omega)$ 以 $\frac{p-1}{2}$ 为 power law index.

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物理学前沿与辐射机制

推导同步辐射的总辐射功率的定量表达式

Spectrum of Synchrotron Radiation

沿方向 $\hat{n}$ 的功率谱

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2c}\left|\int \hat{n}\times(\hat{n}\times\vec{\beta})\exp i\omega\Big(\underset{\uparrow\ \text{retarded time}}{t'-\frac{\hat{n}\cdot\vec{r}(t')}{c}}\Big)dt'\right|^2$$

其中 $t' - \frac{\hat{n}\cdot\vec{r}(t')}{c} = t' - \frac{a}{c}\cos\theta\sin\Delta\alpha$

只考虑小的 $\Delta\alpha$：由于 beam effect，也只考虑小的 θ

$$\simeq t' - \frac{a}{c}\left(1-\frac{\theta^2}{2}\right)\left(\frac{vt'}{a}-\frac{1}{6}\left(\frac{vt'}{a}\right)^3\right)$$

$$= t'\left(1-\frac{v}{c}\right) + \frac{v}{c}\frac{\theta^2}{2}t'$$

并且注意到曲率半径 $a = \frac{3}{2}\gamma^3\frac{v}{\omega_c} \simeq \frac{3\gamma^3c}{2\omega_c}$

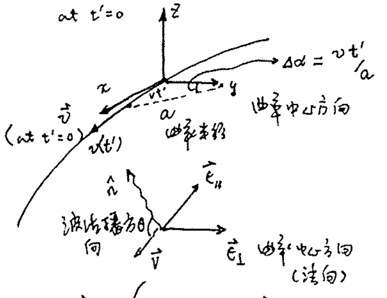
$$\simeq 2\gamma^{-2}t' + \frac{\theta^2}{2}t' = (2\gamma^2)^{-1}\left((1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2t'^3}{3a^2}\right)$$

经过 是个很大的量，指数积分为0（为什么？）

只有 $\gamma\theta \lesssim 1$ 时，积分才非零

另外还要 $\frac{\gamma t'}{a} \lesssim 1$

而 $\hat{n}\times(\hat{n}\times\vec{\beta}) = (\hat{n}\cdot\vec{\beta})\hat{n}-\vec{\beta} = -\hat{\epsilon}_\perp\underbrace{\sin\Delta\alpha}_{小量} + \hat{\epsilon}_\parallel\cos\Delta\alpha\underbrace{\sin\theta}_{小量}$，所以 $\hat{n}\times(\hat{n}\times\vec{\beta})$ 很小。



$\frac{\vec{\beta}(t')}{|\vec{\beta}|} = (\cos\Delta\alpha, \sin\Delta\alpha, 0)$

$\hat{n} = (\cos\theta, 0, \sin\theta)$ 不失一般性，假设它在 x-z 平面

$\hat{\epsilon}_\perp = (0,1,0)$

$\hat{\epsilon}_\parallel = \hat{n}\times\hat{\epsilon}_\perp = (-\sin\theta, 0, \cos\theta)$

功率谱可以写为 ∥ 和 ⊥ 方向的贡献之和

$$\frac{dW}{d\omega d\Omega} = \frac{dW_\parallel}{d\omega d\Omega} + \frac{dW_\perp}{d\omega d\Omega}$$

注意 $v\simeq c$

其中，along $\hat{\epsilon}_\perp$，$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2c}\left|\int\frac{t'c\,dt'}{a}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta^2\gamma t' + \frac{c^2\gamma^2t'^3}{3a^2}\right)\right]\right|^2$

$\downarrow$

$\theta_\gamma = 1+\gamma^2\theta^2$

along $\hat{\epsilon}_\parallel$，$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2c}\left|\int dt'\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2t' + \frac{c^2\gamma^2t'^3}{3a^2}\right)\right]\right|^2$

所以 $\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2c}\left(\frac{a\theta_\gamma^2}{\gamma^2c}\right)^2\left|\int_{-\infty}^{\infty}y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$，let $y = \gamma\frac{ct'}{a\theta_\gamma}$

$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2c}\left(\frac{a^2\theta_\gamma^2}{\gamma^4c}\right)^2\left|\int_{-\infty}^{\infty}\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$

$\eta \equiv \frac{\omega a\theta_\gamma^3}{3c\gamma^3}$

$\searrow$ or γ^2c

because cancelled by γ in y

$$\theta\simeq 0, \quad \eta \simeq \eta(\theta=0) = \frac{\omega}{2\omega_c}$$

所以 $\frac{dW}{d\omega d\Omega}$ depends on $\frac{\omega}{\omega_c}$ and $\gamma\theta$

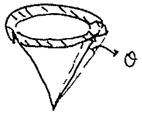
积分后得到

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{3\pi^2c}\left(\frac{a\theta_\gamma^2}{\gamma^2c}\right)^2K^2_{2/3}(\eta) \rightarrow \text{modified Bessel function}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{3\pi^2c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2K^2_{1/3}(\eta)$$

由于粒子做螺旋线运动，速度扫过一个立体角，半角是 pitch angle

$$d\Omega = 2\pi\sin\alpha\, d\theta$$



$\theta \sim \frac{1}{\gamma}$

总的辐射谱

$$\frac{dW}{d\omega} = \int\frac{dW}{d\omega d\Omega}d\Omega = \int\left(\frac{dW_\perp}{d\omega d\Omega}+\frac{dW_\parallel}{d\omega d\Omega}\right)d\Omega = \frac{dW_\perp}{d\omega}+\frac{dW_\parallel}{d\omega}$$

$$\frac{dW_\perp}{d\omega} = \frac{2q^2\omega^2a^2\sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^4K^2_{2/3}(\eta)d\theta = \frac{\sqrt{3}q^2\gamma\sin\alpha}{2c}(F(x)-G(x))$$

$$\frac{dW_\parallel}{d\omega} = \sim\int_{-\infty}^{\infty}\theta_\gamma^2\theta^2K^2_{1/3}(\eta)d\theta = \sim(F(x)-G(x))$$

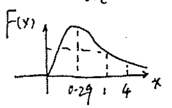
$\frac{dW}{d\omega} = \sim\times 2F(x)$

$\searrow$ or Ω?

其中 $F(x) = x\int_x^\infty K_{5/3}(\xi)d\xi$，$x = \frac{\omega}{\omega_c}$

$\uparrow$ 是 $\frac{\omega}{\omega_c}$ 的函数

$$\begin{cases}\frac{4}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} & x<<1\\ \left(\frac{\pi}{2}\right)^{1/2}e^{-x}x^{1/2} & x>>1\end{cases} \simeq$$



$\omega/\omega_c < 1$ 有显著贡献

在时间 $\frac{2\pi}{\omega_B}=T$ 内，平均功率

$$\frac{dW}{T\,d\Omega} = P(\omega) = \frac{\sqrt{3}q^3B\sin\alpha}{2\pi mc^2}F\left(\frac{\omega}{\omega_c}\right)$$

or ω?

Thomson 散射是弹性散射，只是相对论性情况的近似。

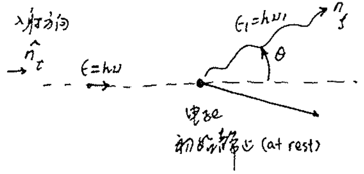
如果入射光的能量很高，那么电子会被冲击，光的频率也会变化；

$h\nu \simeq m_e c^2$

另外，如果入射光强度很大，散射截面的公式也要改变

} 量子效应

Compton Scattering



4-momentum of γ

$P_{\gamma i}^{\mu} = \frac{\epsilon}{c}(1,\hat{n}_i)$ $\qquad P_{\gamma f}^{\mu} = \frac{\epsilon_1}{c}(1,\hat{n}_f)$

4-momentum of electron

$P_{e i}^{\mu} = (mc,\vec{0})$ $\qquad P_{e f}^{\mu} = \left(\frac{E}{c},\ \vec{p}\right)$

metric $(-1,,,)$

运动学

Conservation of momentum

$$P_{ei}^{\mu} + P_{\gamma i}^{\mu} = P_{ef}^{\mu} + P_{\gamma f}^{\mu}$$

$$\Rightarrow \left|P_{ef}^{\mu}\right|^2 = \left|P_{ei}^{\mu} + P_{\gamma i}^{\mu} - P_{\gamma f}^{\mu}\right|^2$$

$$\left|P_{ef}^{\mu}\right|^2 = -\left(\frac{E}{c}\right)^2 + p^2 = -(mc)^2$$

$$\left|P_{ei}^{\mu} + P_{\gamma i}^{\mu} - P_{\gamma f}^{\mu}\right|^2 = \left|\left(\frac{\epsilon}{c}-\frac{\epsilon_1}{c}+mc,\ \frac{\epsilon}{c}\hat{n}_i-\frac{\epsilon_1}{c}\hat{n}_f\right)\right|^2 = -\left(\frac{\epsilon}{c}-\frac{\epsilon_1}{c}+mc\right)^2 + \left|\left(\frac{\epsilon}{c}\hat{n}_i-\frac{\epsilon_1}{c}\hat{n}_f\right)\right|^2$$

$$\Rightarrow -m^2c^4 = -(\epsilon-\epsilon_1+mc^2)^2 + \left|\epsilon\hat{n}_i-\epsilon_1\hat{n}_f\right|^2$$

$$\left|\epsilon\hat{n}_i-\epsilon_1\hat{n}_f\right|^2 = \epsilon^2+\epsilon_1^2-2\epsilon\epsilon_1\underbrace{\hat{n}_i\cdot\hat{n}_f}_{\cos\theta}$$

$$= -(\epsilon-\epsilon_1)^2 - m^2c^4 - 2mc^2(\epsilon-\epsilon_1) + \epsilon^2+\epsilon_1^2-2\epsilon\epsilon_1\cos\theta$$

$$\Rightarrow \epsilon\epsilon_1(1-\cos\theta) = mc^2(\epsilon-\epsilon_1)$$

$$\Rightarrow \epsilon_1 = \frac{\epsilon}{1+\frac{\epsilon}{mc^2}(1-\cos\theta)}$$

此即散射角和散射能量的关系

$\epsilon = \frac{hc}{\lambda}$

$$\Rightarrow \lambda_1 = \lambda\left(1+\left(\frac{\lambda}{\lambda_c}\right)^{-1}(1-\cos\theta)\right)$$

定义 $\frac{\epsilon}{mc^2} = \frac{h/mc}{\lambda} = \frac{\lambda_c}{\lambda}$，$\lambda_c = \frac{h}{mc}$ 为 Compton wave length（是电子的特征波长）

当 $\lambda<\lambda_c$ 即 $\epsilon>mc^2$ 时，Compton 效应明显

电子的质量m很小，$\lambda_c \simeq 0.024$Å，波长较长，实验容易观察到。

另外，λ_c 只与靶的质量有关。

散射引起频率变化

反弹时 $\Delta\lambda = 2\lambda_c$

不变方向时 $\Delta\lambda=0$

$$\underset{\text{loss of energy}}{\Delta\lambda} = \lambda_1-\lambda = \underset{\frac{h}{mc}}{\lambda_c}(1-\cos\theta)$$

只与散射靶的质量有关，和散射角有关（最重要的运动学结论）。

对于长波 $\frac{\lambda_1-\lambda}{\lambda}$ 很小，就回到 Thomson 散射。

之前的 thomson 散射截面 $\sigma_T = \frac{8}{3}\pi r_0^2$，微分散射截面 $\frac{d\sigma}{d\Omega} \cong \frac{1}{2}r_0^2(1+\cos^2\theta)$

Note:

$\iint_{all}\cos^2\theta\, d\Omega = \frac{4}{3}\pi$

$r_0 = \frac{e^2}{mc^2}$

现在，加入 QED 效应，对 Compton 散射

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2}r_0^2\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1}+\frac{\epsilon_1}{\epsilon}-\sin^2\theta\right) \Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{KN} = \frac{1}{2}r_0^2\left(\frac{\epsilon_1}{\epsilon}+\left(\frac{\epsilon_1}{\epsilon}\right)^3-\left(\frac{\epsilon_1}{\epsilon}\right)^2\sin^2\theta\right)$$

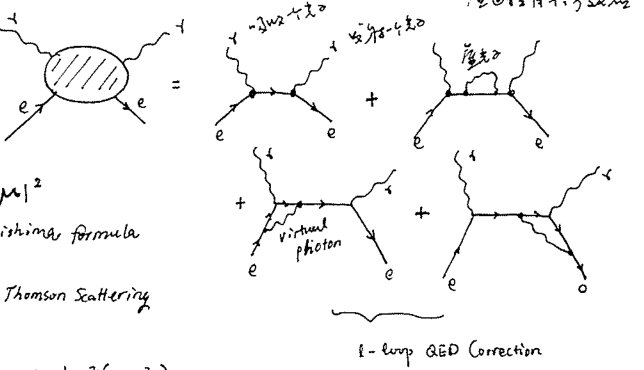
可以有哪些过程影响 Compton 散射？

实过程

虚过程有一些量子效应

$i\mathcal{M} =$

散射幅度矩阵



$$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2$$

Klein-Nishina formula

$\epsilon_1\sim\epsilon$, reduced to Thomson Scattering

$\epsilon\gg\epsilon_1$, $\frac{\epsilon_1}{\epsilon}\ll 1$

$$\left(\frac{d\sigma}{d\Omega}\right)_{KN} \ll \frac{1}{2}r_0^2(1+\cos^2\theta)$$

能量很高时，散射截面反而变小

Compton Scattering becomes less efficient

定义 $x = \frac{h\nu}{mc^2} = \frac{\lambda_c}{\lambda} \gg 1$，可以展开得到截面积。

$$\sigma = \sigma_T\cdot\frac{3}{4}\left(\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x}-\ln(1+2x)\right\}+\frac{1}{2x}\ln(1+2x)-\frac{1+3x}{(1+2x)^2}\right)$$

$$\begin{cases}\sigma_T\left(1-2x+\frac{26}{5}x^2+\cdots\right) & \text{for } x\ll1 \text{ (non-relativistic)}\\ \frac{3}{8}\sigma_T x^{-1}\left(\ln 2x+\frac{1}{2}\right) \propto \frac{\ln 2x}{x} & \text{for } x\gg1 \text{ (ultra-relativistic)}\end{cases}$$

电子处于加热状态时，可以有很高的能量；可以与CMB发生逆Compton散射

$T_e \gtrsim 10^4 K \simeq 1eV$ $T_{CMB} \simeq 3K\underbrace{(1+6)}_{\text{reionization redshift}} \sim 2\times10^{-3}eV \ll T_e$

电子打光子 { 电子损失能量 / 光子获得能量

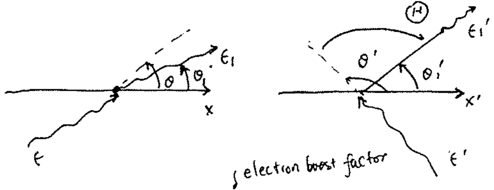
在电子的静止参考系下，仍然是光子损失能量.（电子坐标系下，光子有很高的能量）
下面定性估计光子获得的能量

Inverse Compton Scattering 低能光子和高能电子散射

In the rest frame of electron, $h\nu \ll mc^2$ non-relativistic $\sigma = \sigma_T$ ignore QED correction

高能量是Compton散射，低能是Thomson散射（长波长）

Let K be the lab frame, K' be the electron rest frame



Doppler shift $\epsilon' = \epsilon \gamma(1-\beta\cos\theta)$ (γ: election boost factor)

$\epsilon_1 = \epsilon_1'\gamma(1+\beta\cos\theta_1')$

$$\epsilon_1' = \frac{\epsilon'}{1+\frac{\epsilon'}{mc^2}(1-\cos\Theta)} \simeq \epsilon'\left(1-\frac{\epsilon'}{mc^2}(1-\cos\Theta)\right)$$

for Thomson Scattering

By definition

$$\cos\Theta = \hat{n}_{\gamma i}'\cdot\hat{n}_{\gamma f}' = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')\cdot(\sin\theta_1'\cos\phi_1', \sin\theta_1'\sin\phi_1', \cos\theta_1')$$

$$= \sin\theta'\sin\theta_1'\cos(\phi'-\phi_1') + \cos\theta'\cos\theta_1'$$

In case of relativistic electrons

$\gamma^2-1 \gg \frac{h\nu}{mc^2}$ $\epsilon':\epsilon_1' = 1+\frac{\epsilon'}{mc^2}(1-\cos\Theta) \simeq 1$ $\epsilon' \ll mc^2$（散射前后光子能量）

$\epsilon_1':\epsilon_1 = \frac{1}{\gamma(\beta\cos\theta_1'+1)} \sim \frac{1}{\gamma}$

$E_{\gamma i} : E_{\gamma i}' : E_{\gamma f}$

$\Uparrow$

$\epsilon:\epsilon' = \frac{1}{\gamma(1-\beta\cos\theta)} \overset{\theta=\frac{\pi}{2}}{\simeq} \frac{1}{\gamma}$

（boost 所获得的能量提升）

所以 $\underbrace{E_{\gamma i} : E_{\gamma i}'}_{\frac{1}{\gamma}} : \underbrace{E_{\gamma f}}_{\frac{1}{\gamma}} = 1:\gamma:\gamma^2$，能量获得了两次提升，完全由于Doppler效应

所以，光子能量提升是 γ^2 倍.

低能光子

low energy photon ϵ ⟶ high energy photon $\gamma^2\epsilon$

$m_ec^2 \simeq 500keV$, $\epsilon = 100keV \xrightarrow{\gamma^2}$ 问题出于500keV，此时不是很有效 这时就不是Inverse

γ^2提升只有在非相对论情况下适用
对高能光子不适用
（$\gamma\epsilon \ll m_ec^2$，光子获得的能量显著
$\gamma\epsilon \sim m_ec^2$，光子获得的能量不显著）

Review

$\epsilon : \epsilon' : \epsilon_1' : \epsilon_1 \simeq 1:\gamma:\gamma:\gamma^2$

- ϵ: photon energy before scattering in K
- ϵ': before in K'
- ϵ_1': after in K'
- ϵ_1: after in K

两次boost 获得能量

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Inverse Compton Power for Single Scattering

Want to know the average emitted power for isotropic distribution of photon scattering off an isotropic distribution of electrons.

Let $n(p)$ = photon phase space distribution function

$n\,\underbrace{d^3p\,d^3x}_{\text{invariant}}$ = photon numbers (invariant)

所以 $n=n'$, distribution is invariant

nd^3p = photon number density in $[\epsilon, \epsilon+d\epsilon]$

所以 $\epsilon = \gamma\epsilon'$, $d^3p = \gamma d^3p'$ $\Rightarrow$ $\frac{nd^3p}{\epsilon} = \frac{n'd^3p'}{\epsilon'}$ Lorentz invariant

The total emitted power in K' (rest frame of electron)

假定电子是静止的
光子场在K下是各向同性分布的

$$\frac{dE_1}{dt} = \frac{dE_1'}{dt'} = c\sigma_T\int \epsilon_1'\, n'd^3p'$$

$c\sigma_T$: 单位时间内的散射事件
ϵ_1': 单个光子的能量, emitted photon energy in K'
$n'd^3p'$: 单位体积内的数目

$$= c\sigma_T\int \underset{\epsilon'^2}{\epsilon_1'^2}\frac{n'd^3p'}{\underset{\epsilon'}{\epsilon_1'}} = c\sigma_T\int \underset{\epsilon'^2}{\epsilon_1'^2}\frac{nd^3p}{\underset{\epsilon'}{\epsilon_1}}$$

$$= c\sigma_T\int [\gamma(1-\beta\cos\theta)\underset{\epsilon}{\epsilon_1}]^2\frac{nd^3p}{\underset{\epsilon}{\epsilon_1}}$$

$$= c\sigma_T\gamma^2\int (1-\beta\cos\theta)^2\epsilon\, nd^3p$$

Suppose $\underbrace{\gamma^2-1}_{\text{光子获得能量}} \gg \underbrace{\frac{\epsilon}{mc^2}}_{\text{K系电子损失能量}}$

亦即在K'系的能量交换 ≪ K系的能量交换
即能量交换主要通过Boost产生

i.e. $\epsilon_1' = \epsilon'$

also $\frac{dE_1'}{dt'} = \frac{dE_1}{dt}$ （γ factor 消掉了）

For isotropic photon distribution $\langle(1-\beta\cos\theta)^2\rangle = 1+\frac{1}{3}\beta^2$, $\langle\cos\theta\rangle=0$, $\langle\cos^2\theta\rangle = \frac{1}{3}$ （这是对立体角平均）

defines photon energy density (initial) $u_{ph} = \int\epsilon\, nd^3p$

$$\frac{dE_1}{dt}=\frac{dE_1'}{dt'} = c\sigma_T\gamma^2\left(1+\frac{1}{3}\beta^2\right)u_{ph}$$

The rate of decrease of initial photon energy in K

$$\frac{dE_1}{dt} = -c\sigma_T\int \epsilon n d^3p = -c\sigma_T u_{ph}$$

总的 Compton 散射功率

$$P_{Compton} = \frac{dE_{rad}}{dt} = c\sigma_T u_{ph}\Big[\overset{\text{逆散射. 获能}}{\gamma^2\left(1+\frac{1}{3}\beta^2\right)} \underset{\text{正散射. 耗能}}{-1}\Big]$$

$$\Big\downarrow \; \gamma^2-1=\beta^2\gamma^2$$

$$= \frac{4}{3}c\sigma_T u_{ph}\gamma^2\beta^2$$

注: 这里假设了 Energy Transfer in K' is neglected.

如果不忽略. 只1阶

$$P_{Compton} = \frac{4}{3}c\sigma_T u_{ph}\gamma^2\beta^2\left(1-\frac{63}{10}\frac{\gamma\overset{\text{光子能量}}{\langle\epsilon^2\rangle}}{mc^2\langle\epsilon\rangle}\right)$$

↓

表示了K'下光子的能量损失

$\sim \frac{\langle\epsilon\rangle}{mc^2}$ 若精确

1阶考虑

如果电子的速度有分布

- Power law distribution of electron γ (UltraRelativistic)

$N(\gamma)d\gamma$ = electron number density in $[\gamma, \gamma+d\gamma]$

$$P_{tot} = \int P_{Compton}(\gamma)\, N(\gamma)\, d\gamma \quad \text{(total power per volume)}$$

$$\text{if } N(\gamma) = \begin{cases} C\gamma^{-p}, & \gamma_{min}<\gamma<\gamma_{max} \\ 0, & \text{otherwise} \end{cases}$$

$$P_{tot} \xrightarrow{\beta\simeq1} \overset{\propto}{\cong} \int \gamma^{2-p}d\gamma \propto \gamma^{3-p}\Big|_{\gamma_{min}}^{\gamma_{max}} = \frac{4}{3}\sigma_T c u_{ph} C(3-p)\left(\gamma_{max}^{3-p}-\gamma_{min}^{3-p}\right)$$

- Thermal distribution (of non-rel electrons)

$$\gamma\simeq1,\quad \langle\beta^2\rangle = \langle v^2/c^2\rangle = 3kT/mc^2$$

$$P_{tot} = n_e\langle P_{compton}\rangle = n_e\frac{4}{3}\sigma_T c\frac{3kT}{mc^2}u_{ph} = \left(\frac{4kT}{mc^2}\right)c\sigma_T n_e u_{ph}$$

Energy Transfer for repeated Scatterings in a finite thermal medium

CMB的谱不是完美的黑体谱

其 spectral distortion 是由高温的电子的逆Compton散射获得能量造成
(>10⁴K)

↑ 由 y parameter 描述

define the y parameter

y parameter = whether a photon will significantly change its energy during repeated scattering

= (average fractional energy change per scattering) × (mean number of scatterings)

If $y \gtrsim 1$, total energy of photon and spectrum will be significantly changed

$y \ll 1$, Not much changed

In non-rel ($\gamma\epsilon \ll mc^2$)

在K'系下光子能量变化不显著

$$\text{In } K' \quad \epsilon_1' = \frac{\epsilon'}{1+\frac{\epsilon'}{mc^2}(1-\cos\Theta)} \simeq \epsilon'\left(1-\frac{\epsilon'}{mc^2}(1-\cos\Theta)\right)$$

$$\frac{\Delta\epsilon'}{\epsilon'} = \frac{\epsilon_1'-\epsilon'}{\epsilon'} = -\frac{\epsilon'}{mc^2}(1-\cos\Theta)$$

↑ K'系下的散射角

所有假定散射是各向同性的

average over angle $\frac{\Delta\epsilon'}{\epsilon'} = -\frac{\epsilon'}{mc^2}$ (in K')

In lab frame

$$\frac{\Delta\epsilon}{\epsilon} = \underbrace{-\frac{\epsilon}{mc^2}}_{\text{损失的能量}} + \alpha\frac{kT}{mc^2}$$

↗ 和电子动能有关

↘ we guess, at least to the lowest order

光子获得的能量 不变

↓

只有光子与电子的相互作用. 若光子的分布

$$N(\epsilon) = K\epsilon^2 e^{-\epsilon/kT} \quad \left(\begin{matrix}\text{Non-degenerate} \\ \text{ultra-rel particles}\end{matrix}\right)$$

Planck 分布

相当于化学势为0的Bose分布

$\langle\Delta\epsilon\rangle = 0$ must be held

计算 $\langle\epsilon\rangle = \int\epsilon\frac{dN}{d\epsilon}d\epsilon \Big/ \int\frac{dN}{d\epsilon}d\epsilon = 3kT$

$$\langle\epsilon^2\rangle = \int\epsilon^2\frac{dN}{d\epsilon}d\epsilon \Big/ \int\frac{dN}{d\epsilon}d\epsilon = 12(kT)^2$$

$$\langle\Delta\epsilon\rangle = -\frac{\langle\epsilon^2\rangle}{mc^2} + \alpha\frac{kT}{mc^2}\langle\epsilon\rangle = \left(\frac{kT}{mc}\right)^2(-12+3\alpha) \Rightarrow \alpha=4$$

所以对和对足够多的光子

electrons in thermal equillibrium

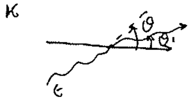
$$(\Delta\epsilon)_{NR} = \frac{\epsilon}{mc^2}(4kT-\epsilon)$$

- High temperature $4kT>\epsilon$, photons gain energy from electrons. (Inverse Compton Scattering)

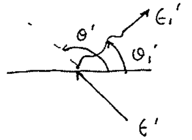
Low temperature $4kT<\epsilon$, photon lose energy (长波获能, 短波失能)

In ultra-relativistic limit ($\gamma \gg 1$) (but $\gamma\epsilon \ll mc^2$)

K



K'



$$\epsilon' = \epsilon\gamma(1-\beta\cos\theta)$$
$$\epsilon_1 = \epsilon_1'\gamma(1+\beta\cos\theta_1')$$

in K' $\epsilon' \simeq \epsilon_1'$ (because $\gamma\epsilon \ll mc^2$)

各向同性的散射 also $\langle\,\rangle$ 均为0

$$\epsilon_1 = \epsilon\gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta_1') = \epsilon\gamma^2\left(1-\underset{\sim}{\beta\cos\theta}+\beta\cos\theta_1'-\beta^2\underset{\sim}{\cos\theta\cos\theta_1'}\right)$$

对于光子是各向同性的 $\langle\,\rangle$ 均为0

$\langle\,\rangle$ 均为 $-1/3$
$\simeq -\langle\cos^2\theta\rangle$
$\theta_1' \simeq 180^\circ - \theta$?

$$(\Delta\epsilon)_R = \epsilon_1 - \epsilon = \epsilon\left(\gamma^2(1+\tfrac{1}{3}\beta^2) - 1\right)$$
$$\Big\downarrow\ \gamma^2-1 = \gamma^2\beta^2$$
$$= \frac{4}{3}\epsilon\gamma^2\beta^2$$
$$\overset{\beta=1}{\simeq} \frac{4}{3}\epsilon\gamma^2$$

$$\langle\gamma^2\rangle = \frac{\langle E^2\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2 ,$$

$$(\Delta\epsilon)_R = 16\epsilon\left(\frac{kT}{mc^2}\right)^2$$

有散射时，不能用直线情况下的光的传播方程
(辐射转移)

结论是

mean number of scattering $\simeq \mathrm{Max}\left(\tau_{es}, \tau_{es}^2\right)$

光厚时 τ_{es}^2
光薄时 τ_{es}

, τ_{es} 是电子散射光深
es ≃ electron scattering

$$\tau_{es} \simeq \rho\,\overset{[cm^{-1}]}{\overbrace{\kappa_{es}}}\,R$$

κ_{es} = electron scattering opacity $= \sigma_T/m_p = 0.40\,\mathrm{cm^2 \cdot g^{-1}}$

R = size of finite medium

$$(y\text{ parameter})_{NR} \cong \frac{4kT}{mc^2}\mathrm{Max}\left(\tau_{es}, \tau_{es}^2\right) \qquad (\text{if } 4kT \gg \epsilon)$$

$$(y\text{ parameter})_{\text{Relativistic}} \cong 16\left(\frac{kT}{mc^2}\right)^2 \max\left(\tau_{es}, \tau_{es}^2\right)$$

要研究相空间分布 $n(p)$ 的演变，需要从第一性原理出发（考虑 Boltzman 方程）
(或 $n(\omega)$)

Kompaneets Equation ~ Boltzmann方程的具体应用
(Repeated Scatterings by non-relativistic electrons)

let $n(\omega)$ = photon phase space density
↑ assume isotropic
$n(\vec{p})$

Reaction Equation $\gamma + e \rightleftharpoons \gamma + e$
for scattering $\omega\ \vec{p}\quad \omega_1\ \vec{p}_1$

let $f_e(\vec{p})$ = phase space density of electrons

Boltzmann equation for $n(\omega)$ due to Scatterings

Similar to that in Saha equation
"+" because Bose-Einstein enhancement

$$\frac{\partial n(\omega)}{\partial t} = c\underset{\substack{\uparrow\\ \text{electron}}}{\int d^3p}\int \frac{d\sigma}{d\Omega}d\Omega\left[\underbrace{f_e(\vec{p}_1)n(\omega_1)}_{\substack{\text{Scattering into }\omega\\ \text{by photons of }\omega_1}}\left(1+n(\omega)\right) - f_e(\vec{p})n(\omega)\left(1+n(\omega_1)\right)\right]$$

electron photon scattering cross-sec

为什么没有电子的 Pauli exclusion?

由于有动量守恒，所以守恒的结果，所以只有 $\int d^3p$ 而没 $\int d\Omega$

Boltzmann eq. can be expanded to 2nd order ⇒ Fokker-Planck eq.
$\Big\downarrow$ non relativistic thermal distribution of electrons
Kompaneets equation (1967)

For non-relativistic electron in thermal equilibrium

$$f_e(E) = n_e(2\pi mkT)^{-3/2}e^{-E/kT}$$

$E = \frac{p^2}{2m}$, n_e 是电子的 numberdensity
(electron number per unit volume)

在 Maxwell 分布里是 $e^{-\frac{1}{2}\frac{mv^2}{kT}}$
现在只是变一般的 Boltzmann 分布

Define energy transfer of electrons

$$\Delta = \frac{\hbar(\omega_1-\omega)}{kT} \ll 1$$

非相对论性电子在 kT 下不会显著改变能量ω
这是一个小量，可以用来展开 Boltzmann eq.

$$n(\omega_1) = n(\omega) + (\omega_1-\omega)\frac{\partial n(\omega)}{\partial\omega} + \frac{1}{2}(\omega_1-\omega)^2\frac{\partial^2 n_1(\omega)}{\partial\omega^2} + \cdots \qquad (*1)$$

$$f_e(E_1) = f_e(E) + (E_1-E)\frac{\partial f_e}{\partial E} + \frac{1}{2}(E_1-E)^2\frac{\partial^2 f_e}{\partial E^2} + \cdots \qquad (*2)$$

Note: $f_e(E)$ 写作 $e^{-E/kT}$ 形式 仅适合 $E \gg kT$

For electron, 由能量守恒 $E_1 - E = -(\omega_1-\omega)\hbar$，又 $f_e(E) \propto e^{-E/kT}$，$\frac{\partial f_e}{\partial E} = -\frac{1}{kT}f_e$

所以 将 $\frac{\partial f_e}{\partial E}$ 代入 (*2) 有

$$(E_1-E)\frac{\partial f_e}{\partial E} = -f_e(E)(-1)\frac{\hbar(\omega_1-\omega)}{kT}$$
$$= f_e(E)\Delta$$

$$f_e(E_1) = f_e(E) + f_e\Delta + f_e\frac{1}{2}\Delta^2 + \cdots = f_e(E)\left(1+\Delta+\frac{1}{2}\Delta^2+\cdots\right)$$

For photons, define $x = \hbar\omega/kT$. So that $n' \equiv \frac{\partial n}{\partial x}$, $n'' \equiv \frac{\partial^2 n}{\partial x^2}$

$$\frac{\hbar(\omega_1-\omega)}{kT}n' = (\omega_1-\omega)\frac{\partial n}{\partial \omega}$$

$$n(\omega_1) = n(\omega)(1+0) + n'\Delta + n''\frac{\Delta^2}{2}$$

Expanding the Boltzmann equation:

$$C^{-1}\frac{\partial n}{\partial t} = \underbrace{\iint}_{\text{all at } \omega, \vec{p}} d^3p\frac{d\sigma}{d\Omega}d\Omega\left[f_e\left(1+\Delta+\frac{1}{2}\Delta^2\right)\left(n+n'\Delta+\frac{1}{2}n''\Delta^2\right)(1+n) - f_e n\left(1+n+n'\Delta+n''\frac{1}{2}\Delta^2\right)\right]$$

note: 0^{th} order $= 0$

1^{th} order $= [f_e(1+n)(n'+n) - f_e nn']\Delta = \Delta f_e(n'+n+n^2)$

2^{nd} order $= \left[f_e(1+n)\left(\frac{1}{2}n''+n'+\frac{1}{2}n\right) - f_e n\frac{1}{2}n''\right]\Delta^2 = f_e\Delta^2\cdot\left(\frac{1}{2}n''+n'+\frac{1}{2}n+nn'+\frac{1}{2}n^2\right)$

总之，保留到2阶有

$$C^{-1}\frac{\partial n}{\partial t} = (n'+n(1+n))\iint \underbrace{d^3p\frac{d\sigma}{d\Omega}d\Omega\, f_e\Delta}_{I_1} + \underbrace{\left[\frac{1}{2}n''+n'(1+n)+\frac{1}{2}n(1+n)\right]}_{\text{all at } \omega}\underbrace{\iint d^3p\frac{d\sigma}{d\Omega}d\Omega}_{I_2}\, f_e\Delta^2$$

Δ与电子的动量有关

I_1: "Secular" shift in energy

I_2: "random walk" change in energy

As defined $\Delta = \frac{x\vec{p}\cdot(\vec{n}_1-\vec{n})}{mc} + O\left(\frac{kT}{mc^2}\right)$

$\vec{p}$: electron momentum before scattering

$\vec{n}$: photon propagation direction before scattering

$\vec{n}_1$: … after scattering

note: $\Delta = -\frac{E_1-E}{kT} = -\frac{1}{kT}(p_1^2-p^2)\frac{1}{2m}$

Cross section $\frac{d\sigma}{d\Omega} = d\sigma_{Thomson}/d\Omega = \frac{1}{2}r_0^2(1+\cos^2\theta)$

$$I_2 = \iint d^3p\int\frac{1}{2}r_0^2(1+\cos^2\theta)\,2\pi\, d\cos\theta\,\frac{n_e}{(2\pi mkT)^{3/2}}e^{-p^2/2mkT}\,x^2\frac{(\vec{p}(\vec{n}_1-\vec{n}))^2}{(mc)^2}$$

$$= 2x^2n_e\sigma_T\left(\frac{kT}{mc^2}\right) + O\left(\frac{kT}{mc^2}\right)^2$$

$kT/mc^2 \ll 1$ for non-rel case

$$I_1 = n_e\sigma_T\, x(4-x)\left(\frac{kT}{mc^2}\right)$$

Refine critical/characteristic time $t_c \simeq n_e\sigma_T c t = \frac{t}{\underbrace{(n_e\sigma_T c)^{-1}}_{\text{mean time before scattering}}}$

$$\frac{\partial n}{\partial t_c} = \left(\frac{kT}{mc^2}\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4(n'+n+n^2)\right]$$ 好看

$$= \left(\frac{kT}{mc^2}\right)\left[4x(n'+n(1+n)) + x^2(n''+n'+2n'n)\right]$$ 计算用 (Kompaneets eq.)

Plasma effect

↳ 整体是中性，但粒子都带电

Plasma = globally netural, ionized gas

- Dispersion in cold, isotropic plasma
  色散

When all qantities $\propto e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, the Maxwell equation

$$i\vec{k}\cdot\vec{E} = 4\pi\rho \qquad i\vec{k}\cdot\vec{B} = 0$$

$$i\vec{k}\times\vec{E} = \frac{i\omega}{c}\vec{B} \qquad i\vec{k}\times\vec{B} = \frac{4\pi\vec{j}}{c} - i\frac{\omega}{c}\vec{E}$$

电流一定改写为电子的运动（质子较重）

另外假定 Assume no external B-field（这样，等离子体才可以说是 isotropic 的）

Consider the motion of electron

$$m\ddot{\vec{x}} = -e\vec{E}$$

$$\Rightarrow -\frac{i}{\omega}m\vec{v} = -e\vec{E} \Rightarrow \vec{v} = \frac{e\vec{E}}{im\omega} \Rightarrow \vec{j} = -ne\vec{v} = \frac{ine^2}{m\omega}\vec{E} = \sigma\vec{E}$$

电导率 $\sigma = \frac{ine^2}{m\omega}$ (note: conductivity is pure imaginary)

Use charge conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

$$\Rightarrow -i\omega\rho + i\vec{k}\cdot\vec{j} = 0 \Rightarrow \rho = \frac{1}{\omega}\vec{k}\cdot\vec{j} = \omega^{-1}\sigma\vec{k}\cdot\vec{E}$$

⇓ 等离子体本身的净电荷密度

$$\Rightarrow i\vec{k}\cdot\vec{E} = 4\pi\rho = \frac{4\pi\sigma}{\omega}\vec{k}\cdot\vec{E} \Rightarrow \overset{want}{i\epsilon\vec{E}\cdot\vec{k} = 0} \Rightarrow \epsilon \equiv 1-\frac{4\pi\sigma}{i\omega}$$

↑ Source free

等离子体整体不带电

最后一个 Maxwell 方程

$$i\vec{k}\times\vec{B} = \frac{4\pi}{c}\sigma\vec{E} - \frac{i\omega}{c}\vec{E} = -\frac{i\omega}{c}\left(1-\frac{4\pi\sigma}{i\omega}\right)\vec{E} = -\frac{i\omega}{c}\epsilon\vec{E}$$

等离子体中的 maxwell 方程与真空中的一样，只要把介电常数换为 ϵ，成为无源方程

Solve the "source-free" Maxwell eqs

$$\vec{E} = \hat{a}_1E_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B} = \hat{a}_2B_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

here $E_0 = \left(\frac{\omega}{kc}\right)^2\epsilon E_0$, $c^2k^2 = \omega^2\epsilon$，即色散关系有变化

Note: 所以 $k = \frac{\omega}{c}\sqrt{\epsilon}$, $\epsilon = 1-\frac{4\pi}{i\omega}\sigma = 1-\frac{4\pi}{i\omega}\frac{ine^2}{m\omega}$

$= 1-\frac{\omega_p^2}{\omega^2}$. $\omega_p = \sqrt{\frac{4\pi ne^2}{m}}$

ω_p is called plasma frequency

色散关系 dispersion relation

$$k = c^{-1}\sqrt{\omega^2-\omega_p^2} \quad 或 \quad \omega^2 = \underset{\sim}{\omega_p^2}+c^2k^2$$

"好像 plasma 提供了一个静止的质量"

注 Plasma 中传播的电磁波可看 $\sqrt{\epsilon}$ 一种来表

For $\omega<\omega_p$, $k = \frac{i}{c}\sqrt{\omega_p^2-\omega^2}$

(pure imaginary)

$$e^{i\vec{k}\cdot\vec{r}} = e^{-\frac{1}{c}\sqrt{\omega_p^2-\omega^2}\cdot r}$$

- 将会指数衰减. Scale of decrease $\sim 2\pi c/\omega_p$
- ω_p can be viewed as a cut-off frequency to forbid $\omega<\omega_p$ propagation.

注意 ω_p 与 m, n 有关

- 这时的电导率 $\sigma = \frac{ine^2}{m\omega}$ 仍是 pure-imaginary

$i \simeq e^{i\frac{\pi}{2}}$. $\vec{j}=\sigma\vec{E}$ 相当于 $\vec{j}, \vec{E}$ 之间存在 $\frac{\pi}{2}$ 的相位差

mechanical work rate $w = -e\vec{v}\cdot\vec{E} \propto \vec{j}\cdot\vec{E}$

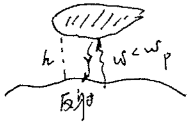
电场对电荷做功率

$$\langle w\rangle = 0 \quad (因为有\ e^{i\frac{\pi}{2}}\ 因子)$$

平均能量不能传递, 波返回在衰减层里面

(被反射)

- 这方法可测 地球的电离层参数



测得 ω_p 后可估计 n_e

脉冲的返回时间可测得云的高度 h

For $\omega>\omega_p$, 电磁波将传播

phase velocity $v_{ph} = \frac{\omega}{k} = \frac{c}{n_r} > c$

$\uparrow$ index of refraction

折射率 $n_r = \sqrt{\epsilon} = \sqrt{1-\frac{\omega_p^2}{\omega^2}} < 1$

group velocity

能量传递速度 $v_g = \frac{\partial\omega}{\partial k} = c^2\frac{k}{\omega} = c n_r < c$

若电磁波有外加磁场时, 将会有 Faraday 旋转效应

Propagation along a magnetic field (Faraday rotation)

Assume external B_0 field >> B-field in E-M wave

Motion of electron

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v}\times\vec{B_0} \qquad (外加磁场远比电磁波大.\ eE \gg e\underset{\underset{\ll 1}{\sim}}{\frac{v}{c}}\times\vec{B})$$

Assume $\vec{B_0}$ along the propagation direction of EM wave

Consider circular propagation

$\vec{E}(t) = E e^{-i\omega t}(\vec{\epsilon_1} \mp i\vec{\epsilon_2})$ "−" right circular polarization

"+" left ...

$\vec{B_0} = B_0\vec{\epsilon_3}$

$\vec{v}$ 的变化总是 $\perp\vec{\epsilon_3}$, 即 $\vec{\epsilon_3}\perp\vec{\epsilon_1},\vec{\epsilon_2}$, let $\vec{v} = v_1\vec{\epsilon_1}+v_2\vec{\epsilon_2}i$, 运动方程写为

$$-im\omega\left(v_1\vec{\epsilon_1}+iv_2\vec{\epsilon_2}\right) = -eE\left(\vec{\epsilon_1}\mp i\vec{\epsilon_2}\right) - \frac{eB_0}{c}\left(v_1\vec{\epsilon_2}+iv_2\vec{\epsilon_1}\right)$$

$$-im\omega v_1 = -eE - \frac{ieB_0}{c}v_2$$

$$m\omega v_2 = \pm ieE + \frac{eB_0}{c}v_1$$

对 "−" 右旋光 : $v_2 = v_1$

由对称性 对 "+" 左旋光 : $v_2 = -v_1$

$\Bigg\}$ $v_1 = \frac{-ie}{m}E\frac{1}{\omega\pm\omega_B}$, $\omega_B = \frac{eB_0}{mc}$ 回旋频率 cyclotron freq.

$$\vec{v} = v_1(\vec{\epsilon_1}\mp i\vec{\epsilon_2}) = \frac{-ie}{m(\omega\pm\omega_B)}E(\vec{\epsilon_1}\mp i\vec{\epsilon_2}) = \frac{e}{im(\omega\pm\omega_B)}\vec{E}$$

Current $\vec{j} = n(-e)\vec{v} = \frac{ine^2}{m(\omega\pm\omega_B)}\vec{E}$, 电导率 $\sigma = \frac{ine^2}{m(\omega\pm\omega_B)}$

Use Current Conservation

$$\rho = \sigma\omega^{-1}\vec{k}\cdot\vec{E}$$

介电常数 $\epsilon_{R\cdot L} = 1-\frac{4\pi\sigma}{i\omega} = 1-\frac{4\pi}{\omega}\frac{ne^2}{m(\omega\pm\omega_B)} = 1-\frac{\omega_p^2}{\omega(\omega\pm\omega_B)}$

同样. group velocity

$v_g = c\sqrt{\epsilon}$, $k_{R\cdot L} = \frac{\omega}{c}\sqrt{\epsilon_{R\cdot L}}$ 对右旋和左旋不一样

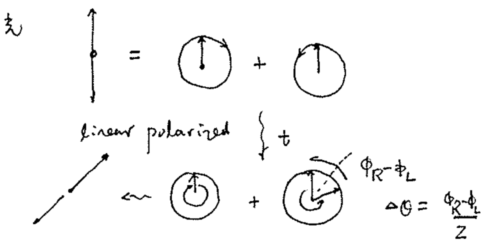
任何线性极化的光可分解为 左旋和右旋的光

由于它们的 k 不一样

(L 的小, 波长更长)

振动面将会有旋转.

旋转大小与路径长度一样有关 (路径长)



$\Delta\theta = \frac{\phi_R-\phi_L}{2}$

Rotation angle

$$\Delta\theta = (\phi_R-\phi_L)/2 = \frac{1}{2}\int_0^d (k_R-k_L)\,ds = \frac{1}{2}\int_0^d (c\omega^2)\,\omega_p^2\omega_B\,ds = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d nB_{\parallel}\,ds = \Delta\theta_F$$

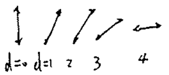
前面算的波是 $\vec{K} \parallel B_{\parallel}$

一般情况下 $\Delta\theta$ 只与 $B_{\parallel}$ 有关

If $\omega \gg \omega_p$, $\omega \gg \omega_B$, $k_{R,L} = \frac{\omega}{c}\left[1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1\mp\frac{\omega_B}{\omega}\right)\right]$ (Faraday Rotation)

$$\Delta k = k_R-k_L = (\omega^2 c)^{-1}\,\omega_p^2\omega_B \left(\text{determine } \Delta\theta\right)$$

随着光的传播, Rotation 逐渐发生:



; 当 n 和 $B_{\parallel}$ 与位置无关时, $\Delta\theta \propto d$.

- Faraday Rotation $\Delta\theta \propto \omega^{-2}$; 测量 $\Delta\theta \sim \omega$ relation 可以估计光经过的 n, B 等信息

  在 Interstellar Medium 里, 看不同 ω 的结果, 可以限制 B

Cherenkov Radiation

真空中的匀速带电粒子 ⇒ no radiation
vacuum uniformly move

在电介质中
dielectric
$$\begin{cases} v < v_{ph} \Rightarrow \text{No} \quad (\text{在 plasma 中, 总是如此}) \\ c > v > v_{ph} \Rightarrow \text{Yes, radiation} \end{cases}$$

If $n_r > 1$, $v_{ph} < c$ (带电粒子可以满足 $v > v_{ph}$)

在真空中的推迟势 $\Phi = \left[\frac{q}{\kappa R}\right]$

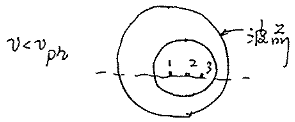
在介质中可得 modified "Lienard-Wiechart" potential, $\kappa = 1-\beta\cos\theta \rightarrow \kappa = 1-\beta n_r\cos\theta$ $(n_r > 1)$

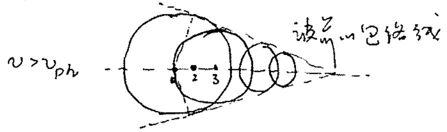
When $\cos\theta = (\beta n_r)^{-1}$. $\kappa = 0$ 而使 L-W 势发散 $\Phi \rightarrow \infty$

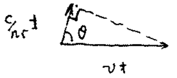
这在物理中不可能

在真空中. $\Phi \propto \frac{1}{R}$, $E \propto \frac{1}{R^2}$ 而无辐射; 但介质中因为 $\Phi \rightarrow \infty$ invalidates the argument against $1/R^2$

⇒ 可以产生 radiation







$$\cos\theta = \frac{c}{n_r\cdot v} = \frac{1}{\beta n_r}$$

If $n_r < 1$. $v_{ph} > c$ 不可能. 没有 Cherenkov radiation.

2018 0514　　6B211　　天体物理中的辐射机制

SR也是经典联
Quiz 经典部分 6.4 | 2018
R&L 第一章到第八章，量子部分之前

量子跃迁后的辐射 ~ 分立谱 { 原子,分子的转动谱线 / 电子能级跃迁辐射 ，谱线的 致宽 / 加宽 / 展宽.

经典力学的辐射 ~ 连续谱.

本节的内容见Frank & Shu 的讲义，网络上可下载.

Non-relativistic Quantum theory of Radiative Process (Shu Ch21)

- Justification of Non-rel treatment

Bohr Model　$\frac{1}{2}mv^2=\frac{e^2}{2a_0}$, where $a_0\equiv\frac{\hbar^2}{m_e e^2}$ is Bohr radius.

$$\Rightarrow \frac{v}{c}\simeq\frac{1}{137}=\alpha=\frac{e^2}{\hbar c}$$

$\Rightarrow \frac{v}{c}\ll1$，所以非相对论近似成立

Note: 在Bohr model下:

$$r_n = n^2\frac{(\hbar c)^2}{mc^2\cdot e^2}\Rightarrow a_0=\frac{\hbar^2}{me^2}$$

$$v_n=\frac{1}{n}\cdot\alpha c$$

$$T_n=\frac{1}{n^2}\frac{1}{2}m(\alpha c)^2\ ,\ \underline{V_n}=-2T$$

For typical radiative process/transition, $\hbar\omega$ "$\simeq$" $\frac{e^2}{2a_0}$

$$k=\frac{\omega}{c}\simeq\frac{e^2}{2a_0\hbar c}=\frac{1}{2}\frac{\alpha}{a_0}$$

$\Rightarrow ka_0=\frac{1}{2}\alpha\ll1$，所以经典电动力学的多极展开可成立
multiple expansion hold
for radiative transition in typical atom and neucleus.

For many nuclei atoms. system (heavy atoms)　effective coupling, $Z\alpha$,
Z = nucleus charge

when $Z\simeq100$, $Z\alpha\sim O(1)$, multiple expansion fail.

这里不讨论重元素的情况.

- For a single particle in EM field

$$\underset{(t,\vec{x},\vec{v})}{L}=\frac{1}{2}m\dot{\vec{x}}^2-q\phi(\vec{x},t)+q\frac{\dot{\vec{x}}}{c}\cdot\vec{A}(\vec{x},t)$$

Proof: Solve the Euler-Lagrangian eq. $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{x}}}\right)-\frac{\partial L}{\partial\vec{x}}=0$
to obtain Lorentz force $m\ddot{\vec{x}}=q(\vec{E}+\frac{\dot{\vec{x}}}{c}\times B)$

Define Canonical momentum

$$\vec{p}=\frac{\partial L}{\partial\dot{\vec{x}}}=\underbrace{m\dot{\vec{x}}}_{\text{机械动量}}+\frac{q\vec{A}}{c}\leftarrow\text{电磁动量}$$

Lagrangian transformation

$$\underset{(t,\vec{x},\vec{p})}{H}=\dot{\vec{x}}\cdot\vec{p}-L=\frac{1}{2}m|\dot{\vec{x}}|^2+q\phi=\frac{1}{2m}\left|\vec{p}-\frac{q}{c}\vec{A}(\vec{x},t)\right|^2+q\phi(\vec{x},t)$$

$q=-e$ for electrons: $H(t,\vec{x},\vec{p})=\frac{1}{2m}\left|\vec{p}+\frac{e}{c}\vec{A}(\vec{x},t)\right|^2-e\phi(\vec{x},t)$

The electrons interact only with

(a) an electrostatic field associated with an external charge distribution (e.g. nucleus or the rest of atom)

(b) Vacuum radiation fields　　忽略核的反冲

(i.e.) neglect the self-interaction of an electron, external B-field, etc)
自旋,因是量子电动力学的高阶效应

The gauge $\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0\Leftrightarrow\partial_\mu A^\mu=0$ (Lorentz gauge, 可以写成协变形式)

$\nabla\cdot\vec{A}=0$ (Coulomb gauge, 不是协变的，所以选了特殊参考系)

对非相对论性的情况，选后者有好处.

Helmholtz's Theorem

For any vector field $\vec{A}\overset{\text{can be decomposed to}}{=}\vec{A}_\parallel+\vec{A}_\perp$

$$\vec{A}_\parallel=\nabla\phi,\ \text{i.e. } \nabla\times\vec{A}_\parallel=0\ \text{(无旋)}$$

$$\vec{A}_\perp=\nabla\times\vec{Q},\ \text{i.e. } \nabla\cdot\vec{A}_\perp=0\ \text{(无源)}$$

We can always make EM field $\vec{A}$ satisfies $\vec{A}_\parallel=0$. s.t.　　φ完全描述了静电的部分

(1) Scalar potential φ carries the Coulomb filed due to the static charge distribution $\rho_e(\vec{x})$

(2) Vector potential $\vec{A}$ carries the vacuum radiation fields, i.e. $\vec{A}$ represents a radiation field.　　$\vec{A}$完全描述了外辐射场

Proof: suppose $\phi,\vec{A}$ do NOT fill the bill, make a gauge transformation

$$\begin{cases}\vec{A}'=\vec{A}+\nabla\psi\\ \phi'=\phi-\frac{1}{c}\frac{\partial\psi}{\partial t}\end{cases}$$

希望使 $\nabla\cdot\vec{A}'=0$

$$lhs=\nabla\cdot\vec{A}+\nabla^2\psi=0\Rightarrow\nabla^2\psi=-\nabla\cdot\vec{A}$$

是Poisson方程，解是 $\psi(\vec{x},t)=\frac{1}{4\pi}\int\frac{\nabla'\cdot\vec{A}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$

So we always find such a scalar field ψ.

For such a choice of gauge

$$\nabla^2\phi' = \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla^2\psi) = \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\nabla\cdot\underbrace{\left(-\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}_{\vec{E}} = -\nabla\cdot\vec{E}$$

$$= -4\pi\rho_e(\vec{x},t) = -4\pi\underset{\text{static}}{\rho_e(\vec{x})}$$

So ϕ' has the Coulomb solution $\phi'(\vec{x}) = \int \frac{\rho_e(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$

(i.e. Static)

$$\nabla^2\vec{A}' = -\nabla\times(\nabla\times\vec{A}') + \nabla(\underset{\text{Coulomb gauge}}{\cancel{\nabla\cdot\vec{A}'}}) = -\nabla\times\vec{B} = -\frac{1}{c}\frac{\partial\vec{E}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t}\right)$$

$$= \frac{1}{c^2}\frac{\partial^2\vec{A}'}{\partial t'^2}, \text{ since } \frac{\partial\phi'}{\partial t} = 0$$

$$\nabla^2\vec{A}' - \frac{1}{c^2}\frac{\partial\vec{A}'}{\partial t^2} = 0$$

$\vec{A}'$ Satisfies a wave equation, and represents a vacuum radiation field.

$$\left.\begin{array}{l}\nabla\cdot\vec{A}'=0 \Rightarrow \nabla\cdot\vec{A}_{/\!/}' = 0 \\ \qquad\qquad\quad \nabla\times\vec{A}_{/\!/} = 0\end{array}\right\} \Rightarrow \begin{array}{l}\vec{A}_{/\!/}' = 0 \\ \vec{A}_{\perp}' = \vec{A}'\end{array}$$

- Structure and interaction Hamiltonians

In Coulomb gauge

$$H = \frac{1}{2m_e}\left|\vec{p} + \frac{e}{c}\vec{A}(\vec{x},t)\right|^2 - e\phi(\vec{x})$$

$$= \underbrace{\frac{1}{2m_e}|\vec{p}|^2 - e\phi(\vec{x})}_{\substack{\text{与时间无关的部分} \\ H_{st} = H_0 \\ (\text{static})}} + \underbrace{\frac{e}{2m_e c}(\vec{p}\cdot\vec{A} + \underbrace{\vec{A}\cdot\vec{p}}_{\substack{\text{交叉项} \\ H_1 \propto \vec{A}}}) + \frac{e^2}{2m_e c^2}\underbrace{\vec{A}\cdot\vec{A}}_{H_2 \propto \vec{A}^2}}_{\substack{H_{int} \\ (\text{interaction})}}$$

Here H_{st} : static part
that determines the structure
of the electronic equilibrium state.

H_{int} : perturbation part due to interaction with the radiation field.

$H_1 \propto \vec{A}$, leads to 1 photon processes

$H_2 \propto \vec{A}^2$, leads to 2-photon processes

(很少被讨论，但很常见)

两光子的散射过程

单光子过程在某些情况是被禁止的，双光子
可以完全完成这一过程

Note $\frac{H_1}{H_0} \sim \frac{e\,pA/2m_e c}{\frac{1}{2m_e}p^2} \sim \underbrace{\frac{\sqrt{e^2/2a_0\cdot 2m_e}\;\frac{e}{2m_e c}\cdot k^{-1}E}{\frac{1}{2}m_e v^2}}_{k\sim\frac{\alpha}{2a_0}} \quad \frac{2a_0}{c}\sqrt{\frac{1}{4m_e a_0}}\;\frac{\alpha}{2a_0}E \sim$ see below

$\frac{1}{2}m_e v^2 \simeq \frac{e^2}{2a_0}$

$\because \frac{H_2}{H_1} \ll 1$: $\frac{H_2}{H_1} \sim \frac{\frac{e^2}{2m_e c^2}A^2}{\frac{epA}{m_e c}} = \frac{eA}{2pc} = \frac{eA}{2m_e vc} = \frac{\alpha^2 a_0 A}{2e\,v/c}$, $\alpha = \frac{e^2}{\hbar c}$

$\alpha^2 a_0 = \frac{e^2}{m_e c}$

$\sim \frac{\alpha a_0 A}{2e}$

$\frac{v}{c} \simeq \alpha$

From $\vec{B} = \nabla\times\vec{A}$, $B = kA$, $A = k^{-1}B$. $k \sim \frac{\alpha}{2a_0}$

$A \sim 2a_0\alpha^{-1}B$

$\sim 2a_0\alpha^{-1}E$ ($B = E$ for vacuum radiation field)

$$\frac{H_2}{H_1} \sim \frac{\alpha a_0}{2e}\,\frac{2a_0 E}{\alpha} = \frac{a_0^2 E}{e}$$

For photon absorption processes, energy density in vacuum radiation field

$$\frac{1}{8\pi}(B^2+E^2) = \frac{1}{4\pi}E^2 = \underset{\substack{\uparrow \\ \text{photon number density}}}{n_{ph}} \times \hbar\omega, \quad \hbar\omega \sim \frac{e^2}{2a_0} \text{ typical photon energy}$$

$$E^2 \sim 4\pi n_{ph}\frac{e^2}{2a_0}$$

$$= 2\pi n_{ph}\frac{e^2}{a_0}$$

$$\frac{H_2}{H_1} = \frac{a_0^2}{e}E \sim \frac{a_0^2}{e}\sqrt{\frac{2\pi n_{ph}e^2}{a_0}} = \sqrt{2\pi n_{ph}a_0^3}$$

$\frac{H_2}{H_1} \ll 1$, if $2\pi n_{ph}a_0^3 \ll 1$

i.e. $n_{ph} \ll \frac{a_0^{-3}}{2\pi} \sim 10^{24}\,cm^{-3}$

In Comparison, $n_{ph} \sim \frac{aT^4}{k_B T}$ ← Stefan law $= aT^3/k_B \sim 10^{14}\,cm^{-3} \ll 10^{24}\,cm^{-3}$

For hydrogen atoms in a star / hydron

$T > 10^4 K$

$\frac{H_2}{H_1} \ll 1$ always satisfied

Note: $\rho_r \simeq aT^4$

$\simeq \frac{5.6\times10^{-8}\,W\cdot m^{-2}}{c}\left(\frac{T}{K}\right)^4$

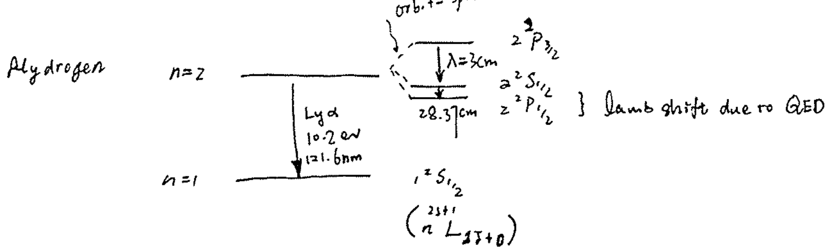
$\because \frac{H_1}{H_0} = \frac{\frac{e}{m_e c}pA}{\frac{p^2}{2m_e}} = \frac{2eA}{pc} = \frac{4H_2}{H_1} \ll 1$

所以，可以先讨论结构部分，把与外界的作用视作微扰.

Note: Exception when $H_1 \simeq 0$ 单光子跃迁被压低 (对称性), 双光子跃迁才突出

(两场跃迁可能不如回旋跃迁)

orbit-spin Coupling 导致 J不同, 能量不同; J不同决定.



selection rules

单光子跃迁(电偶极辐射)(electric dipole) 选择定则

$$\Delta L = 0, \pm1 \quad (\text{except for } \Delta L = 0 \text{ not allowed})$$

$$\Delta J = 0, \pm1 \quad (\text{except for } J = 0 \to 0)$$

$$\Delta S = 0$$

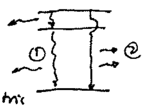
所以 2p→1s 可以 ($2^2P_{3/2} \to 1^2S_{1/2}$, $2^2P_{1/2} \to 1^2S_{1/2}$)

2s→1s 不可以

但是, 光子总是有办法从 2s→1s 过程产生, 不能使它完全处在 2s 态.

方法1 $2^2S_{1/2} \to 2^2P_{1/2} \to 1^2S_{1/2}$

方法2 $2^2S_{1/2} \to 1^2S_{1/2}$



而 $w \propto \Delta E^3$ for electric dipole transitions

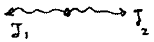
$W_{①} \simeq W_{2s\to2p} \simeq 10^{-21} W_{2p\to1s}$ (第一步能差太小)

$W_{②} \simeq \alpha^3 W_{2p\to1s} \simeq 10^{-7} W_{2p\to1s} \gg W_{①}$

而 $\propto \alpha^3$ for quadrapole transitions

在这种情况下 2s→1s Double photon decay wins.

由于这两个光子满足
$$\begin{cases} J_1+J_2=0 \\ S_1+S_2=0 \\ E_1+E_2=10.2\text{eV} \end{cases}$$
} Quantum entanglement.
能量可以连续分配.
产生连续谱



一般 $2s\to1s+2\gamma$ (contributes ~10% continuum background

- 在量子力学框架下求解 Hamiltonian, Radiation Transitions

$$H = H_{st} + H_1 + H_2$$

H_{st} = H_0 (unperturbed)

$$H_{st} \gg H_1 \gg H_2$$

这里记 $H = H^0 + H'(t)$

H^0: atomic Hamiltonian indep of t: $H^0 = \frac{|\vec p|^2}{2m} + e\phi(\vec x)$

$H'(t)$: perturbation due to external radiation field $H'(t) = -\frac{e}{2m}(\vec A\cdot\vec p + \vec p\cdot\vec A)$

For H^0, write $H^0 \phi_k = E_k \phi_k$

Time evolution $\psi(t) = \sum_k a_k \phi_k e^{-\frac{i}{\hbar}E_k t}$ 静态解的叠加, 但是在结构中的演化; 没有外加场

From state i to state f

$$H'_{fi}(t) = \int d^3x\, \phi_f^*(\vec x) H'(t,\vec x) \phi_i(\vec x) = \langle \phi_f | H' | \phi_i \rangle$$ 矩阵元 ← H'的矩阵元

$$H'_{fi}(\omega) = \frac{1}{2\pi}\int_0^T H'_{fi}(t') e^{i\omega t'} dt'$$ ← 假设一切波的发射/吸收是在一段时间存在 或者说扰动只在 0~T 内是非零的 T is the duration of perturbation

$$\omega = \frac{E_f - E_i}{\hbar}$$

Probability per unit time for a transition from i→f

$$W_{fi} = \frac{4\pi^2}{\hbar^2 T} | H'_{fi}(\omega_{fi}) |^2$$ (量子力学微扰论的基本结果)

↓ 由 H'_{fi} 积分上 $\int_0^T$ 的计算决定

这部分与 T 相关才对

下面的工作是 将 $H' = \frac{e}{mc}\vec A\cdot\vec p$ 代入 $H'_{fi}(t)$ 的表达式进行求解.

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For External EM field,

$$\vec A(\vec r,t) = \vec A(t) e^{i\vec k\cdot\vec r}$$ (假定平面波入射)

$\vec p_j \leftrightarrow -i\hbar\nabla_j$ j是电子个数的求和

Hamiltonian

$$H' = -\frac{e}{mc}\vec A\cdot\sum_j \vec p_j = \frac{ie\hbar}{mc}\vec A\cdot\Sigma\nabla_j = \frac{ie\hbar}{mc}e^{i\vec k\cdot\vec r}A(t)\sum_j\nabla_j$$

→ 一般地电荷e

(忽略 $\vec A\cdot\vec p$ 对易导致的常数项 对辐射无影响

$$H'_{fi}(t) = \frac{ie\hbar}{mc}\vec A(t)\cdot\langle f | e^{i\vec k\cdot\vec r}\sum_j\nabla_j | i\rangle$$

$$H'_{fi}(\omega) = \frac{ie\hbar}{mc}\vec A(\omega_{fi})\langle f \left| e^{i\vec k\cdot\vec r}\sum_j\nabla_j \right| i\rangle$$

Suppose $\vec{A} = A\vec{I}$, $\vec{I}$ is a unit vector

$$W_{fi} = \frac{4\pi^2 e^2}{T m^2 c^2} \left| A(\omega_{fi}) \right|^2 \left| \langle f | e^{i\vec{K}\cdot\vec{r}} \vec{I}\cdot\sum_j q_j | i \rangle \right|^2$$

下面表达出可观测量

Intensity $I = \langle \vec{S}\cdot\hat{n} \rangle = \frac{c}{4\pi T}\int_{-\infty}^{\infty} |E(t)|^2 dt = \frac{c}{2T}\int_{-\infty}^{\infty} \left|\tilde{E}(\omega)\right|^2 d\omega = \frac{c}{T}\int_0^{\infty} |\tilde{E}(\omega)|^2 d\omega$

$\tilde{E}(-\omega) = E(\omega)$ for real fourier transform

Define $\mathcal{J}(\omega) = \frac{dW}{dt\, dA\, d\omega} = \frac{c}{T}\left|\tilde{E}(\omega)\right|^2 \underset{\uparrow}{=} \frac{\omega^2}{Tc}|A(\omega)|^2$

$$\vec{E} = \underset{\uparrow}{-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}} \Rightarrow \widetilde{E}(\omega) = \frac{i\omega}{c}\widetilde{\vec{A}}(\omega) = \frac{i\omega}{c}\widetilde{A}(\omega)\vec{I}$$

没有静电势部分

所以几率可以写为

$$W_{fi} = \frac{4\pi^2 e^2}{m^2 c\, \omega_{fi}^2}\mathcal{J}(\omega_{fi}) \left| \langle f | e^{i\vec{K}\cdot\vec{r}} \vec{I}\cdot\sum_j q_j | i \rangle \right|^2$$

← 定态的部分，由原子物理给出 (单光子迁移的跃迁几率) *1

ω间隔内 J 越大，跃迁速率越快

可以发现 $W_{fi} = W_{if}$

因为 $\langle i | e^{i\vec{K}\cdot\vec{r}} \vec{I}\sum_j q_j | f \rangle = \int d^3x\ \phi_i^* e^{i\vec{K}\cdot\vec{r}}\, \vec{I}\cdot\sum q_j\, \phi_f$

$= \int d^3x\ \vec{I}\cdot\sum q_j\,(\phi_i^* e^{i\vec{K}\cdot\vec{r}}\phi_f) - \int d^3x\ \phi_f e^{i\vec{K}\cdot\vec{r}}\, \vec{I}\cdot\sum q_j \phi_i^*$

Surface term $\simeq 0$

$- \int d^3x\ \phi_i^*\phi_f \sum \underbrace{\vec{I}\cdot i\vec{K}}_{\text{后面项} \simeq 0} e^{i\vec{K}\cdot\vec{r}}$

$= -\int d^3x\ \phi_f\, e^{i\vec{K}\cdot\vec{r}}\,\vec{I}\cdot\sum q_j\,\phi_i^*$

$= - \langle f | e^{-i\vec{K}\cdot\vec{r}}\vec{I}\cdot\sum q_j | i \rangle^*$

K的大小与方向都相同
一个是incoming 一个是outgoing

$e^{i\vec{K}\cdot\vec{r}}$ 不能写成一般形式 可以做展开得到dipole项

$$e^{i\vec{K}\cdot\vec{r}} = \underset{\text{dipole}}{1} + \underset{\text{quadrapole}}{i\vec{K}\cdot\vec{r}} - \underset{\substack{\text{octupole} \\ \text{八极矩}}}{\frac{1}{2}(\vec{K}\cdot\vec{r})^2} + \cdots$$

Since $\vec{K}\cdot\vec{r} \sim Z\alpha \ll 1$

其中的首项阶次是电偶极跃迁 lowest order → electric dipole trisition

1st higher order : electric quadrupole + magnetic dipole trasition

为什么电四极和磁偶极量级相同?

洛伦兹力 $\vec{F} \propto -\frac{e}{c}\vec{v}\times\vec{B} \sim O(\frac{v}{c})$ 本来就是小量

电力
$\vec{F} = -e(\nabla\phi_{static} + E_{em})$
$= -eE_{em}$

Electric dipole $e^{i\vec{K}\cdot\vec{r}} \simeq 1$,

讨论最简单的情况

$$\langle f | e^{i\vec{K}\cdot\vec{r}}\vec{I}\cdot\sum_j q_j | i\rangle = \int d^3x\ \phi_f^*(\vec{I}\cdot\sum_j q_j)\phi_i = \sum_j i\hbar^{-1}(\vec{I}\cdot\vec{P}_j)_{if}$$

↑ 矩阵元

$= \sum_j \frac{m}{\hbar^2}\int \phi_f^*\vec{I}\cdot(\underbrace{\vec{r}_j H^0}_{\text{动量的形式}} - H^0\vec{r}_j)\,\phi_i\, d^3x$ （→作用到 ϕ_i）

$= \sum_j \frac{m}{\hbar^2}(-E_f + E_i)\int \phi_f^*\,\vec{I}\cdot\vec{r}_j\,\phi_i\, d^3x$

$= \sum_j \frac{m}{\hbar}(\omega_i - \omega_f)\,(\vec{I}\cdot\vec{r}_j)_{fi}$

其中 $H^0 = \frac{\sum p_j^2}{2m} - e\phi(\vec{r}_1 \cdots \vec{r}_N)$

而 $[\vec{r}_j, \vec{p}_j^{\,2}] = 2i\hbar\vec{p}_j \Rightarrow \hbar^{-2}[\vec{r}_j, H^0] = \frac{i}{m\hbar}\vec{p}_j$

注：这里求和和平方的次序可能有误

跃迁几率

$$W_{fi} = \frac{4\pi^2}{m^2 c}\frac{\mathcal{J}(\omega_{fi})}{\omega_{fi}^2}\frac{m^2}{e^2\hbar^2}\omega_{fi}^2\left|(\vec{I}\cdot\vec{d})_{fi}\right|^2 = \frac{4\pi^2}{c\hbar^2}\left|\left(\vec{I}\cdot\frac{\vec{d}}{e}\right)_{fi}\right|^2\mathcal{J}(\omega_{fi})$$

注：可能没有e

where $\vec{d} \equiv e\sum_j\vec{r}_j$

For unpolarized radiation , and for atoms with random orintations.

$$\left|\langle(\vec{I}\cdot\vec{d})_{fi}\rangle\right|^2 = \frac{1}{3}|d_{fi}|^2$$

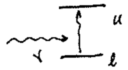
因为 $\vec{I}\cdot\vec{d} \propto \mu$, $\langle\mu^2\rangle = \frac{1}{3}$

这里 $|d_{fi}|^2 = |(d_x)_{fi}|^2 + |(d_y)_{fi}|^2 + |(d_z)_{fi}|^2$

$$\langle W_{fi}\rangle = \frac{4\pi^2}{3c\hbar^2}\left|\frac{d_{fi}}{e}\right|^2\mathcal{J}(\omega_{fi})$$

可以和 Einstein - AB 系数连系在一起
(假设 波是连续的
原子能级不连续)
(相当于从能量到单光子的量子力学情况)

Einstein B - coefficient



定义 $\langle W_{lu}\rangle = B_{lu}\mathcal{J}_{\nu_{ul}}$ （↑ 频率元素的；low到upper 吸收一个光子）

A ~ 是量子力学的 (自发辐射)
不能用半经典描述 (但可由细致平衡得出)
B 是半经典的, 可以用量子力学/半经典描述

$\mathcal{J}_{\nu_{ul}} = \frac{1}{4\pi}\mathcal{J}(\nu_{ul}) = \frac{1}{2}\mathcal{J}(\omega_{ul})$

per unit frequency interval

$\mathcal{J}(\omega) = \frac{dW}{dt\,dA\,d\omega} = \frac{1}{2\pi}\mathcal{J}(\nu)$

所以 $\langle W_{lu}\rangle = \frac{1}{2}B_{lu}\mathcal{J}(\omega_{ul}) \Leftrightarrow B_{lu} = \frac{2}{\mathcal{J}(\omega_{ul})}\langle W_{lu}\rangle = \frac{2}{\mathcal{J}(\omega_{ul})}\frac{4\pi^2}{3c\hbar^2}\left|\frac{d_{lu}}{e}\right|^2\mathcal{J}(\omega_{ul})$

$= \frac{8\pi^2}{3c\hbar^2 e^2}|d_{lu}|^2 = \frac{32\pi^4}{3ch^2e^2}|d_{lu}|^2$

For non-degenerate levels

$$B_{ul} = B_{lu} \qquad A_{ul} = \frac{4\omega_{fi}^3}{3c^3\hbar e^2}|d_{ul}|^2$$

[illegible]

For degenerate 簡併的話會有係數g

各個次能級的子能級都會要考慮

$$B_{lu} = \frac{8\pi^2}{3c\hbar^2e^2}\frac{1}{g_l}\sum_l\sum_u|d_{lu}|^2$$

不對 upper 平均，因為是從 l 出發（只對 g_l 做平均）

$$B_{ul} = \frac{8\pi^2}{3c\hbar^2e^2}\frac{1}{g_u}\sum_l\sum_u|d_{ul}|^2$$

$$A_{ul} = \frac{64\pi^4\nu_{ul}^3}{3hc^3}\frac{1}{g_u}\sum_u\sum_l|d_{ul}|^2$$

Define Absorption Oscillator strength f_{lu} such that

$$B_{lu} = \frac{4\pi^2e^2}{h\nu_{ul}mc}f_{lu}$$

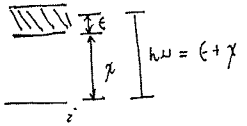
$$f_{lu} = \frac{2m}{3\hbar^2g_le^2}(E_u-E_l)\sum|d_{lu}|^2e^{-2}$$

Emission oscillator strength $f_{ul} < 0$

$$g_lB_{lu} = g_uB_{ul} \qquad B_{ul} = \frac{4\pi^2e^2}{h\nu_{lu}mc}f_{ul}$$

$$g_lf_{lu} = -g_uf_{ul} \qquad (\nu_{ul} = -\nu_{lu})$$

For transition between l and the Continuum



probability

$$\frac{df}{d\epsilon}d\epsilon$$

Continuum Oscillator Strength

$$f_c = \int_0^\infty\frac{df}{d\epsilon}d\epsilon = \int_{\nu_0}^\infty\frac{df}{d\nu}d\nu$$

$h\nu_0 = \chi$

- Selection Rules

$$W_{fi} \propto |\langle f|e^{i\vec{k}\cdot\vec{r}}\,\vec{\epsilon}\cdot\sum_j q_j|i\rangle|^2$$

electric dipole $\vec{d}_{fi} = e\int\phi_f^*\sum_j\vec{r}_j\,\phi_i\,d^3x$ ．要求計出 W_{fi}．$\vec{d}_{fi}$ 非零的項

① Laporte's rule : $\vec{d}_{fi} = 0$ if state i and f have the same parity

$$\phi(\vec{r}) \to \phi(-\vec{r}) = \pm\phi(\vec{r})$$

$$\sum\vec{r}_j \to \sum(-\vec{r}_j) \qquad \phi_f^*\phi_i(-\vec{r}) = \phi_f^*\phi_i(\vec{r})$$

parity $(-1)^{\sum l_i} = (-1)^L$

所以 $\Delta L \neq 0$ 才能發生躍遷

② change by only one orbital (one-electron jump rule)

各個電子的波函數 1~k 各個電子

$$\vec{d}_{fi} = e\sum_j\int u_a^*u_b^*\cdots u_k^*\,\vec{r}_j\,u_{a'}u_{b'}\cdots u_{k'}\,d^3x$$

$$= e\sum_j\int u_j^*\vec{r}_ju_j\,(\text{the rest})\,d^3x$$

If only one-electron jump allowed,

To evaluate $\vec{d}_{fi}$ between one-electron states

⟨1⟩ $[\vec{r},\vec{S}] = 0 \Rightarrow m_s = m_{s'}$ 自旋量子數不改變

⟨2⟩ $\int rR_{nl}R_{n'l'}\,dr \propto d_{fi}$

⟨3⟩ $d_{fi} \propto \int\mu\,Y_{lm}^*Y_{l'm'}\,d\Omega \Rightarrow d_{fi}$ is non-zero if $\Delta l = \pm1$, $\Delta m = 0, \pm1$

③ For many electron atoms, total quantum number

electric dipole for L-S Coupling
$\Delta S = 0$
$\Delta L = 0, \pm1$ ($L=0$ is not allowed)
$\Delta J = 0, \pm1$ (exept $J = 0 \to 0$)

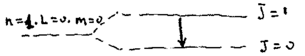
higher multiple transition

$$\Delta J = 0, \pm1 \quad (\text{except } J = 0 \to 0)$$

electric quadrupole
magnetic dipole $i\vec{k}\cdot\vec{r}$ } require parity is changed. 是否要改變 L

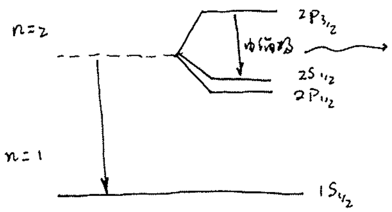
$\frac{\vec{v}\times\vec{B}}{c}$ 提供額外的 no parity

e.g. 21cm 譜線



21cm is forbidden by electric dipole transition, but allowed by magnetic dipole transition.

e.g.



→ 3cm ≃ 21cm

但這也是 radio 波段的重要譜線

可探測 super massive 黑洞附近吸積盤的模型

而 Transition Rate 則決定譜線的強度．下面介紹

Hw. R8L
10.4
10.5
10.6

Transition Rates 类氢粒子为例

Hydrogen-like irons (HI, HeII, LiIII, etc)

Bound-bound transition $\hbar\omega = Z^2 R_y\left(\frac{1}{n^2}-\frac{1}{n'^2}\right)$, $R_y = \frac{e^2}{2a_0}$ ≫ Rydeberg Constant 13.6eV

Bound-free transition $\hbar\omega = R_y/n^2 + \frac{1}{2}mv^2$ ↘ free electron kinetic energy

Bound-bound transition for Hydrogen

$$B_{lu} = \frac{8\pi^2}{3c\hbar^2}|\vec{d}_{lu}|^2 \quad \text{(dipole approx)}$$

lower to upper absorption B-coeff
(吸收率 ∝ B_{lu})

$$\vec{d}_{lu} \equiv e\int \phi_l^* \Sigma\, \vec{r}_j \phi_u\, d^3\vec{x} \propto \int R_{n\ell}^{(r)} R_{n'(\ell-1)}^{(r)}\, r\, dr \quad (\Delta\ell = 1)$$

$$\phi(\vec{x}) = R_{n\ell}(r) Y_{\ell m}(\hat{n})$$

↑ known function

absorption oscillator strength f_{lu}

$$B_{lu} = \frac{4\pi^2 e^2}{h\nu_{ul} m c} f_{lu}$$

For Lyman Series (n=1)

$$g_1 f_{1n} = \frac{2^9 n^5 (n-1)^{2n-4}}{3(n+1)^{2n+4}}$$

Bound-free transition for Hydrogen (连续吸收过程)

a.k.a = photonionization
(光致电离)

Differential trasition Rate
表征了粒子吸收截面

$$dW = \frac{4\pi^2 e^2}{m^2 c}\frac{J(\omega)}{\omega^2}\left|\langle f | e^{i\vec{k}\cdot\vec{r}}\,\vec{\ell}\cdot\nabla | i\rangle\right|^2 \times \frac{dn}{dp\,d\Omega}dp\,d\Omega$$

dW ↘ 总跃迁率

J(ω) ↑ 入射强度

↑ 连续谱的 continuum state f

the number of free electrons in (p, p+dp) and solid angle dΩ
(对于电子, 都可散射)

$$\hbar\omega = R_y + \frac{p^2}{2m}$$

↑ incident photons ↘ free electron

By definition $J(\omega) = \frac{dW}{dt\,dA\,d\omega} = \frac{\hbar\omega\, dN}{dt\,dA\,d\omega} = \frac{\hbar^2\omega\, dN}{dt\,dA(\hbar d\omega)} = \frac{\hbar^2 m\,\omega\, dN}{p\,dt\,dA\,dp} = \frac{m\hbar^2\omega}{p}\frac{dN}{dt\,dA\,dp}$

dW → 能量

对频度积分

$$\hbar\, d\omega = \frac{p}{m}dp$$

利用周期性边界条件下, $p_x = \frac{h}{L}n_x$

$$p^2\, dp\, d\Omega = d^3p = \frac{h^3}{V}dn \Rightarrow \frac{dn}{dp\,d\Omega} = \frac{V}{h^3}p^2$$

最后可以将 dW 写为

$$dW = \frac{4\pi^2 e^2}{m^2 c}\frac{m\hbar^2}{\omega p}\frac{dN}{dt\,dA\,dp}\left|\langle f|e^{i\vec{k}\cdot\vec{r}}\,\vec{\ell}\cdot\nabla|i\rangle\right|^2 \frac{Vp^2}{h^3}dp\,d\Omega$$

$$\underset{[M]^{-2}}{\frac{d\sigma_{bf}}{d\Omega}} = \frac{\overset{[T]^{-1}}{dW}}{\left(\frac{dN}{dt\,dA}\right)d\Omega} = \frac{\alpha v V}{2\pi\omega}\left|\langle f| e^{i\vec{k}\cdot\vec{r}}\,\vec{I}\cdot\nabla|i\rangle\right|^2$$

↑ incident photon → γ

$\alpha = \frac{e^2}{\hbar c}$, $v = \frac{p}{m}$ velocity of free electron

可以积分得到

$$\sigma_{bf} = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{(2\pi)^{9/2}\pi Z^5 c^{7/2}}{3a_0^{3/2}\omega^{7/2}} \quad \text{for } \hbar\omega \gg R_y$$

(总是和在小能级吸收无关, n=1, n=2, … 一样)

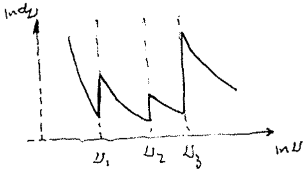
In general

$$\sigma_{bf} = \frac{5/2\,\pi^7 m e^{10} Z^4}{3\sqrt{3}\,c h^6 n^5}\frac{1}{\omega^3} g(\omega, n, \ell, Z)$$

↘ bound-free Gaunt factor

$$\alpha_\nu = N_n \sigma_{bf}$$

↑ atomic density at the absorption level n



光致电离的逆过程 (Recombination)

$$\gamma + H(n) \underset{\leftarrow}{\overset{\text{电离}}{\rightleftharpoons}} e + H^+$$

photoionization
可能是一次一次, 逐次进行
Radiative Recombination 辐射复合

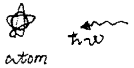
满足 Milne relation : Connection between the rates for photoionization and recombination

Line Broadening Mechanism $\phi(\nu)$

Mechanisms
- Doppler/thermal broadening
- Natural broadening
- Collisional broadening
- Combined broadening

Doppler (thermal) broadening

由于有相对运动的运动, 光子一进来就会有shift



atom
(bulk Velocity or thermal velocity)

无gradient 不展宽 只移动
有gradient 展宽+移动

→ 不影响total 强度
只是拉大线宽
(因为是随机的)

Thermal broadening Spreads the line out, but does not change total strenghoh.

$\nu_{RF} \leftrightarrow \nu$ for an observer due to Doppler shift

$$\nu - \nu_0 = \nu_0\frac{v_\parallel}{c}$$

～ 视线方向的

热运动致宽

number of atoms $n\,dv_z \propto \exp\left(-\frac{m_a v_z^2}{2kT}\right) dv_z$ (Maxwellian distribution)

m_a: atom mass

$v_z = \frac{c}{\nu_0}(\nu-\nu_0)$, $dv_z = \frac{c\,d\nu}{\nu_0}$

The strength of emission in $[\nu, \nu+d\nu]$ $\propto \exp\left[-\frac{m_a c^2}{2\nu_0^2 kT}(\nu-\nu_0)^2\right] d\nu$

Line profile $\phi(\nu) = \exp\left[\frac{-(\nu-\nu_0)^2}{(\Delta\nu_D)^2}\right]\frac{1}{\sqrt{\pi}\,\Delta\nu_D}$. Doppler width $= \Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m_a}} \propto \sqrt{T/m_a}\cdot\nu_0$

$\Delta\nu_D$: Doppler

$\frac{1}{\sqrt{\pi}\,\Delta\nu_D}$: normalization $\int\phi(\nu)d\nu = 1$

$\frac{\Delta\nu_D}{\nu_0} \propto \sqrt{T/m_a}$

中心处的吸收截面大小
Line center cross section

$$\sigma_{\nu_0} = B_{12}\frac{h\nu_0}{4\pi}\phi(\nu_0) = \frac{\pi e^2}{mc}f_{12}\frac{1}{\sqrt{\pi}\Delta\nu_D} = 1.16\times10^{-14}\left(\frac{\lambda_0}{1Å}\right)\sqrt{A/T}\,f_{12}\,(\text{cm}^2)$$

See Page 6

f_{12}: absorption oscillator strength

A: atom number

Non-thermal Doppler broadening due to bulk velocity (e.g. Turbulence)

When the scale of turbulance << m.f.p
远小于平均自由程

$\phi_{D(NT)}(\nu)$ is gaussian, $\Delta\nu_D = \frac{\nu_0}{c}\xi$

ξ: R.M.S of turbulance velocities

$$\Delta\nu_D(\nu) = \left(\Delta\nu^2_{D(T)} + \Delta\nu^2_{D(NT)}\right)^{1/2} = \frac{\nu_0}{c}\left(\frac{2kT}{m_a} + \xi^2\right)^{1/2}$$

$\Delta\nu_{D(T)}$: thermal

Note:
可以看成两个 gaussian profile 的卷积
可以计算得到 $\Delta\nu_0 = [\Delta\nu_1^2 + \Delta\nu_2^2]^{1/2}$

When the Scale of turbulance >> m.f.p

$\nu - \nu_{RF} = \nu_{RF}\frac{V_z(\vec{x})}{c}$ $\vec{x} \to V_z(\vec{x})$

包含到线展宽的贡献

$d\nu = \frac{\nu_{RF}}{c}\frac{dV_z}{dz}dz$

$$\nu_{RF} = \frac{\nu}{1+\frac{V_z}{c}} \approx \nu\left(1-\frac{V_z}{c}\right) \Rightarrow \frac{d\nu_{RF}}{d\nu} = 1-\frac{V_z}{c} - \nu\frac{1}{c}\frac{dV_z}{dz}\frac{dz}{d\nu},\quad \frac{d\nu_{RF}}{dz} = -\frac{\nu}{c}\frac{dV_z}{dz}$$

平移项 梯度项

$$\phi(\nu) = \delta(\nu_{RF}-\nu_0) = \frac{\delta(\nu-\nu')}{\left|d\nu/d\nu_{RF}\right|}$$

$\nu' = \nu_0\left(1+\frac{V_z}{c}(\vec{x})\right)$

$$= \frac{\delta(z-z_0)}{\left|\frac{d\nu_{RF}}{dz}\right|_{z_0}}$$

z_0: the possition of emission NOT Redshift

$\vec{z} \leftrightarrow V_z \leftrightarrow \nu_{RF}$
给定ν，可以对应到一个 z 或 V_z

$$= \frac{c}{\nu_0}\frac{\delta(z-z_0)}{\left|dV_z/dz\right|_{z_0}}$$

Natural Broadening (due to uncertainty Principle)

$\Delta E\Delta t \sim \hbar$

ΔE: spread in energy

Δt: duration

For spontaneous decay. rate $\gamma_n = \sum_{n'} A_{nn'}$

n': all lower energy state n'

$\Delta t_n \simeq \frac{1}{\gamma_n}$ $\Delta E_n \simeq \frac{\hbar}{\Delta t} = \hbar\gamma_n$

对于波函数 $\psi = \psi_n e^{-(E_n+\Delta E_n)t/\hbar} \propto e^{-\gamma_n t/2}$

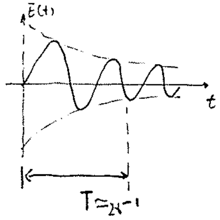
$\gamma_n/2$: spread 的一半

偶极矩 $d_{fi} \propto \langle f|e^{i\vec{k}\cdot\vec{r}}\,\vec{l}\cdot\vec{r}|i\rangle \propto e^{-\gamma t}$

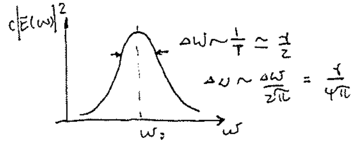
辐射电场 $E(t) \propto e^{-\gamma t/2}\sin\omega_0 t$

辐射电场: electric field

ω_0: line central frequency



变换到 Fourier space



$\Delta\omega \sim \frac{1}{T} \simeq \frac{\gamma}{2}$

$\Delta\nu \sim \frac{\Delta\omega}{2\pi} = \frac{\gamma}{4\pi}$

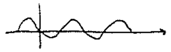
所以 $\phi(\nu) = \frac{(\Delta\nu)^2/\Delta\nu}{(\nu-\nu_0)^2+(\Delta\nu)^2}\frac{1}{\pi} = \frac{1}{4\pi^2}\gamma\frac{1}{(\nu-\nu_0)^2+(\frac{\gamma}{4\pi})^2}$ (Lorentz profile)

$\frac{1}{\pi}$: 归一化系数

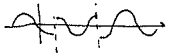
$1 = \int\phi(\nu)d\nu$

Collisional Broadening

在辐射时,若发生了碰撞, phase 可能突变



sinusoidal



Subject to random phase interprupted by atomic collisions
是一个完全随机的过程

ν_{col} = collisional frequency

on average, each atom experience ν_{col} collisions per unit time.

$\Delta t = \frac{1}{\nu_{col}}$

$\phi(\nu)$ 与 natural broadening 类似. $\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2+(\frac{\Gamma}{4\pi})^2}$, $\Gamma = \gamma + 2\nu_{col}$

Combined broadening (Doppler + Lorentz profile)

綜合起來. broadening 將是 Lorentz 和 Gaussian 的合成.

$$\phi(\nu)=\tilde{\phi}(\nu-\nu_0) = \frac{\Gamma^2}{4\pi^2}\frac{1}{(\nu-\nu_0)^2+(\Gamma/4\pi)^2}$$

Note: $\frac{v_z}{c} = \overset{\text{Doppler}}{\frac{\nu'-\nu_0}{\nu_0}}$ 再經 natural broadening $\nu' \to \nu$

$$\to \int_{-\infty}^{\infty} \frac{\Gamma^2}{4\pi^2}\frac{1}{(\nu-\nu_0-\nu_0\frac{v_z}{c})^2+(\frac{\Gamma}{4\pi})^2}\cdot\left(\frac{m}{2\pi kT}\right)^{1/2}\exp\left(\frac{-mv_z^2}{2kT}\right)dv_z$$

(即假定ν_0由于$\overset{\text{random}}{\text{速度}}$有移動)

i.e. The average of the Lorentz profile over the random velocity states.

(或稱之理解為 $\Phi(\nu)_{Thermal}$ 与 $\phi(\nu)_{Natural}$ 的疊加)

Define the Voigt function

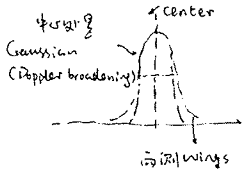
$$H(a,u) \equiv \frac{a}{\pi}\int\frac{e^{-y^2}dy}{a^2+(u-y)^2}\ ,\quad \text{記}\ \phi(\nu) = (\Delta\nu_D)^{-1}\pi^{-1/2}\,\underset{\uparrow}{H(a,u)}$$

$a = \frac{\Gamma}{4\pi\Delta\nu_D}$, $u = \frac{\nu-\nu_0}{\Delta\nu_D}$

$\Gamma = \gamma + 2\nu_{col}$

For $a \ll 1$, i.e. $\Gamma \ll \Delta\nu_D$

但展寬主要作用



by Lorentz profile, 比 exponential 慢

2018/0604 天体物理中的辐射机制

下一周 Quiz (建议先看作业题)
期末考试 预计 6月25日(18周)
在课堂考试, 2hr

Molecular structure

Bohr-Oppenheimer Approximation : sperate the motion of electrons and nuclei.

$$\frac{m_{electron}}{m_{nuclei}} \sim \frac{0.5\,MeV}{>1\,GeV} \sim (10^{-4} \sim 10^{-5})$$

在讨论电子的运动时，可以认为电子是在静止的核周围运动；

电子的运动对原子核的影响则可以用有效势来描述，总的来说是使核被束缚在一起.

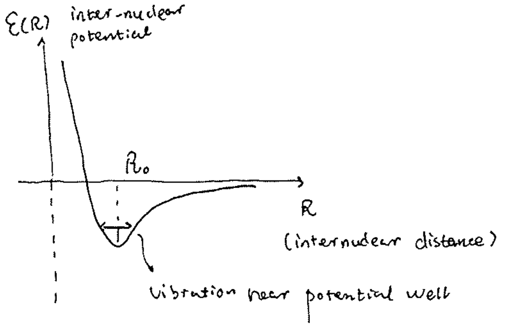
分子体系的总能量 = 电子能量 + 振动能 + 转动能
(核-核) (核-核)
(径向自由度) (角向自由度)

- Electron energy

$p \sim \frac{\hbar}{a}$, $E_{electron} \simeq \frac{\hbar^2}{2ma^2} \sim O(eV)$

a: typical molecular size ~ 1Å

- Vibration energy



$\mathcal{E} \sim \frac{1}{2} M \omega^2 \xi^2$

ω: vibration frequency　$\xi = R - R_0 \sim a$

Solved by $\frac{1}{2} M\omega^2 a^2 \simeq \frac{\hbar^2}{2ma^2}$ (经典的能量相等)

$\Rightarrow \omega = \frac{\hbar}{\sqrt{Mma^2}}$

$\Rightarrow E_{vib} \simeq \hbar\omega \simeq \frac{m}{M}\frac{\hbar^2}{ma^2} \sim \sqrt{\frac{m}{M}} E_{electron} \sim 10^{-2} eV$

量子力学的振动基态能量　质量很大

- Rotational Energy

$L \sim l\hbar$, $l = 0, 1, 2, \cdots$

$$E_{rot} \simeq \frac{L^2}{2I} = \frac{l(l+1)\hbar^2}{2I} \sim \frac{l(l+1)}{2}\frac{m}{M}\frac{\hbar^2}{ma^2} \underset{l \simeq 1}{\overset{\text{for small } l}{\sim}} \frac{m}{M} E_{electron} \sim 10^{-4} eV$$

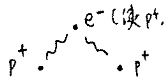
$I = Ma^2$　neglect a 2

Total Energy　$E = E_{electron} + E_{vibration} + E_{rotation}$

$1 : \sqrt{\frac{m}{M}} : \frac{m}{M}$

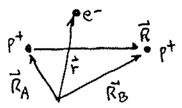
高阶量　更高阶量

⟨e.g⟩ for H_2^+ ion : 2 protons held together by 1 electron



e^- (使 p^+, p^+ 间稳地结合起来)

在欧氏坐标系下，Hamiltonian can be written by



$$\mathcal{H} = -\frac{1}{2}\nabla^2 - \frac{1}{|\vec{r}-\vec{R}_A|} - \frac{1}{|\vec{r}-\vec{R}_B|} + \frac{1}{|\vec{R}_A-\vec{R}_B|}$$

electron Kinetic energy

这里都是无量纲的量.
$m \xrightarrow{\text{记为}} m/m_e$
$\hbar \to \hbar/e$
$r \to r/a$
$E \to E/2R_y$

$a \simeq \frac{\hbar^2}{me^2}$
$R_y \simeq \frac{e^2}{2a_0}$

Assume　$\psi(\vec{r}) = \alpha\psi_A(\vec{r}) + \beta\psi_B(\vec{r})$

Two hydrogen atomic states
Bohr-Oppenheimer Approx.

$$\left.\begin{aligned}\psi_A(\vec{r}) &= \frac{1}{\sqrt{\pi}} e^{-|\vec{r}-\vec{R}_A|}\\ \psi_B(\vec{r}) &= \frac{1}{\sqrt{\pi}} e^{-|\vec{r}-\vec{R}_B|}\end{aligned}\right\} \text{for ground state(s)}$$

The probability function is symmetric about A ↔ B (so are the wave functions $|\psi(\vec{r})|^2$)

because the observable $\langle\mathcal{H}\rangle$, should be symmetric under A ↔ B

$\Rightarrow$ $\alpha = \beta$ or $\alpha = -\beta$.

$\Rightarrow$ Only allowed prob function

$\psi_\pm(\vec{r}) = C_\pm\left(\psi_A(\vec{r}) \pm \psi_B(\vec{r})\right)$

Normalization $\int |\psi_\pm|^2 d^3r = 1$

$\Rightarrow C_\pm = [2 \pm 2S(R)]^{-1/2}$

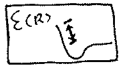
$S(R) = Re\int \psi_A^* \psi_B d^3r \underset{=\!=\!=\!=}{\text{for ground state}} (1+R+\frac{1}{3}R^2)e^{-R}$

$R = |\vec{R}_A - \vec{R}_B|$ fixed (把两核看作静止)

The Mean of Energy :

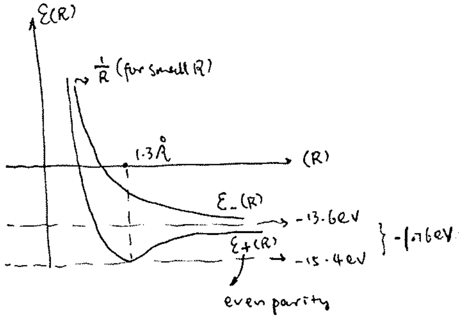
$$\langle H_\pm\rangle = \langle\psi_\pm|H|\psi_\pm\rangle = C_\pm^2\left[\langle\psi_A|H|\psi_A\rangle + \langle\psi_B|H|\psi_B\rangle \pm 2\langle\psi_A|H|\psi_B\rangle\right]$$

$$\langle\psi_A|H|\psi_A\rangle = \langle\psi_B|H|\psi_B\rangle \overset{\text{for ground state}}{=\!=} -\frac{1}{2}+\underset{\sim}{(1+R^{-1})}e^{-2R}$$

当R很小时，这一项提供很大的值，导致排斥 

$$\langle\psi_A|H|\psi_B\rangle = \mathrm{Re}\int\psi_A^*H\psi_B\,d^3r$$

$$= \left(-\frac{1}{2}+\frac{1}{R}\right)S(R)-(1+R)e^{-R}$$



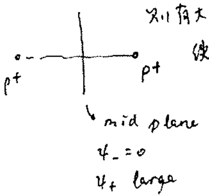
In experiment $R_0 \simeq 1.03$Å

$\Delta E \simeq -2.8$ ev ~ consistent with this solution

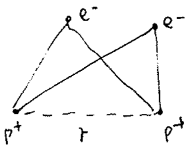
for anti-symmetric solution $\varepsilon_-(R)$,

no bound-state (old parity)

对于反对称解，由对称性，在 mid-plane 的波函数一定是0，所以电子很容易就跑到一边去了，不够稳定；对于对称解，电子则有大几率处在中间，可以使原子稳定下来



<e.g.> For H_2 molecule : $2H_2^+$ orbital states form H_2 wave function (of electron)



$$\underset{\uparrow\nearrow\ 2e^-}{\psi_s(1,2)} = \underbrace{\underset{\substack{\uparrow\\ e^-(1)\text{ in }H_2^+\\ \psi_+}}{[\psi_A(\vec r_1)+\psi_B(\vec r_1)]}\underset{\substack{\uparrow\\ e^-(2)\text{ in }H_2^+\\ \psi_-}}{[\psi_A(\vec r_2)+\psi_B(\vec r_2)]}}_{\text{Symmetric orbital wave function}}\frac{1}{2(1+S(r))}\times\chi_s$$

anti-symmetric spin wave function

$\chi_s = \frac{1}{2}\left[-|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle\right]$

(Singlet state)

由泡利不相容原理，ψ_s 关于 1↔2 交换应是反对称的 (Fermions)，所以自旋部分必须反对称

$$\text{orbital part} \propto \underbrace{[\psi_A(\vec r_1)\psi_A(\vec r_2)+\psi_B(\vec r_1)\psi_B(\vec r_2)]}_{\text{a }H^-\text{ iron + a proton}} + \underbrace{[\psi_A(\vec r_1)\psi_B(\vec r_2)+\psi_A(\vec r_2)\psi_B(\vec r_1)]}_{\text{two seperated }H\text{ atoms}}$$

可能性不大，此种状态易形成自由电子 + 自由 proton + H，所以往往忽略

两个分立的氢原子，加上交换项

忽略掉第一项，则有

$$\psi_s(1,2) \equiv \underset{\substack{\swarrow\\ \text{new normalization in terms of }S.\\ \text{不重要}}}{\frac{1}{\sqrt{2(1+S^2)}}}\left(\psi_A(\vec r_1)\underbrace{\psi_B(\vec r_2)+\psi_A(\vec r_2)}_{\text{even parity}}\psi_B(\vec r_1)\right)\times\underset{\substack{\uparrow\\ \text{odd parity}}}{\chi_s}$$

(Valence bond or London-Heiter method)

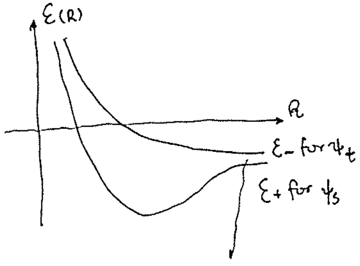
因为第一项是小量，

同样，构造 ψ(1,2) 也可以用两个 ψ_-，这样就有

$$\psi_t(1,2) \propto \left[\psi_A(\vec r_1)\underbrace{\psi_B(\vec r_2)-\psi_A(\vec r_2)}_{\text{odd parity}}\psi_B(\vec r_1)\right]\times\underset{\substack{\uparrow\\ \text{even parity}}}{\chi_t}$$

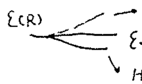
$\chi_t = |\uparrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle$ or $\frac{1}{2}\left[|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right]$

最后的有效势 ε(R) 是和 H_2^+ 类似的



(为什么自旋对称态的能量反而高一些？和单原子系统不一样？是因为泡利原理)

考虑更高阶项以后，对于很大的R



→ 若他原子有提升，也可能存在 bound-state

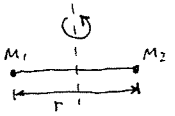
ε_- for ψ_t

H_2原子有下降

前面已经考虑到 $E_{electron} > E_{vir} > E_{rot}$，可以知道 rotation 的能级跃迁是最容易激发的，所以可能存在纯转动、振动激发。

Pure rotation Spectra for diatomic molecule

H_2, CO, OH 等都是典型的双原子分子



Orbital angular momentum quantum number

转动能量 $E_k = \frac{\hbar^2}{2I} K(K+1)$

或者可以写成 E_{nk}，因为K的取值与n有关.

radial quantum number

For dipole radiation (dipole approx.), need $d \neq 0$

d = permanent dipole moment $= Z_1 e r_1 + Z_2 e r_2 + d_e$

electron contribution

For homonuclear diatomic molecule (H_2, Cl_2, O_2), $d = 0$, 不能由纯转动产生偶极跃迁. ⇒ No pure rotational spectra

In general, $\Delta K = \pm 1$

$$\omega_K = \frac{E_{n,K+1} - E_{n,K}}{\hbar} = \frac{\hbar(K+1)}{\mu r_0^2}\left[1 - \frac{4\hbar^2(K+1)^2}{k_n \mu r_0^4}\right]$$

有效质量

r_0, 分子处在最低能量的距离

当温度更高，转动和振动能级跃迁都可以激发，但是 NO pure vibration spectra, because $E_{vib} > E_{rot}$

Rotation - Vibration Spectra

$$E_{nv} = \hbar \omega_{nk} (v + \tfrac{1}{2})$$

E: vibrational energy

v: vibrational quantum number 0, 1, 2, ⋯

偶极跃迁选择定则

$d \neq 0$

$\frac{d}{dr}(d)\Big|_{r=r_0} \neq 0$

$\Delta v = \pm 1$

$$\begin{cases} K = \pm 1, & \Lambda = 0 \\ K = \pm 1, 0, & \Lambda \neq 0 \end{cases}$$

这里 Λ 是电子的轨道角动量的轴向分量 (electronic orbital angular momentum along the internuclear axis)

Λ = 0, 1, 2

Σ, Π, Δ, 四等谱线，名字非常奇怪.

Electron - vibration - rotation Spectra

总的来讲，分子系统的能量 ($E_{electron} + E_{vib} + E_{rot}$)

电子的自旋角动量未考虑

total angular momentum

$\vec{J} = \vec{K} + \vec{L}$ = nuclear rotation + electron orbital

$$E_{nvJ} = V_{n0} + \gamma_n \hbar^2 \Lambda^2 + \alpha_n \hbar^2 J(J+1) + (v + \tfrac{1}{2}) \hbar \omega_n$$

component of $\vec{L}$ along the internuclear axis

Selection Rules

$$\begin{cases} \Delta\Lambda = 0, \pm 1 \\ \Delta J = 0, \pm 1 \ (\text{except } 0 \to 0) \\ \Delta J = 0 \text{ is not allowed if } \Lambda = 0 \to 0 \\ \Delta v \neq 0 \end{cases}$$

更进一步，计算各种跃迁的几率，就可以得到谱线强度. 一般来讲，转动能量能级差小，更容易激发，所以观测上 CO 的转动线更为重要.

Final exam Monday 25 8:30 - 12:00 6B211
(planned for 2.5 hr, just in case)

Late homework, no later than Monday June 18 11:59 a.m.

散射.

Recap 光沿直线传播, 只有发射.吸收时

Radiative transfer function $\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$

而散射不属于吸收/发射. 因方向随机

Pure scattering $I_\nu = I_\nu(t, \vec{x}, \hat{n}, \nu)$

Most generally $\left\langle \begin{array}{l} \text{direction } \hat{n} \text{ changes} \\ \text{frequency } \nu \text{ changes (eg. Compton)} \end{array}\right.$ 非弹性散射

Redistribution function $R(\hat{n}_1, \hat{n}_2, \nu_1, \nu_2)$ 较复杂

若只考虑 elastic scattering ν unchanged.
(also known as coherent scattering, monochromatic scattering)

再假设 散射后的发射光 各向同性

i.e. "scattering coefficient" σ_ν

$$\underset{\substack{\uparrow \\ \text{各向同性的} \\ \text{出射光}}}{j_\nu} = \sigma_\nu \underset{\substack{\searrow \\ \text{入射光}}}{J_\nu} \qquad J_\nu = \frac{1}{4\pi}\int I_\nu \, d\Omega$$

若只有散射. 无纯吸收/发射.

$$\frac{dI_\nu}{ds} = -\sigma_\nu I_\nu + j_\nu = -\sigma_\nu(I_\nu - \overset{\nearrow \text{入射光. unknown}}{J_\nu})$$

∴无法解出形式解.

可处理如随机游走 或 特殊情况的近似解.

随机游走 Random walk.

感谢梁赋玲同学

$$\vec{R} = \sum_i \vec{r}_i$$

$$l_*^2 \equiv \langle R^2 \rangle = \langle r_1^2 \rangle + \langle r_2^2 \rangle + \cdots + \langle r_N^2 \rangle + \underbrace{2\langle r_1 \cdot r_2 \rangle + \cdots}_{\substack{\text{cross-term} \\ \text{为0 for forward-backward} \\ \text{symmetry}}}$$

$$= N \cdot l^2 \quad (l: \text{mean free path})$$

l_* : root mean square of net displacement
also called effective mean path / diffusion length / thermal

$\Rightarrow l_* = \sqrt{N}\, l$ 即光子从产生到湮灭的路径

需估算 N : 媒介长为 L, 则 $\tau = L/l$ (i.e. $l \equiv \frac{L}{\tau}$)

For $\tau \gg 1$, $N = (l_*/l)^2 \doteq (L/l)^2 = \tau^2$

For $\tau \ll 1$, 有散射几率为 $(1-e^{-\tau}) \doteq \tau$ $\quad N \simeq \tau > \tau^2$

In general $N \doteq \max\{\tau, \tau^2\}$

若散射同时有吸收/发射 $\quad (j_\nu = \alpha_\nu B_\nu)$

$$\frac{dI_\nu}{ds} = \cancel{\alpha_\nu I_\nu + j_\nu} - \alpha_\nu I_\nu + \alpha_\nu B_\nu - \sigma_\nu(I_\nu - J_\nu)$$

$$= -(\alpha_\nu + \sigma_\nu) I_\nu + \alpha_\nu B_\nu + \sigma_\nu J_\nu$$

$$= -\underbrace{(\alpha_\nu + \sigma_\nu)}_{\text{extinction coefficient}}(I_\nu - S_\nu)$$

形式上定义 source function $S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$

For pure scattering l_ν (m.f.p) $= \sigma_\nu^{-1}$, pure absorption/emission $l_\nu = \alpha_\nu^{-1}$

For combined case $l_\nu = (\alpha_\nu + \sigma_\nu)^{-1}$

$$\underset{\text{纯吸收比例}}{\varepsilon_\nu \equiv \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}}, \quad \underset{\text{simple scattering albedo}}{\underline{1-\varepsilon_\nu}} = \underset{\text{纯scattering比例}}{\frac{\sigma_\nu}{\alpha_\nu + \sigma_\nu}}$$

相当于是 $\tau = \alpha_\nu l_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$

$\hookrightarrow N \simeq \varepsilon^{-1}$

$l = \frac{1}{\alpha_\nu + \sigma_\nu}$

$$l_*^2 = l^2 N = \frac{l^2}{\varepsilon}$$

$$\Rightarrow l_* = [\alpha_\nu(\alpha_\nu + \sigma_\nu)]^{-1/2}$$

$$\Rightarrow S_\nu = \underset{\text{散射贡献}}{(1-\varepsilon_\nu) J_\nu} + \underset{\text{吸收贡献}}{\varepsilon_\nu B_\nu}$$

τ_* : effective optical depth $\tau_* = L / l_*$

$\tau_* \ll 1$ $(l_* \gg L)$ effectively thin 逃逸前只散射无吸收

Luminosity $L_\nu = 4\pi j_\nu \cdot V = 4\pi \alpha_\nu B_\nu V$ 热平衡辐射

$\tau_* \gg 1$ $(l_* \ll L)$ effectively thick 大部分光子被媒介吸收

则 l_* 尺度范围内媒介达到热平衡

Luminosity $L_\nu = 4\pi (\alpha_\nu B_\nu) \cdot A \cdot l_*$ A 截面积

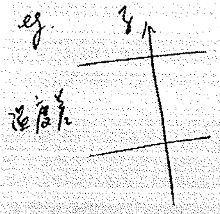
$$\alpha_\nu l_* = \left(\frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu} \right)^{1/2} = \sqrt{\varepsilon} \Rightarrow L_\nu = 4\pi \sqrt{\varepsilon_\nu} B_\nu A$$

以上是随机游走的处理

以下是近似解（之二）

Radiative Diffusion

eg.



光：假设 I_ν 等只与 z 有关 且变化慢

Rosseland Approximation

$$ds = dz / \mu$$

$$\Rightarrow \mu \frac{\partial I_\nu (z, \mu)}{\partial z} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu) \Rightarrow I_\nu = \cdots$$

$$F = \int_0^\infty F_\nu d\nu \qquad F_\nu = \int I_\nu \mu d\Omega$$

近似解 $F(z) = \underset{\Delta}{-} \frac{16\sigma T^3}{3\alpha_R} \cdot \frac{\partial T}{\partial z}$ eg. of radiative diffusion

光流的能流密度 温度梯度

Eddington approximation