研究生课程

天体物理中的辐射机制　　茅奕 ymao@tsinghua.edu.cn

- 电动力学 → 物质的辐射(机制) 经典
  的基本理论　动力学…
  ↘ 量子: 谱线 ↗ 半经典: Compton (Inverse)
  跃迁. f. Δf. 高中的原子. 双原子分子的谱线

- 教材: Rybicki & Lightman (内容精) follow, 跳过不重要部分
  Frank H. Shu　　盗版书网站 http://gen.lib.rus.ec/
  参考: 朗道 (二卷, 非常详实清楚)

- Grades:
  30%, 书上的题, 有solution 但不许抄 (一周后, 按时交)
  30%. Quiz, 3-in-class
  40%, Final Exam } open, one page of formulae
  期末, 或者回家自己考

Lecture 1

Fundamentals of radiation and radiation transfer

作业:
R&L 1.1, 1.2, 1.3, 1.4
下周一交

天体物理的类型
{ 电磁辐射 (光子) ~ 本课程
引力波辐射 (引力子)
中微子辐射 (中微子) 不成熟, 引力更小得多
宇宙线辐射 (高能粒子) 不是严格的辐射 但也是传递信息的媒介

Electromagnetic Spectrum
电磁辐射谱

prism
{ grating
slit
…
分光

~ $c=\lambda\nu$, $E=h\nu$, $T=\frac{E}{k}$

e.g.
硬X与γ有重叠
软X与紫外交叠

分光 → 中国的望远镜
FAST, HXMT, LAMOST

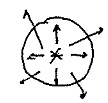
↘ 大气层对波的吸收
地面天文 (光学, 射电)
空间天文 (紫外, X, γ)
红外, 亚毫米

e.g. 不同波段的太阳 / MW 图像
(天体, 不同波段的能谱 不一定是黑体谱, 有结构, 如quasar) 太阳

1. Elementary properties of radiation

- radiation flux

假设光是直线传播 (忽略衍射等…)
Assume light travels in a straight line. Flux $F=\frac{dE}{dt\,dA}$



各向同性的发光体: $F\cdot 4\pi r^2=\frac{dE}{dt}$ = power

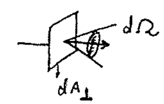
对power不含时时, $F\propto\frac{1}{r^2}$ (能量守恒的结果)

我们希望描述光的物理量是恒定的 (与传播距离无关), F 不是很好

- Specific intensity (brightness), for individual rays
  比强度

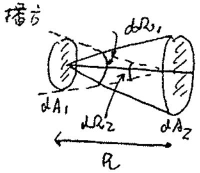
$$I_\nu(t,\vec{x},\hat{n},\nu)=\frac{dE}{dt\,dA\,d\Omega\,d\nu}$$

位置　传播方向 与 $d\Omega$ 有关



比强度的"比" $d\nu$, 没有 $d\nu$ 时称 intensity

prove: Constancy of I_ν along rays in free space, $\frac{dI_\nu}{ds}=0$ (沿传播方向) — 无吸收, 发射, 散射



经过 dA_1 的光也必经过 dA_2

$$dE=I_{\nu_1}dt_1\,dA_1\,d\Omega_1\,d\nu_1=I_{\nu_2}dt_2\,dA_2\,d\Omega_2\,d\nu_2$$

$dt_1=dt_2$ 不考虑相对论效应
$d\nu_1=d\nu_2$ 不考虑红移

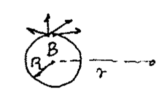
$d\Omega_1=\frac{dA_2}{R^2}$
$d\Omega_2=\frac{dA_1}{R^2}$

$$\frac{dA_2}{d\Omega_1}=\frac{dA_1}{d\Omega_2}\Rightarrow dA_1\,d\Omega_1=dA_2\,d\Omega_2$$

$$\Rightarrow I_{\nu_1}=I_{\nu_2},\ 即\ I_\nu=\text{Constant}$$

说明: 不与平方反比律矛盾, 可以导出平方反比律 (它们之间的一致性)
The inverse square law from the Constancy of I_ν

$$\frac{dF}{d\nu}=\int I\cos\theta\,d\Omega=B\int_0^{2\pi}d\phi\int_0^{\theta_c}\cos\theta\sin\theta\,d\theta$$
$$=2\pi B\sin^2\theta_c\cdot\frac{1}{2}$$
$$=\pi B\left(\frac{R}{r}\right)^2\propto\frac{1}{r^2}$$

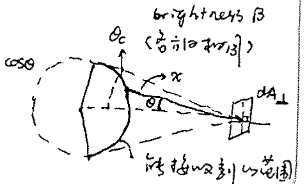


说明: 与前面的差别是, 这里是有限大的球体, 不是点状发光体, 但仍是 $\frac{1}{r^2}$

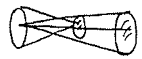
而
$$I=\begin{cases}B, & \text{if ray intersects with sphere}\\ 0, & \text{without}\end{cases}$$

对x光束
$dA_\perp=dA\cdot\cos\theta$

$$\delta\left(\frac{dF}{d\nu}\right)=\frac{dE}{dt\,dA\,d\nu}=I\cos\theta\,d\Omega$$



Note: (1) at large r，可以简单地证明 constancy I_ν 与单位立体角的关系



$I = \dfrac{dE}{dt\,dA\,d\Omega\,d\nu}$

同一束光，在不同距离 $dA \propto r^2$

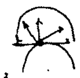
$d\Omega \approx \pi\left(\frac{R}{r}\right)^2 \propto \frac{1}{r^2}$

所以 $I = \text{const}$

(2) at the surface $r = R$

$\frac{dF}{d\nu} = \pi B \neq 2\pi B$ 　和光源的直觉不同

$\neq 2\pi B \times 2$ 　因为只有一半的光能接收到



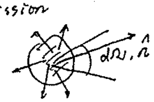
2. Radiative transfer

Along the ray, $\frac{dI_\nu}{ds} = 0$ in free transfer

ds = differential element of length along the ray

有哪些因素会改变辐射的强度？ Emission, Absorption, Scattering ...

- Emission



不一定各向同性

$dE = j_\nu dV\, d\Omega\, dt\, d\nu$

$\uparrow$ 辐射系数

spontaneous emission coefficient

(自发，与后面的受激发射对应，只与自发辐射率有关)

定义 $j = \int j_\nu d\nu$ 也叫自发辐射系数

For a ray traveling through the volume

dA cdt

体积 $dV = c\,dt\,dA = ds\,dA$

经过辐射，比强度增量 $dI_\nu = \dfrac{dE}{dt\,dA\,d\Omega\,d\nu} = \dfrac{j_\nu ds\,dA\,dt\,d\Omega\,d\nu}{dt\,dA\,d\Omega\,d\nu} = j_\nu \cdot ds$

或 $\frac{dI_\nu}{ds} = j_\nu$

另外 emissivity (按质量来 specify)

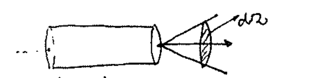
$$dE = \epsilon_\nu \underbrace{\rho\, dV}_{dm}\, dt\, \underbrace{\frac{d\Omega}{4\pi}}_{\text{the fraction of energy radiated into } d\Omega}\, d\nu$$

而 j_ν 与 ϵ_ν 的关系 $j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$

- Absorption

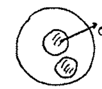
(自发)发射只与发光率有关

吸收与入射光有关，也与吸收率有关



dA ds

toy 经典 model: consider a beam of I_ν travels through a cloud of gas with number density n



σ_ν 某个粒子的截面，与 ν 有关

粒子的截面足够小(间距足够大)，σ_ν 不重叠

gas particle presents an effective absorbing area σ_ν

(cross section)

number of absorber $= n\,dA\,ds$

Total absorbing area $= n\sigma_\nu dA\,ds$

Total energy absorbed

$$dE = -I_\nu \underbrace{(n\sigma_\nu dA\,ds)}_{\text{粒子的总截面积}} d\Omega\, dt\, d\nu$$

而 $dE = dI_\nu \cdot dA \cdot dt\, d\Omega\, d\nu$

combine of two $\frac{dI_\nu}{ds} = -I_\nu \cdot n\sigma_\nu$

* independent of incoming ray

* only dep of gas

α_ν

In general, we define α_ν as absorption coefficient ($[L]^{-1}$)

$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$

只与吸收率有关。

在 toy model 里，与 n, σ_ν 有关

一般地，与 n, ν, 材料有关

Note: (1) assumption: (在 toy model 里 粒子间有足够大的距离)

- linear scale of $\sigma_\nu \ll$ mean interparticle distance

$\sqrt{\sigma_\nu} \ll d \sim n^{-1/3}$

亦即 $\left(\frac{\alpha_\nu}{n}\right)^{1/2} \cdot n^{1/3} \ll 1 \Rightarrow \frac{\alpha_\nu}{n} \cdot n^{2/3} = \alpha_\nu \cdot n^{-1/3} \sim \alpha_\nu \cdot d \ll 1$

- random distribution

(吸收的粒子应该随机分布，对热平衡的系统一般是满足的)

(2) 这里的 "Absorption" = true absorption (纯吸收的)

$+$

stimulated emission

(受激发射，也有同样的关系，也被包含在内)

有时，可以有 $\alpha_\nu < 0$

(3) 这里的 "Absorption" does NOT include scattering

("吸收"是减少入射的能量
"散射"只改变 ν, angle 的分布, 没有减少能量, 不能用直线传播来描述)

- Radiation transfer equation (No scattering)
辐射转移方程

$$\frac{dI_\nu(s)}{ds} = j_\nu(s) - \alpha_\nu(s) I_\nu(s)$$

emission absorption

$S_\nu(s')$

s_0 s' s

$\tau_\nu(s')$ $\tau_\nu(s)$

$I_\nu(0)$ $\underbrace{I_\nu(0)e^{-\tau_\nu}}_{吸收} + 发射$

可以有形式解, 但无直解, define optical depth 光深 $d\tau_\nu(s) = \alpha_\nu(s)\,ds$

(when $\alpha_\nu(s) > 0$, τ_ν monotonically increasing along s. $\tau_\nu(s) \leftrightarrow s$)

$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\,ds'$ (无量纲的数), 光深是一个积分量, 注意它是积分变数后, 不只与某点的结果有关

求解 $I_\nu(s)$ 等价于求解 $I_\nu(\tau_\nu)$

将上式两边除以 α_ν, 有

$$\frac{dI_\nu}{d\tau_\nu} = \frac{dI_\nu}{\alpha_\nu ds} = \frac{j_\nu(s)}{\alpha_\nu} - I_\nu(s)$$

$S_\nu = \frac{j_\nu}{\alpha_\nu}$ source function

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu(s) + S_\nu$$

常数变易法和 Green 函数法?

求解 $\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = S_\nu e^{\tau_\nu}$ 得 $I_\nu(s)e^{\tau_\nu(s)} = \underbrace{I_\nu(s_0)e^{\tau_\nu(s_0)}}_{1} + \int_0^{\tau_\nu} \underbrace{e^{-(\tau_\nu-\tau_\nu')}}_{设为\ e^{\tau_\nu'(s')}} S_\nu(\tau_\nu')\,d\tau_\nu'$

也就是

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu} + \underbrace{\int_0^{\tau_\nu} e^{-(\tau_\nu-\tau_\nu')} S_\nu(\tau_\nu')\,d\tau_\nu'}_{发射}$$

入射光经 brightness at ∞

$I_\nu(0)e^{-\tau_\nu}$ 为入射的吸收后能量

$S_\nu(s')e^{-(\tau_\nu-\tau_\nu')}$ 为重发射的光被吸收后的能量 (新产生光被吸收后的结果)

[formal solution
并非 $I_\nu \sim s$ 关系,
而是 $I_\nu \sim \tau_\nu$ 关系, 只是形式上解决了, $\tau_\nu \sim s$ 关系仍未知] 往往需要数值模拟, 沿光线方向 tracing, 积分

Note: (1) for case $\tau_\nu > 1$, optically thick or opaque
$\tau_\nu < 1$, optically thin or transparent

(2) for only emission, $\alpha_\nu = 0$ (没有光深)

$$\frac{dI_\nu}{ds} = j_\nu$$

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\,ds'$$

(3) for only absorption $j_\nu = 0$, $S_\nu = 0$

$$I_\nu(s) = I_\nu(s_0)\,e^{-\tau_\nu}$$

τ_ν 是积分量.

(4) S_ν is a constant

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1-e^{-\tau_\nu}\right)$$

also $I_\nu - I_\nu(0) = (S_\nu - I_\nu(0))(1-e^{-\tau_\nu})$

有时, 观测到的光有背景光 $I_\nu(0)$ background, 例如 21cm, 需要的是 $I_\nu - I_\nu(0)$

(5) 极特殊的情况

when $S_\nu \neq 0$ only at a point (忽略热展宽, 量子展宽的理想化情况, 如 21cm 谱中)

可证明也有 $I_\nu(s) - I_\nu(s_0) = (S_\nu(s) - I_\nu(s_0))(1-e^{-\tau_\nu})$

往往只在光学薄的情况 $\tau_\nu \ll 1$, 或者忽略后面经过全吸收的发射, 此时

$$I_\nu(s) - I_\nu(s_0) = [S_\nu(s) - I_\nu(s_0)]\cdot\tau_\nu$$

- Mean free path (MFP)

有吸收时, 光不可能无限传播; 在能传播的有限距离, 给平均自由程;

Mean free path is defined as the average distance a photon can travel thru an absorbing material w/o (without) being absorbed.

$$I_\nu = I_\nu(0)\,e^{-\tau_\nu}$$

mean $\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu \underbrace{e^{-\tau_\nu}}_{光子前进 \tau_\nu 的几率} d\tau_\nu = 1$

probability of a photon travelling τ_ν

homogeneous medium

$$\langle\tau_\nu\rangle \equiv \alpha_\nu l_\nu = 1$$

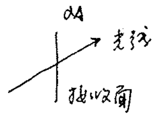
$$\Rightarrow \text{MFP } l_\nu = \frac{1}{\alpha_\nu}$$

在非均匀介质中, 可定义某地点 local MFP $l_\nu = \frac{1}{\alpha_\nu}$ at the point in an inhomogenous medium

注: 经常用来估计光子的传播尺度, 作不同尺度的近似 (如再电离光子)

Some more definitions

（与ν也有关）



- $dF_\nu = \frac{dE}{dt\,dA\,d\nu} = I_\nu \cos\theta\, d\Omega$

如果只在乎经过截面的总的光子数，不在乎方向，则定义 net flux

$$F_\nu = \int_{\text{all direction}} I_\nu \cos\theta\, d\Omega$$

- Mean intensity （不在乎 intensity 的方向）

$$J_\nu = \frac{1}{4\pi}\int I_\nu\, d\Omega$$

- Radiation energy density

$$dE = I_\nu\, dt \cdot dA \cdot d\Omega \cdot d\nu$$
$$= u_\nu^{(\hat n)} \cdot \underbrace{c\,dt \cdot dA}_{dV}\, d\Omega\, d\nu$$

则有 $u_\nu^{(\hat n)} = \frac{I_\nu}{c}$

同样可对方向积分 $u_\nu = \int u_\nu(\hat n)\, d\Omega = \frac{4\pi}{c} J_\nu$，称为 Radiation energy density

在读文献时，有的叫法可能不一样，要仔细 check

Lecture 2 03/05/2018 六教 6B211

Review:

Fundamentals of radiations $I_\nu = \frac{dE}{dt\, dA_\perp\, d\Omega\, d\nu}$ （光线传播过程中不变的 I_ν）

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \quad \text{（转移方程）}$$

j_ν：发光率；α_ν：吸光率/吸收率；入射

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu\, ds \quad \text{（光深）} \Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

形式解 $I_\nu = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - \tau_\nu')} d\tau_\nu'$

→ 本身平衡的

Blackbody radiation is a radiation which is itself in thermal equilibrium. $I_\nu = B_\nu(T)$

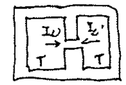
Thermal radiation is a radiation from some material which is at thermal equilibrium with itself.

发光体是平衡的 (but not necessarily with the radiation) $S_\nu = B_\nu(T)$ (only at $\tau \gg 1$)

Blackbody Spectrum $I_\nu = B_\nu(T)$ Planck function, $I_\nu = I_\nu(t, \vec{x}, \nu, \hat n, T) = I_\nu(\nu, T)$

$I_\nu(t, \vec{x}, \nu, \hat n)$ 由于是热平衡，与 $t, \vec{x}, \hat n$ 无关

Proof for $I_\nu = I_\nu(T, \nu)$



if $I_\nu \neq I_\nu'$, energy flows

but T is the same, no flows (2^{nd} law)

$\Rightarrow$ So that I_ν is indep of $t, \vec{x}, \hat n$

腔可开在任意位置，任意方向

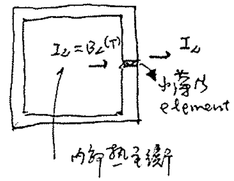
Thermal Radiation

辐射不一定热平衡，所以 $I_\nu \neq B_\nu(T)$

但是 source function $S_\nu = \frac{j_\nu}{\alpha_\nu} = B_\nu(T)$ (Kirchhoff's law)

描述物体的发射/吸收性质

Proof for $S_\nu = B_\nu(T)$



（足够大/黑，完全吸收）

Assume $\tau \to \infty$, at the element

$I_\nu = S_\nu$

If $S_\nu \neq B_\nu \Rightarrow I_\nu \neq B_\nu$

which is impossible, because for the element absorption must be equal to radiation.

由于薄片可以任取，所以 $S_\nu = B_\nu(T)$ 是整个黑体的性质

Planck Spectrum $B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$ (Planck's law)

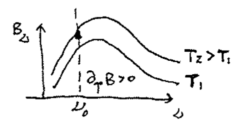
Properties (1) $h\nu \ll kT$, Rayleigh–Jeans law (typically radio radiation)

$I_\nu(T) = \frac{2\nu^2}{c^2} kT$ ~ 线性近似

(2) $h\nu \gg kT$. Wien's law

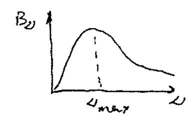
$I_\nu(T) = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$

(3) $\frac{\partial B_\nu(T)}{\partial T} > 0$



两条曲线一定不相交

(4) 黑体谱是 ν 的单峰函数 $\frac{\partial B_\nu}{\partial \nu}(\nu_{max}) = 0$



$\frac{h\nu_{max}}{kT} = 2.82$

$\frac{\nu_{max}}{T} = 5.88 \times 10^{10}\ Hz \cdot K^{-1}$

所以 ν_{max} 可估算用来判断温度.

(5) $F = \int F_\nu\, d\nu = \sigma T^4$, $\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}$

亮温度

Brightness temperature 对于射电来说重要

For any value of I_ν, we can fit $I_\nu = B_\nu(T_b)$

Radio Astronomy (for $h\nu \ll kT$)

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

也是 ν 的函数

所以，例如 CMB，总是可以用温度表示强度

The transfer function equation:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \Rightarrow \frac{dT_b}{d\tau_\nu} = -T_b + T$$

temperature of the material

有时，不能测到全波段，只能测到 ν_{max} 附近

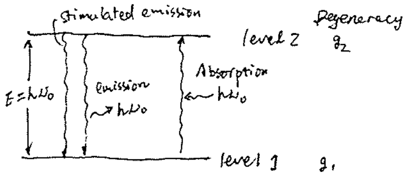
由 ν_{max} 确定的温度 T_c，称为 color temperature

有时，只能得到辐射总量 F，利用 $F = \sigma T_{eff}^4$，称为 effective temperature.

T_b, T_{eff} depend only on the magnitude of the intensity, but T_c depend on the shape.

Einstein Coefficient

$$\frac{j_\nu}{\alpha_\nu} = S_\nu = B_\nu(T) \rightarrow j_\nu = \alpha_\nu B_\nu(T)$$, Einstein Consider



Spontaneous emission: occurs even in the absence of radiation

(电子是自发的)

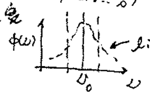
A_{21}: transition probability per unit time for Spontaneous emission

$[A_{21}] = sec^{-1}$

Absorption:

probability $\propto$ photon density or mean intensity J_ν at ν_0

但事实上，能级间的吸收/发射有展宽 (对于谱线很多)



line profile function $\phi(\nu)$

描述多大的范围内能吸收

$\int_0^\infty \phi(\nu) d\nu = 1$. 类似于 δ-function

Assume, if J_ν changes slow over $\Delta\nu$

$$\phi(\nu) \sim \delta(\nu - \nu_0)$$

$B_{12}\bar{J}$ = transition probabilities per unit time for absorption

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu \xrightarrow{\phi(\nu) \approx \delta(\nu-\nu_0)} \bar{J} = J_{\nu_0}$$

Stimulated emission:

$B_{21}\bar{J}$ = transition probability per unit time for stimulated emission

由于这组系数描述原子在相关能级之间跃迁的性质，所以这组系数之间应该有相互关系

(应可以从原子物理导出，实际上 Einstein 从热力学导出)

Relations between Einstein A, B Coefficients

假如系统处于热平衡，由 Detailed balance relation:

$$\underbrace{n_1 B_{12}\bar{J}}_{\text{absorption rate}} = \underbrace{n_2 A_{21}}_{\text{spontaneous emission}} + \underbrace{n_2 B_{21}\bar{J}}_{\text{stimulated emission}}$$

$$\Rightarrow \bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

In thermal equilibrium

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E/kT}}{g_2 e^{-(E+h\nu_0)/kT}} = \frac{g_1}{g_2} e^{h\nu_0/kT}$$

代入 $\bar{J}$ 的表达式得

$$\bar{J} = \frac{A_{21}/B_{21}}{\frac{g_1 B_{12}}{g_2 B_{21}} e^{h\nu_0/kT} - 1}$$ (从系统的原子热平衡)

In thermal equilibrium

$I_\nu = B_\nu$, $J_\nu = B_\nu$, 所以 $\bar{J} = B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu_0/kT} - 1}$ (the Black-body law)

在任意温度，以上温度都成立，因此

$$\left.\begin{aligned} g_1 B_{12} &= g_2 B_{21} \\ A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \end{aligned}\right\} \text{Einstein relations}$$

① irrelevant with T

② irrelevant/holds whether or NOT in thermal equilibrium

虽是从热平衡热力学导出，但它描述原子本身的性质，在非热平衡时也应成立。

③ 所以，为了给出 Planck function，必须有受激发射的过程

Stimulated emission is required to get Planck function.

在短波，低温时，$n_1 \gg n_2$，可不考虑受激发射

应能得到 Wien law.

Absorption and emission Coefficients:

spontaneous emission: amount of energy

$$dE = j_\nu \, dV \, dt \, d\Omega \, d\nu = n_2 \cdot dV \cdot A_{21} \, dt \cdot \phi(\nu) d\nu \cdot \frac{d\Omega}{4\pi} \cdot h\nu_0$$

自发辐射是 isotropic 的

每个光子的能量

所以自发辐射系数 $j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)$

Absorption: total energy absorbed

$$h\nu_0 \cdot n_1 B_{12} \cdot \bar{J} \, dt \cdot dV = h\nu_0 \, n_1 B_{12} \int \frac{d\Omega}{4\pi} \int d\nu \, \phi(\nu) I_\nu \times dt \, dV$$

考虑还要减去受激辐射

改为 $n_1 B_{12} - n_2 B_{21}$

Stimulated emission

$n_1 B_{12} - n_2 B_{21}$

$c \, dt \, dA = ds \, dA$

$$\alpha_\nu I_\nu \, dt \, dA \, ds \, d\Omega \, d\nu = \text{上面的} \; dV \, dt \cdot d\Omega \cdot d\nu \cdot \frac{h\nu_0}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu) I_\nu$$

最终可以得到 $\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) (n_1 B_{12} - n_2 B_{21})$

由于A·B系数与热平衡状态无关. α_ν, j_ν与A·B的关系是一般的. 所以这里的结果是正确的

Source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

代入AB关系

$$= \frac{2h\nu^3/c^2}{\frac{g_2}{g_1}\frac{n_1}{n_2} - 1}$$

只与Planck函数差一点点

在 thermal equilibrium, $\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_0/kT}$

代入S_ν的表达式即可得 $S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu_0/kT} - 1}$ (Planck function)

可得此关系称为

Generalized Birchhoff's law

在热平衡下. 仍然有 $S_\nu = B_\nu$

Thermal emission (LTE)

if $\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_0/kT}$, then $S_\nu = B_\nu(T)$

No thermal emission, $n_1/n_2 \neq \frac{g_1}{g_2} e^{h\nu_0/kT}$

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{n_1 g_2}{n_2 g_1} - 1} \neq B_\nu(T)$$

所以, 是否热平衡并不决定 Birchhoff's law 是否成立

Normally, $\frac{n_1}{g_1} > \frac{n_2}{g_2}$, if existing inverted population $\frac{n_1}{g_1} < \frac{n_2}{g_2}$. 此时 $\alpha_\nu \propto (n_1 B_{12} - n_2 B_{21}) < 0$

粒子数反转

Absorption　Stimulated emission

So we have $\alpha_\nu \begin{cases} > 0 & \text{normal} \\ < 0 & \text{inverted population (masers)} \end{cases}$

Classical Theory of Radiative Processes

HW2
R&L 1.7 1.9
2.1 2.2 2.4

- Maxwell's equations (Gaussian units)

$\nabla\cdot\vec{D} = 4\pi\rho$ $\qquad \nabla\cdot\vec{B}=0$ 无磁单极子

$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad \nabla\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ (*4)

法拉第定律

偏微分方
（线性、耦合、一阶方程组）
why?

$\vec{D}=\varepsilon\vec{E}$ $\qquad \vec{B}=\mu\vec{H}$

介电常数 dielectric constant

磁导率 magnetic permeability

微观描述 $\rho = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i$

$\vec{j} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i\vec{v}_i$

Lorentz force $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})$

$\vec{f} = \rho\vec{E} + \frac{\vec{j}}{c}\times\vec{B}$

force density

charge conservation $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$

（可从 *4 式散度导出）

力学功率密度 $\frac{du_{mech}}{dt} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i \vec{F}_i\cdot\vec{v}_i$

$= \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})\cdot\vec{v}_i$

$= \vec{j}\cdot\vec{E}$

u_{mech} = mechanical energy per unit volume

energy conservation

$\vec{j}\cdot\vec{E} + \frac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S}$

由 $\vec{j}\cdot\vec{E} + \frac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S}$

($\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ Poynting vector ~electromagnetic flux vector)

EM momentum per unit volume

$\vec{P}_{em} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{\vec{S}}{c^2}$

为什么 $\vec{P}_{em}$ 和 $\vec{S}$ 是同向的呢？因为光是直线传播的？

energy-momentum tensor

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & \text{energy flux} \\ \hline T_{0i} \\ T_{i0} & \text{momentum flux}\end{array}\right)$$

momentum conservation

$\frac{\partial\vec{P}_{em}}{\partial t} - \nabla\cdot\overleftrightarrow{\sigma} + \rho\vec{E} + \vec{j}\times\vec{B} = 0$

用四维方程可写得简单一些
$\partial T_{\mu\nu} = 0$

证明 $\vec{j}\cdot\vec{E} = \frac{1}{4\pi}\left(c(\nabla\times\vec{H})\cdot\vec{E} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)$

$\nabla\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot(\nabla\times\vec{H}) + \vec{H}\cdot(\nabla\times\vec{E})$

↑ 代入 Faraday law

$-\nabla\cdot\left(\frac{c}{4\pi}\vec{E}\times\vec{H}\right) = \vec{j}\cdot\vec{E} + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$

if ε and μ indep of time

use $E^2 = \vec{E}\cdot\vec{E}$, $\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} = \frac{1}{2}\frac{\partial(E^2)}{\partial t}$

$\vec{j}\cdot\vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\varepsilon E^2 + \frac{B^2}{\mu}\right) = -\nabla\cdot\left(\frac{c}{4\pi}\vec{E}\times\vec{H}\right)$

$\frac{\partial u_{mech}}{\partial t}$ $\qquad u_{em}$ EM energy density $\qquad$ EM energy flux $\vec{S}$

$u_{em} = \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{B^2}{\mu}\right)$

03/2/2018 Week3

- Plane waves

In vacuum, $\rho=0$, $\vec{j}=0$, $\varepsilon=\mu=1$

$$\begin{cases}\nabla\cdot\vec{E}=0,\ \nabla\times\vec{B}=0 \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\ \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\end{cases}$$

(linear, coupled, first-order partial differential equations)

↓ generally

linear 2nd-order ordinary differential equations

↓

wave solutions (e.g. G.R.)

it is invariant under $\vec{E}\to\vec{B}$, $\vec{B}\to-\vec{E}$
（有什么含义）

平面波解:

$\nabla\times\nabla\times\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \to \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$

$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$, $\nabla\cdot\vec{E}=0$

同样 $\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$

Plane wave solution: $\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{B} = \hat{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

(E_0, B_0 are complex → 实际物理量取实部
monochronic, linearly polarized)
单色平面波

传播方向: $\vec{k}=k\hat{n}$, $\hat{n}$ = direction of propagation

ω = circular frequency
圆频率

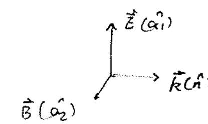
实际上为: E is the real part of $\vec{E}$!

计算: $\nabla\to i\vec{k}$, $\frac{\partial}{\partial t}\to -i\omega$

将平面波解代入 Maxwell eq.

① $i\vec{k}\cdot\hat{a}_1 E_0 = 0$, ② $i\vec{k}\cdot\hat{a}_2 B_0 = 0$

③ $i\vec{k}\times\hat{a}_1 E_0 = \frac{i}{c}\omega\hat{a}_2 B_0$, ④ $i\vec{k}\times\hat{a}_2 B_0 = -\frac{i}{c}\omega\hat{a}_1 E_0$



①② → $\vec{k}\perp\hat{a}_1$, $\vec{k}\perp\hat{a}_2$
+
③④ → $\hat{k}(\hat{n})$, $\hat{a}_1$, $\hat{a}_2$ 构成正交 set.

也可以得到 $kE_0 = \frac{\omega}{c}B_0$ $\quad kB_0 = \frac{\omega}{c}E_0 \Rightarrow E_0 = \left(\frac{\omega}{kc}\right)^2 E_0$

得以对所有频率 $\omega = ck$, and $E_0 = B_0$ (B, E 同频, 同速)

Phase velocity $v_{ph} = \frac{\omega}{k} = c$

Group velocity $v_g = \frac{\partial\omega}{\partial k} = c$ } identical

↳ 信息传播速度, 无论如何也 $<c$.

能量的密度 $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$E_0 = B_0$

时均值才有意义 $\langle\vec{S}\rangle = \frac{c}{4\pi}\left\langle \mathrm{Re}\vec{E}\times\mathrm{Re}\vec{B}\right\rangle = \frac{c}{8\pi}|E_0|^2$

$\langle u\rangle = \frac{1}{8\pi}\left[\left\langle \mathrm{Re}\vec{E}\cdot\mathrm{Re}\vec{E}\right\rangle + \left\langle \mathrm{Re}\vec{B}\cdot\mathrm{Re}\vec{B}\right\rangle\right]$

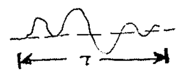
$= \frac{1}{16\pi}\left[|E_0|^2 + |B_0|^2\right] = \frac{1}{8\pi}|E_0|^2 = \frac{\langle S\rangle}{c}$

定理: $A(t) = Ae^{i\omega t}$
$B(t) = Be^{i\omega t}$ (同频)
$\Rightarrow \langle \mathrm{Re}A(t)\,\mathrm{Re}B(t)\rangle$
$= \frac{1}{2}\mathrm{Re}(AB^*)$
$= \frac{1}{2}\mathrm{Re}(A^*B)$

上面都是在真空中而言; 若在介质中, ε, μ 都可以是 ω 的函数.
平面波

- 实际的波不可能是单色平面波，或是有空间宽度
  或是有时间宽度 （对应光子的动量有不确定性）

$\Delta\omega\,\Delta\tau > 1$, for any wave theory of light.



偏振的分解由 Maxwell 方程的线性性决定。

$\vec{E}(t)$ can only have two $\underset{\text{(independent)}}{}$ components $\perp \vec{k}$

联系着 $m_\gamma = 0$ （光子的质量为 0） 如果质量是非零，则一定可以变换到静止参考系
且 spin $= 1$ 在此参考系下没有波动，可以有三个分量。

联系着 (helicity)

parity conservation for EM interaction. （左右对称性）

既然光有两个独立分量，可以讨论一个，然后叠加

光场的傅利叶分解

$$\tilde{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$$

$$E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{-i\omega t}\, dt$$

$\uparrow$ real $\qquad \uparrow$ complex

$\Downarrow$

$\tilde{E}(-\omega) = \tilde{E}^*(\omega)$ i.e. negative frequencies are NOT indep modes

正频率. 所以只讨论正频率，然后叠加

（一般不讨论光的负频率）

能流 $S = \frac{dW}{dt\,dA} = \frac{c}{4\pi}E^2(t)$

那么，在频率空间 $\frac{dW}{d\omega\, dA} = ?$ energy per unit area per unit frequency. for the entire pulse

对整个波而有意义

利用 Parseval's thm. $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty}\tilde{f}(\omega)\,\tilde{g}^*(\omega)\, d\omega$

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\, dt = \frac{c}{4\pi}2\pi\int_{-\infty}^{\infty}|\hat{E}(\omega)|^2\, d\omega = c\int_0^{\infty}|\hat{E}(\omega)|^2\, d\omega$$

$$\frac{dW}{dA\,d\omega} = c\,|\hat{E}(\omega)|^2 \quad \text{(for the entire pulse)}$$

"power spectrum" 称为功率谱

相比于宇宙学的功率谱 $P(k) = |\hat{\delta}(k)|^2$，这个才是真实的功率谱

能否定义 $\frac{dW}{dt\,dA\,d\omega}$? 不行，因为与 $\Delta\omega\Delta\tau > 1$ 违背.

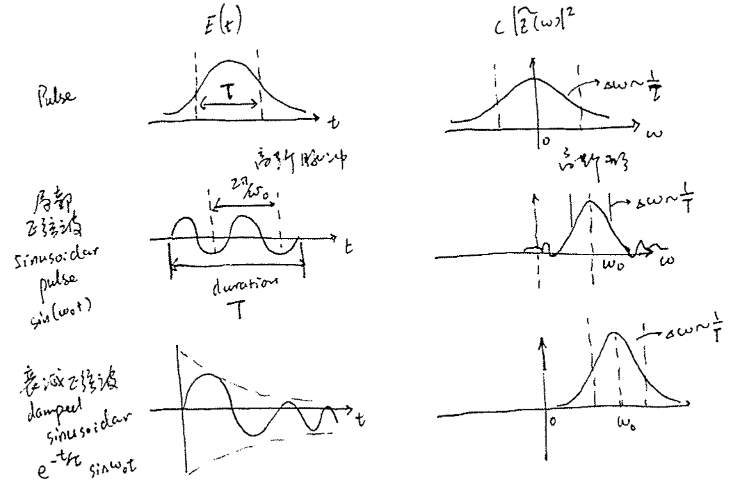
BUT, formally,

对某种形式的波 $\frac{dW}{dt\,dA\,d\omega} := \frac{1}{T}\frac{dW}{dA\,d\omega} = \frac{c}{T}|\hat{E}(\omega)|^2$, if a long signal has same period over its entire length.

在某段时间内有稳定
特征

此时有 $\frac{dW}{dt\,dA\,d\omega} = c\lim_{T\to\infty}\frac{1}{T}|\hat{E}_T(\omega)|^2$ indep of T for large T.

典型的光场的 Fourier 变换



generally, for a pulse with time extent T

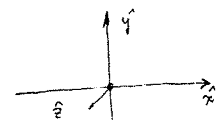
$\rightarrow$ the width $\Delta\omega \doteq \frac{1}{T}$ of the first feature

(sinusoidal) time dependence

$\rightarrow$ Spectrum concentrated at ω_0

- Polarization 极化

$\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ linearly polarized i.e. $\vec{E}(t) \parallel \hat{a}_1$



光沿 $\hat{z}$ 方向传播

由于光场有两个独立分量，还可以有另一个分量，At $\vec{r}=0$，看相对时间的依赖

$$\hat{x}E_1 e^{i(-\omega t)} + \hat{y}E_2 e^{i(-\omega t)} = \vec{E}_0 e^{-i\omega t}$$

E_x, E_y 是 complex，且

可以有相对的相位，e.g. $E_1 = \mathcal{E}_1 e^{i\phi_1}$ $E_2 = \mathcal{E}_2 e^{i\phi_2}$

$$E_x(t) = \mathrm{Re}\left(\mathcal{E}_1 e^{i\phi_1} e^{-i\omega t}\right) = \mathcal{E}_1\cos(\omega t - \phi_1)$$

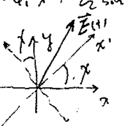
$$E_y(t) = \mathrm{Re}\left(\mathcal{E}_2 e^{i\phi_2} e^{-i\omega t}\right) = \mathcal{E}_2\cos(\omega t - \phi_2)$$

实际的电场 $\vec{E}(t) = \mathcal{E}_1\cos(\omega t - \phi_1)\hat{x} + \mathcal{E}_2\cos(\omega t-\phi_2)\hat{y}$

$$= \cos\omega t\left(\mathcal{E}_1\cos\phi_1\hat{x} + \mathcal{E}_2\cos\phi_2\hat{y}\right) + \sin\omega t\left(\mathcal{E}_1\sin\phi_1\hat{x} + \mathcal{E}_2\sin\phi_2\hat{y}\right)$$

可以定义 $\mathcal{E}_0$ 和 β, χ s.t. $\mathcal{E}_0\cos\beta\cos\chi = \mathcal{E}_1\cos\phi_1$ $\quad \mathcal{E}_0\sin\beta\sin\chi = \mathcal{E}_1\sin\phi_1$

$\mathcal{E}_0\cos\beta\sin\chi = \mathcal{E}_2\cos\phi_2$ $\quad -\mathcal{E}_0\sin\beta\cos\chi = \mathcal{E}_2\sin\phi_2$

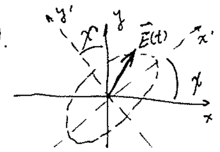


$\vec{E}(t) = \cos\omega t\;\mathcal{E}_0\cos\beta\;\hat{x}' - \sin\omega t\;\mathcal{E}_0\sin\beta\;\hat{y}'$ （沿着新的坐标轴的分解）

立即知道 $\left(\frac{E_{x'}}{\cos\beta\cdot\mathcal{E}_0}\right)^2+\left(\frac{E_{y'}}{\mathcal{E}_0\sin\beta}\right)^2=1$

这是一个主轴为 $\mathcal{E}_0\cos\beta,\ \mathcal{E}_0\sin\beta$ 的椭圆 (ellipse of principal axes)

所以当 E_1, E_2 存在相位差时，一般来说 E 的轨迹为椭圆。



- linear polarized $\beta=0,\ \pm\frac{\pi}{2}$
- circular polarized $\beta=\pm\frac{\pi}{4}$
- general case, elliptically polarized

$0<\beta\le\frac{\pi}{2}$ clock-wise = right-handed polarization = negative helicity

$-\frac{\pi}{2}\le\beta<0$ counter clockwise = left-handed polarization = positive helicity

如果相位完全随机 (ϕ_1, ϕ_2 无关联)，则是无偏振的。

之前是用四参数 $\left(\mathcal{E}_1, \mathcal{E}_2, \underbrace{\phi_1, \phi_2}_{\text{相差才有意义}}\right)$ ↔ 现在用三参数 $(\mathcal{E}_0, \beta, \chi)$

与物理量关联不紧密，且计算不方便

引入 Stokes's parameters for monochromatic waves.

$I\equiv\mathcal{E}_1^2+\mathcal{E}_2^2=\mathcal{E}_0^2 \propto$ total energy flux

$$\begin{cases} Q\equiv\mathcal{E}_1^2-\mathcal{E}_2^2=\mathcal{E}_0^2\cos2\beta\cos2\chi \\ U\equiv2\mathcal{E}_1\mathcal{E}_2\cos(\phi_1-\phi_2)=\mathcal{E}_0^2\cos2\beta\sin2\chi \end{cases}$$

orientation of the ellipse

$I^2=Q^2+U^2+V^2$ (总有3个量是独立的)

$V\equiv2\mathcal{E}_1\mathcal{E}_2\sin(\phi_1-\phi_2)=\mathcal{E}_0^2\sin2\beta$

β. V 描述椭圆的圆度 circularity parameter

$$\begin{cases} \mathcal{E}_0=\sqrt{I} \\ \sin2\beta=\frac{V}{I} \\ \tan2\chi=\frac{U}{Q} \end{cases}$$

- Electromagnetic potentials

vector potential $\vec A=\nabla\times\vec B$, s.t. $\nabla\cdot\vec B=0$

scalar potential ϕ s.t. $\nabla\times\vec E+\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec A=0$

$\nabla\times\left(\vec E+\frac{1}{c}\frac{\partial\vec A}{\partial t}\right)=0 \nearrow \vec E=-\nabla\phi-\frac{1}{c}\frac{\partial\vec A}{\partial t}$

现在，将剩下的方程写为 $\nabla\cdot\vec E=4\pi\rho \Rightarrow \nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec A)=-4\pi\rho$

将位移电流方程写为 $\nabla\times\vec H=\nabla\times(\nabla\times\vec A)=-\nabla^2\vec A+\nabla(\nabla\cdot\vec A) \Rightarrow \nabla^2\vec A-\frac{1}{c^2}\frac{\partial^2\vec A}{\partial t^2}-\nabla\left(\nabla\cdot\vec A+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)=-\frac{4\pi}{c}\vec j$

Gauge invariance $\vec A\to\vec A+\nabla\psi$ 保证 $\vec B\to\vec B$

$\phi\to\phi-\frac{1}{c}\frac{\partial\psi}{\partial t}$ 保证 $\vec E\to\vec E$

所以 $\vec A, \phi$ 的选择有一定的自由度

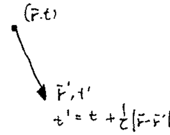
So, free to chose the gauge

choose Lorentz gauge $\nabla\cdot\vec A+\frac{1}{c}\frac{\partial\phi}{\partial t}=0 \longrightarrow \begin{cases}\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho \\ \nabla^2\vec A-\frac{1}{c^2}\frac{\partial^2\vec A}{\partial t^2}=-\frac{4\pi}{c}\vec j\end{cases}$

(是一个Lorentz不变的条件。而库仑 gauge condition 不是 Lorentz 不变的)

上述方程的解 $\phi(\vec r,t)=\int\frac{d^3r'[\rho]}{|\vec r-\vec r'|}$, $\vec A(\vec r,t)=\int\frac{d^3r'[\vec j]}{|\vec r-\vec r'|}$ 称为 retarded potential (推迟势)

其中 $[\rho]=\rho\left(\vec r', t-\frac{|\vec r-\vec r'|}{c}\right)$, $[\vec j]=\vec j\left(\vec r', t-\frac{1}{c}|\vec r-\vec r'|\right)$



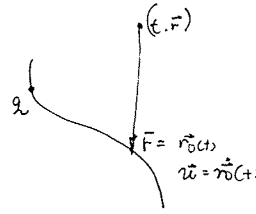
$t'=t+\frac{1}{c}|\vec r-\vec r'|$

考虑Green函数解法?

例2: Radiation from moving charges

$\rho=q\delta(\vec r-\vec r_0(t))$

$\vec j=q\vec u(t)\delta(\vec r-\vec r_0(t))$



因而 $\phi(\vec r,t)=\int d^3r'\rho\left(\vec r', t-\frac{1}{c}|\vec r-\vec r'|\right)/|\vec r-\vec r'|$

$=\int d^3r'\underbrace{\int dt'\delta\left(t'-t+\frac{1}{c}|\vec r-\vec r'|\right)}_{\text{加了一项，值为1}}\frac{\rho\left(\vec r', t-\frac{1}{c}|\vec r-\vec r'|\right)}{|\vec r-\vec r'|}$ $\rightarrow q\delta(\vec r'-\vec r_0(t'))$ $\uparrow$ $t-\frac{1}{c}|\vec r-\vec r'|$

$=q\int dt'\frac{\delta\left(t'-t+\frac{1}{c}|\vec r-\vec r_0(t')|\right)}{|\vec r-\vec r_0(t')|}=q\int dt' R^{-1}(t')\delta\left(t'-t+\frac{R(t')}{c}\right)$

这里 $\vec R(t')=\vec r-\vec r_0(t')$

类似的 $\vec A(\vec r,t)=\frac{q}{c}\int dt'\vec u(t')R^{-1}(t')\delta\left(t'-t+\frac{R(t')}{c}\right)$

定义 $t''=t'-t+\frac{R(t')}{c}$, $dt''=dt'+\frac{\dot R(t')}{c}dt'=\left(1+\frac{\dot R(t')}{c}\right)dt'$

利用关系 $R^2(t')=\vec R(t')\cdot\vec R(t')$

$2R(t')\dot R(t')=2\vec R(t')\cdot\dot{\vec R}(t')=-2\vec R(t')\dot{\vec r}_0(t')$

$=-2\vec R(t')\vec u(t')$

$=-2R(t')\hat n\cdot\vec u(t')$, 这里 $\hat n(t)=\frac{\vec R(t)}{R(t)}$

最后 $\dot R(t')=-\hat n(t')\cdot\vec u(t')$

所以 $dt''=\kappa(t')dt'$, 其中 $\kappa(t')\equiv1-\frac{1}{c}\hat n(t')\cdot\vec u(t')$

所以 Scalar potential

$\phi(\vec r,t)=q\int dt''\kappa^{-1}(t')R^{-1}(t')\delta(t'')$ when $t''=0$, $t'=t_{retard}$ s.t. $c|(t_{ret}-t)|=R(t_{ret})$

$=\frac{q}{\kappa(t_{ret})R(t_{ret})}=\left[\frac{q}{\kappa R}\right]$ (Liénard-Wiechart potential)

同样的 $\vec A(\vec r,t)=\frac{q\vec u(t_{ret})}{c\kappa(t_{ret})R(t_{ret})}=\left[\frac{q\vec u}{\kappa R}\right]$

[…] 指的是取 t_{ret} 时的值

κ 是一个修正项，来源于 dt'' 和 dt' 的关系，代表时间的

Note 1. $\phi\sim\frac{1}{R}$ static case $E\sim\frac{1}{R^2}$

Now $\phi\sim1/R$ at t_{ret}, implicitly depend on R

E falls slower than $1/R^2$.

i.e. $E\neq0$ at $r=\pm\infty$ (EM waves cases)

2. $\kappa=1-\frac{\vec n\cdot\vec u}{c}$ $\phi\sim1/\kappa$ (极端相对论的情况) (就是辐射粒子 beam, 称 "beaming effect")

when $\beta=u/c\to1$ $\kappa\to0$ along $\vec u$, $\kappa\to1 \perp \vec u$, ϕ is concentrated into a narrow cone along $\vec u$.

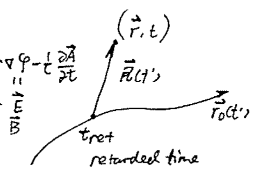
0319/2018 Week 4

HW R.L. 3.1 3.2 3.3

Summary: define $\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')$

$c(t - t_{ret}) = R(t_{ret})$

Solution $\phi(\vec{r},t) = \left[\frac{q}{\kappa R}\right]$ $\quad \vec{A}(\vec{r},t) = \left[\frac{q\vec{u}}{\kappa R}\right] \Rightarrow$ to get $\vec{E}$, $\vec{B}$ $\quad \vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$



其中 κ 称为 Doppler 因子 , $\kappa(t') = 1 - \frac{\hat{n}(t')}{c}\cdot\vec{u}(t') = 1 - \hat{n}\cdot\vec{\beta}$, $\vec{\beta} = \vec{u}/c$

电场强度不一定很大，也可能是因为 Doppler 因子

[...] denotes the evaluation at t_{ret}

$\hat{n}(t') = \vec{R}(t') / |\vec{R}(t')|$

$$\vec{E}(\vec{r},t) = q\underbrace{\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]}_{\substack{\text{velocity field} \\ (\text{dept. on } \vec{\beta})}} + \frac{q}{c}\underbrace{\left[\frac{\hat{n}}{\kappa^3 R}\times\left\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}\right]}_{\substack{\text{acceleration field} \\ (\text{dept. on } \vec{\beta}, \dot{\vec{\beta}})}}$$

acceleration field $\longrightarrow E_a \propto \frac{1}{R}$, $S \propto \frac{1}{R^2}$, $P \propto \int \vec{S}\cdot d\vec{A} \propto$ const

$\Rightarrow$ RADIATION FIELD

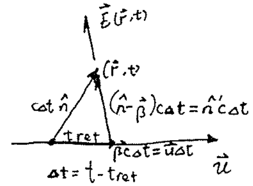
$\vec{E}(\vec{r},t) \perp \hat{n}(t_{ret})$

$\vec{B}(\vec{r},t) = \hat{n}\times\vec{E}(\vec{r},t)$

$\kappa(t') = 1 - \frac{1}{c}\hat{n}(t')\cdot\vec{u}(t')$

Case 1. particles move with constant velocity

$$\vec{E}(\vec{r},t) = \frac{q(1-\beta^2)\hat{n}'}{\kappa^3 R^2}, \quad \hat{n}' = \hat{n} - \vec{\beta} \ (\text{Not a unit vector})$$

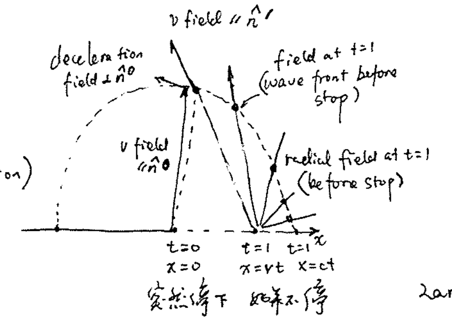


- $\vec{E}(\vec{r},t) \parallel \hat{n}'$, i.e. 'radial' to $\hat{n}'$, along the line toward the current position of the charge.
- When $\beta \ll 1$, $\vec{E} \to$ Coulomb's law
- $E \propto \frac{1}{R^2}$, $S \propto |E^2| \propto \frac{1}{R^4}$, $P \propto \int \vec{S}\cdot d\vec{A} \propto \frac{1}{R^2} \xrightarrow[R\to\infty]{} 0$

  i.e. No radiation at this case. (away to infinity)

  即，有电场，但没有辐射

Example. Charged particle moving at uniform velocity (in +x direction) is stopped at $x=0$ and $t=0$.



$\nabla\cdot\vec{E} = 0$, No Source, $\vec{E}$ preserve flux

power spectrum $\frac{dW}{d\omega dA} = c|\tilde{E}(\omega)|^2$

$dA = R^2 d\Omega$

$$\Rightarrow \frac{dW}{d\omega d\Omega} = c\left|R\tilde{E}(\omega)\right|^2, \quad \tilde{E}(\omega) = \frac{1}{2\pi}\int \vec{E}(t)e^{i\omega t}dt$$

$$= \frac{c}{4\pi^2}\left|\int R\vec{E}(t)e^{i\omega t}dt\right|^2 = \frac{q^2}{4\pi^2 c}\left|\int \left[\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\kappa^{-3}\right]_{t'} e^{i\omega t}dt\right|^2$$

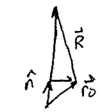
对时间的积分

because $t' = t - \frac{R(t')}{c}$, $t = t' + \frac{R(t')}{c}$, $dt = dt'\left(1 + \frac{\dot{R}(t')}{c}\right) = \kappa(t')dt'$

$R^2 = \vec{R}\cdot\vec{R} \to R\dot{R} = \vec{R}\cdot\dot{\vec{R}} = -\vec{R}\cdot\vec{u} \to \dot{R} = -\hat{n}\cdot\vec{u}$

$e^{i\omega t}$ can be written to $e^{i\omega t} = e^{i\omega(t' + \frac{R(t')}{c})}$

For large distance $|\vec{r}_0| \ll |\vec{r}|$ (range of motion of charge) $\quad R(t') \cong |\vec{r}| - \hat{n}\cdot\vec{r}_0(t')$



So $e^{i\omega t} = e^{i\omega(t' + \frac{R(t')}{c})} = e^{i\omega(t' + \frac{|\vec{r}|}{c} - \frac{\hat{n}\cdot\vec{r}_0(t')}{c})}$

$= e^{i\omega\frac{|\vec{r}|}{c}} \times e^{i\omega(t' - \frac{\hat{n}}{c}\cdot\vec{r}_0(t'))}$

Substitute $e^{i\omega t}$, dt into $\frac{dW}{d\omega d\Omega}$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \hat{n}\times\left\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}\kappa^{-2}e^{i\omega(t' - \frac{\hat{n}}{c}\cdot\vec{r}_0(t'))}dt'\right|^2$$

where $e^{i\omega\frac{|\vec{r}|}{c}}$ is canceled after $|\ |^2$ (just a phase!)

Use identity $\frac{d}{dt'}\left[\kappa^{-1}\hat{n}\times(\hat{n}\times\vec{\beta})\right] = \hat{n}\times\left\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}\kappa^{-2}$ ... (*1)

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\text{surface term} - \int \kappa^{-1}\hat{n}\times(\hat{n}\times\vec{\beta})\,i\omega\kappa\, e^{i\omega(t' - \frac{\hat{n}}{c}\vec{r}_0(t'))}dt'\right|^2$$

surface term: 0 at $t \to \pm\infty$ 因为运动不会无穷

$\kappa = \frac{d}{dt'}\left(t' - \frac{\hat{n}\cdot\vec{r}_0(t')}{c}\right)$

$$= \frac{q^2\omega^2}{4\pi^2 c}\left|\int \hat{n}\times(\hat{n}\times\vec{\beta})\exp\left[i\omega(t' - \hat{n}\cdot\vec{r}_0(t')/c)\right]dt'\right|^2$$

Proof for (*1)

$\hat{n}$ fixed, $\kappa(t')$, $\vec{\beta}(t')$

$$\text{LHS} = \kappa^{-2}\left[(\hat{n}\cdot\dot{\vec{\beta}})\hat{n}\times(\hat{n}\times\vec{\beta}) + \kappa\,\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right] = \kappa^{-2}\left\{(\hat{n}\cdot\dot{\vec{\beta}})\left[(\hat{n}\cdot\vec{\beta})\hat{n} - \vec{\beta}\right] + (1-\hat{n}\cdot\vec{\beta})\left[(\hat{n}\cdot\dot{\vec{\beta}})\hat{n} - \dot{\vec{\beta}}\right]\right\}$$

$$= \kappa^{-2}\left\{(\hat{n}\cdot\dot{\vec{\beta}})\left[(\hat{n}\cdot\vec{\beta})\hat{n} - \vec{\beta}\right] + \cdots\right\} = \kappa^{-2}\left\{(\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}-\vec{\beta}) - \dot{\vec{\beta}}(1-\hat{n}\cdot\vec{\beta})\right\} = \text{RHS}$$

$\cdots + (\hat{n}\cdot\dot{\vec{\beta}})\hat{n} - (\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}\cdot\vec{\beta})\hat{n} + \cdots$

$\cdots - \dot{\vec{\beta}} + (\hat{n}\cdot\vec{\beta})\dot{\vec{\beta}}$

$t' \simeq t_{ret}$

结论: Radiation from non-relativistic system of particles.

$$\frac{E_{rad}}{E_{vel}} \cong \frac{R\dot{u}}{c^2} \cong \frac{Ru\nu}{c^2} = \beta\frac{R}{\lambda}, \quad \beta = \frac{u}{c}$$

If there exists a characteristic frequency of oscillation ν, $\dot{u} \sim u\nu$

"Near Zone" $\quad R \lesssim \lambda$, $E_{vel} \gg E_{rad}$, $\frac{E_{vel}}{E_{rad}} \propto \beta^{-1}$

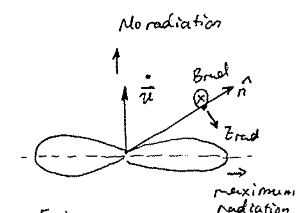
"Far Zone" $\quad R \gg \lambda\beta^{-1}$, $E_{rad} \gg E_{vel}$, $\frac{E_{rad}}{E_{vel}} \propto R$

Larmor's Formula: for a single accelerated charge

$$\vec{E}_{rad} = \frac{q}{Rc^2}\underbrace{\hat{n}\times(\hat{n}\times\dot{\vec{u}})}_{\dot{u}\sin\theta}$$

for non-relativistic case $\kappa \to 1$, $|\vec{\beta}| \ll 1$, $\vec{\beta}\times\dot{\vec{\beta}} \to 0$

$\vec{B}_{rad} = \hat{n}\times\vec{E}_{rad}$



直线运动 → linear polarized

因为 $\hat{n}$, $\dot{\vec{u}}$ 决定了 $\vec{E}_{rad}$ 的方向

$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$, $S = \frac{c}{4\pi}\left(\frac{q}{Rc^2}\right)^2\left[\dot{u}\sin\theta\right]^2 = \frac{dW}{dt\cdot dA} = \frac{dW}{dt\cdot d\Omega\cdot R^2}$

$\frac{dW}{dt\cdot d\Omega} = \frac{c}{4\pi}\left(\frac{q}{c^2}\dot{u}\sin\theta\right)^2 = \frac{q^2\dot{u}^2\sin^2\theta}{4\pi c^3}$

note: $\int_0^{\frac{\pi}{2}}\sin^3\theta\, d\theta = \frac{2}{3}$

$\underbrace{\frac{dW}{dt}}_{\text{辐射功率}} = \int \frac{dP}{d\Omega}d\Omega = \frac{2q^2\dot{u}^2}{3c^3}$ (Larmor's Formula) $\frac{dP}{d\Omega} \propto q^2, \dot{u}^2, \sin^2\theta$, $P \propto (\underbrace{q\dot{u}}_{\text{偶极矩的二阶导}})^2$

Dipole approximation

忽略各粒子之间 t_{ret} 的差别，以整个系统的 t_{ret} 代替

引入 dipole moment $\vec{d} = \sum_i q_i \vec{r}_i$

The radiation field

$$\vec{E}_{rad} = \sum_i \frac{q_i}{c^2} \frac{\hat{n}\times\hat{n}\times\dot{\vec{u}}_i}{R_i} \quad (\text{at } t_{ret})$$

for 'far zone' radiation $|R_i - R_0|/R_0 \ll 1$, $R_i \to R_0$

$$\vec{E}^{(t)}_{rad} = \frac{\hat{n}\times(\hat{n}\times\ddot{\vec{d}})}{c^2 R_0} \quad (\text{like single particle } qr \to d)$$

The radiation power

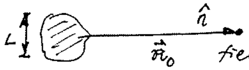
$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\theta$$

$$P = \frac{2|\ddot{\vec{d}}|^2}{3c^3} \quad (\text{dipole approximation})$$

综合 $(t, \vec{r})$，不同的粒子的 $t_{retarded}$ 有差别

But for the case $\tau \gg \frac{L}{c}$ — L = typical size of system

τ: typical time scale for changes in $\vec{E}_{rad}$



we may neglect the difference in t_{ret}.

① $\tau \sim \frac{1}{\nu} \sim \lambda/c$

The condition can be written $\lambda \gg L$

② $\tau \sim \frac{l}{u}$ (l → orbit, u → velocity) $\quad \frac{l}{u} \gg \frac{L}{c} \to \frac{u}{c} \ll \frac{l}{L} < 1$

The condition can be written $u \ll c$ (non-relativistic)

The dipole can be decomposed to Fourier component

$$d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\tilde{d}(\omega)\,d\omega$$

$$\ddot{d}(t) = \int_{-\infty}^{\infty} -\omega^2 \tilde{d}(\omega) e^{-i\omega t}\,d\omega \quad \mathcal{F}[\ddot{d}(t)]$$

So as the field

$$\tilde{E}_{rad}(\omega) = \frac{\sin\theta}{c^2 R_0}\mathcal{F}[\ddot{d}(t)] = \frac{-\omega^2}{c^2 R_0}\tilde{d}(\omega)\sin\theta$$

$$\frac{dW}{d\omega\, d\Omega} = c R_0^2 |\tilde{E}_{rad}(\omega)|^2 = \frac{1}{c^3}\omega^4 |\tilde{d}(\omega)|^2 \sin^2\theta$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 \qquad \frac{8}{3}\pi$$

偶极辐射是可以看成来自辐射的最低阶近似，因而简单而有用.

例如：如果有相距不近的两个偶极，他们之间的 t_{ret} 是有差别的（相位差），要叠加电场，然后求 P.

Thomson Scattering (Electron Scattering)

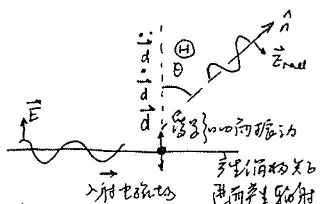
A free charge radiates in response to an incident EM wave in a non-relativistic limit.

Suppose $\vec{E} = \hat{\epsilon} E_0 e^{i(\vec{k}\cdot\vec{r} - \omega_0 t)}$ is the incident field, the equation of motion

$$m\ddot{\vec{r}} = \vec{F} = e\hat{\epsilon}E_0\sin\omega_0 t \quad (\text{at } \vec{r} = \vec{0})$$

$$\vec{d} = e\vec{r}, \quad \ddot{\vec{d}} = e\ddot{\vec{r}} = \frac{e^2 E_0}{m}\hat{\epsilon}\sin\omega_0 t$$

$$\vec{d}(t) = -\left(\frac{e^2 E_0}{m\omega_0^2}\right)\sin\omega_0 t\,\hat{\epsilon} = -\vec{d}_0\sin\omega_0 t \quad \text{where } \vec{d}_0 = \frac{e^2 E_0}{m\omega_0^2}\hat{\epsilon}$$



表示单的 Thomson Scattering (electron scattering)

The incident flux $\langle S\rangle = \frac{c}{8\pi}E_0^2$

The scattering EM field

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\frac{e^2 E_0}{m}\right)^2 \sin^2\omega_0 t \sin^2\Theta$$

or

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^4 E_0^2}{8\pi m^2 c^3}\sin^2\Theta$$

or total power

$$\langle P\rangle = \frac{e^4 E_0^2}{3m^2c^3}$$

Define the differential cross section (微分)散射截面

$$\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega\rangle}{\langle S\rangle} = \frac{e^4\sin^2\Theta}{m^2c^4} = r_0^2\sin^2\Theta,$$

the probability of the incident EM wave being scattered by the point source charge.

where $r_0 = \frac{e^2}{mc^2}$ denote the 'size' of the pointer charge ($\frac{e^2}{r_0} = mc^2$)

The total cross section $\sigma = \frac{8\pi}{3}r_0^2$. If r_0^2 is larger, so is the σ.

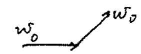
So for electron and proton, $r_0'/r_0 \sim 10^3$, $\sigma'/\sigma_2 \sim 10^6$

在CMB的辐射中，电子的散射远比质子强，质子的贡献被忽略.

$r_0 \sim 2.82\times10^{-13}\,\text{cm}$

$\sigma_T = 0.665\times10^{-24}\,\text{cm}^2$

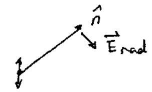
Note: ① 散射功率 $dP/d\Omega$ 和散射截面 σ 都与入射频率 (frequency) 无关.



This is elastic scattering.

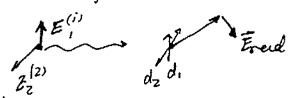
- This is valid for sufficient low ν, such that $h\nu < mc^2 = 0.511\,\text{MeV}$ for electron → non relativistic
- Valid for sufficiently low intensity of radiation field.
  Otherwise the charge may move relativisticly, the dipole approx fails.

②



散射波是 linearly polarized (if incident wave is linearly polarized)

如果入射波是非偏振的



$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_{polarized}(\Theta) + \left(\frac{d\sigma}{d\Omega}\right)_{polarized}\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}r_0^2(1+\sin^2\theta) = \frac{1}{2}r_0^2(1+\cos^2\alpha)$$

α → 散射角

- $\sigma_{unpol} = \frac{8}{3}\pi r_0^2 = \sigma_{pol}$
- But here are 2 polarized direction, degree of polarization $\Pi = \frac{1-\cos^2\alpha}{1+\cos^2\alpha} = \begin{cases} 0\%, & \alpha = 0 \text{ along } \hat{k} \\ 100\%, & \alpha = \frac{\pi}{2}\ \perp \hat{k}\end{cases}$
  That means unpolarized incident wave produces a partially polarized outgoing wave.

对于CMB而言，入射是非偏振的，Thomson散射后，变为部分偏振的.

入射是 T. E. B 皆非偏振 → 散射后：$E \neq 0$，$B = 0$ → B型偏振不产生；E → 产生出E型偏振

0326/2018 week 5 6B211 狭义相对论复习

free-free emission

HW
R & L
4.3; 4.7; 4.14

Relativistic Covariance and Kinematics

Contra-variant Vector $x^\mu = (ct, x, y, z)$, $\mu = 0,1,2,3$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Covariant vector $x_\mu = \eta_{\mu\nu} x^\nu$

Invariant 不变量 $s^2 = x_\mu x^\mu$ under $K \to K'$

Lorentz transformation along X-axis

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-1/2}$$

contra-variant under Λ: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Invariant $s^2 = \eta_{\mu\nu} x^\mu x^\nu = \eta_{\sigma\tau} x'^\sigma x'^\tau = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu$

s^2 is invariant for arbitrary x^μ, only if $\eta_{\mu\nu} = \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \eta_{\sigma\tau}$

or $\eta = \Lambda^T \eta \Lambda$ $\qquad = (\Lambda^T)_\mu{}^\sigma \eta_{\sigma\tau} (\Lambda)^\tau{}_\nu$

or $\det\eta = (\det\Lambda)^2 \det\eta$

$\Rightarrow \det\Lambda = \pm 1$ $\begin{cases} +1, & \text{proper Lorentz transformation} \quad \cdots ① \\ -1, & \text{reflection } \vec{x} \to -\vec{x} \end{cases}$

也就是4维宇称 $\cdots ②$

另外，时序不改变要求 $\Lambda^0{}_0 \geq 1$ isochronous (i.e. the flow of time is the same in K and K')

NOT CONSIDERED HERE

Covariant Vector $x_\mu \to x'_\mu = \tilde{\Lambda}_\mu{}^\nu x_\nu$

it can also put constraint on $\eta_{\mu\nu}$:

$s^2 = x'_\mu x'^\mu = \tilde{\Lambda}_\mu{}^\sigma \Lambda^\mu{}_\tau x_\sigma x^\tau = \delta^\sigma_\tau x_\sigma x^\tau = x_\sigma x^\sigma$

$\Rightarrow \tilde{\Lambda}_\mu{}^\sigma \Lambda^\mu{}_\tau = \delta^\sigma_\tau$

$\Rightarrow \tilde{\Lambda} = \Lambda^{-1}$

we conclude the transformation of 4-vector satisfies

$$\begin{cases} \text{Contra-variant} & A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu & A_\mu = \eta_{\mu\nu} A^\nu \\ \text{Covariant} & A_\mu \to A'_\mu = \tilde{\Lambda}_\mu{}^\nu A_\nu & A^\mu = \eta^{\mu\nu} A_\nu \end{cases}$$

for any A^μ, B_μ, their inner product $A^\mu B_\mu$ is invariant. (4-scalar).

four velocity $\underline{U}^\mu \equiv \frac{dx^\mu}{d\tau}$, $\underline{U}^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = c\gamma_\mu$

$= \gamma_u (c, \vec{u})$

$\gamma_\mu = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$, $\vec{u} = \frac{d\vec{x}}{dt}$ ～ 和坐标系之间变换没有关系 $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$\underline{U}^i = \frac{dx^i}{d\tau} = \gamma_\mu \frac{dx^i}{dt} = \gamma_u u^i$

EM wave phase $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\vec{k}\cdot\vec{x} - \omega t$ should be invariant

$\vec{k}\cdot\vec{x} - \omega t = \eta_{\mu\nu} k^\mu x^\nu$

where $k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$

$x^\nu = (ct, \vec{x})$

Thus k^μ should be a 4-vector, and

$k^2 = -\left(\frac{\omega}{c}\right)^2 + \vec{k}^2 = 0$

$\frac{\omega}{c} = |\vec{k}|$ for plane-wave

i.e. k^μ is a null-vector.

Tensor analysis

zeroth-rank tensor = Lorentz scalar (invariant)

1st-rank tensor = 4-vector

2nd-rank tensor $T^{\mu\nu} \to T'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}$

$T_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\tau} T^{\sigma\tau}$

$T_{\mu\nu} \to T'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\sigma \tilde{\Lambda}_\nu{}^\tau T_{\sigma\tau}$

Tensor satisfies:

1. Addition: $A^{\mu\nu} \pm B^{\mu\nu}$ is a tensor of the same rank with $A^{\mu\nu}$, $B^{\mu\nu}$

2. Multiplication: $A^\mu B^\nu G_{\sigma\tau}$ is a 4th-rank tensor

3. Raising and lowering indices using $\eta^{\mu\nu}$, $\eta_{\mu\nu}$, respectively

4. Contraction e.g. $A^\mu A_\mu$, $A^\mu B_\mu$ is a scalar; $T^{\mu\nu}{}_\mu$ is a 4-vector

5. Gradients: if λ is a 4-scalar:

$\partial_\mu \lambda \equiv \frac{\partial\lambda}{\partial x^\mu} = \lambda_{,\mu}$ is a covariant vector.

similarly for tensor $T^{\mu\nu}$:

$T^{\mu\nu}{}_{,\sigma} \equiv \frac{\partial T^{\mu\nu}}{\partial x^\sigma}$ is a 3rd order tensor.

tensor 的好处是，可以方便地构建不变量；

所有的物理方程均为 tensor equation. ← if it is true in frame K then it is true in all frames K'

That means the LHS, RHS should has same rank.

Maxwell equations in covariant form

charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0 \quad \Rightarrow \quad j^\mu = (\rho c, \vec{j})$$

$$\partial_\mu j^\mu = 0$$

Gauge potential form of Maxwell equation

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$$

$$\Rightarrow \quad A^\mu = (\phi, \vec{A})$$

$$\partial^2 A^\mu = \eta^{\alpha\beta}\partial_\alpha\partial_\beta A^\mu = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) A^\mu = -\frac{4\pi}{c} j^\mu$$

under Lorentz gauge $\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \Rightarrow \partial_\mu A^\mu = 0$

What about $\vec{E}$ and $\vec{B}$?

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a 2^{nd}-rank tensor

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell equations

$$\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

e.g. Use $\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$

$$\underbrace{\partial^\mu\partial^\nu}_{\text{symmetric } \mu\leftrightarrow\nu} \underbrace{F_{\mu\nu}}_{\text{anti } \mu\leftrightarrow\nu} = \frac{4\pi}{c}\partial^\mu j_\mu \quad \Rightarrow \quad \partial^\mu j_\mu = 0$$

Transformation under Λ

$$F_{\mu\nu} \to F'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta F_{\alpha\beta}$$

e.g.

$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0$ automatically satisfied for antisymmetric $F_{\mu\nu}$

$$\Leftrightarrow \quad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

For a boost along $\vec{v} = c\vec{\beta}$

$$\vec{E}'_\parallel = \vec{E}_\parallel, \quad \vec{B}'_\parallel = \vec{B}_\parallel$$

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}), \quad \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E})$$

• 在一个坐标系下，没有磁场，则 Boost之后可能会产生磁场

例如. A pure E and/or B field is NOT Lorentz invariant.

Useful invariant

① $F^{\mu\nu}F_{\mu\nu} = 2(|\vec{B}|^2 - |\vec{E}|^2)$

Lorentz invariant

不变量很重要

② For any 2^{nd}-rank tensor $A_{\mu\nu}$, its determinant $\det A$ is a scalar

$$\det A'_{\alpha\beta} = \det\left(A_{\mu\nu}\tilde{\Lambda}_\alpha{}^\mu\tilde{\Lambda}_\beta{}^\nu\right) = \det\left(\tilde{\Lambda}_\alpha{}^\mu A_{\mu\nu}(\tilde{\Lambda}^T)^\nu{}_\beta\right) = \det(\tilde{\Lambda}A\tilde{\Lambda}^T)$$

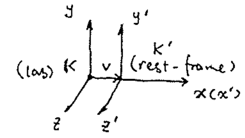
$$= (\det\tilde{\Lambda})^2 \cdot \det A = \det A$$

For EM field

$\det F_{\mu\nu} = (\vec{E}\cdot\vec{B})^2$ is invariant

$\vec{E}\cdot\vec{B}$ is also invariant.

Example: fields of a uniformly moving charge



In the Rest frame $\vec{B} = 0$

$$\begin{cases} E_x' = q x'/r'^3 \\ E_y' = q y'/r'^3 \\ E_z' = q z'/r'^3 \end{cases} \qquad r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

In the lab frame

$$E_x = q x'/r'^3, \quad B_x = 0$$

$$\vec{E}_\perp = \gamma\vec{E}'_\perp, \quad \vec{B}_\perp = \gamma\vec{\beta}\times\vec{E}'_\perp \qquad \left.\begin{cases} B_y = -q\gamma\beta z/r^3 \\ B_z = q\gamma\beta y/r^3 \end{cases}\right\} \vec{B} = \frac{q\gamma\vec{\beta}\times\vec{r}}{r^3}$$

$$\begin{cases} E_x = \frac{q\gamma}{r^3}(x - vt) \\ E_y = q\gamma y/r^3 \\ E_z = q\gamma z/r^3 \end{cases} \qquad r = \sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}$$

• [illegible]

Relativistic Mechanics

4-momentum $\quad p^\mu = m_0 u^\mu = \left(\frac{E}{c}, \vec{p}\right)$

$$p^0 = c m_0 \gamma_u, \quad p^i = m_0\gamma_u \vec{u}$$

$$p^2 = p^\mu p_\mu = -m_0^2\gamma_u^2(-c^2 + u^2) = -m_0^2 c^2 \quad \Rightarrow \quad |\vec{p}|^2c^2 + m_0^2c^4 = E^2$$

$$\| \quad -\frac{E^2}{c^2} + |\vec{p}|^2$$

For photons. $p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right) = \hbar k^\mu$

$$\omega = c|\vec{k}|$$

4-acceleration

$$a^\mu = \frac{du^\mu}{d\tau}$$

4-force $\quad F^\mu \equiv m_0 a^\mu = \frac{dp^\mu}{d\tau}$

Lorentz force $\quad \vec{F} = e\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$

$$\underbrace{F^\mu}_{\text{Lorentz force}} = \frac{e}{c}\underbrace{F^\mu{}_\nu}_{\text{anti-symmetric tensor}} u^\nu \quad \text{or} \quad a^\mu = \frac{e}{m_0 c}F^\mu{}_\nu u^\nu$$

For $\mu = 0$ $\quad \underbrace{\frac{dW}{dt}}_{\text{power}} = e\vec{E}\cdot\vec{v}$

$$\mu = i \quad \frac{dp_x}{dt} = e\left(E_x + \left(\frac{\vec{v}\times\vec{B}}{c}\right)_x\right)$$

$$F^\mu u_\mu = \frac{e}{c}F^\mu{}_\nu u^\nu u_\mu$$

$$= \frac{e}{c}\underbrace{F_{\mu\nu}}_{\text{anti } \mu\leftrightarrow\nu}\underbrace{u^\mu u^\nu}_{\text{sym. } \mu\leftrightarrow\nu} = 0$$

例如. 在四维时空之下 F^μ, u^μ 是正交向量.

Emission From Relativistic Particles

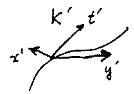
非相对论性的粒子，可以用 Larmor 公式求解

相对论性的粒子，求解复杂些（思路：在静止系下用 Larmor 公式，再变回来）

· 总的辐射功率

Instantaneous rest frame K' : In K', particle has zero velocity at a certain time.

For infinitesimally neighboring time the particle moves NON-relativistic



$\Rightarrow$ we can use Larmor's (or dipole) Formula in K'

In K', total amount of energy dW'

K (lab frame) moves $-\vec{v}$ w.r.t K', $dW = \gamma\, dW'$ (from the transformation of p^μ)

$dt = \gamma\, dt'$

这里的"dW'"是什么意义？以什么速度运动

$$P = \frac{dW}{dt} = P' \quad (\text{功率是 Lorentz 不变的})$$

From Larmor's formula $P' = \frac{2q^2}{3c^3}\left|\vec{a}'\right|^2$

是个 Lorentz 不变量，很好写

Easy to show $a_\mu u^\mu = \frac{du_\mu}{d\tau} u^\mu = \frac{1}{2}\frac{d}{d\tau}\left(u_\mu u^\mu\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^2\right) = 0$

$\frac{dx^\mu}{d\tau} = \gamma_\mu(c, \vec{u})$

In K', $\underline{u}'^\mu = \left(c, \vec{0}\right)$

$a'_\mu u'^\mu = a_0' u'^0 = 0 \Rightarrow a_0' = 0$

$a'_\mu a'^\mu = a_0' a'^0 + |\vec{a}'|^2 = |\vec{a}'|^2$

So the Power $P = \frac{2q^2}{3c^3} a^\mu a_\mu$ (Covariant form)

In K', $a'_\parallel = \gamma^3 a_\parallel$, $a'_\perp = \gamma^2 a_\perp$ $\Rightarrow$ $P = \frac{2q^2}{3c^3}\vec{a}'\cdot\vec{a}' = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$

(textbook p140)

· 辐射的角分布

Angular distribution of emitted and received power.

In K', $d\Omega' = \sin\theta' d\theta' d\phi'$

$\mu' = \cos\theta'$

Since 4-momentum p^μ is a 4-vector, $dW = \gamma\left(dW' + v\, dp_x'\right) = \gamma\, dW'\left(1+\beta\mu'\right)$

$\gamma\, dW'\left(1 + \beta c \frac{dp_x'}{dp'}\underbrace{\frac{dp'}{dW'}}_{\frac{1}{c}}\right)$, μ'

$$\mu = \frac{\mu' + \beta}{1+\beta\mu'} \quad (\text{see R.L. eq 4.8b}) \quad \Rightarrow \sin^2\theta' = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2}$$

and $d\mu = \frac{d\mu'}{\gamma^2(1+\beta\mu')^2}$

$d\phi = d\phi'$

$\Rightarrow d\Omega = \frac{d\Omega'}{\gamma^2(1+\beta\mu')^2}$

$$\frac{dW}{d\Omega} = \gamma^3(1+\beta\mu')^3\frac{dW'}{d\Omega'}$$

To divide dt

$\rightarrow dt = \gamma\, dt'$ emitted power

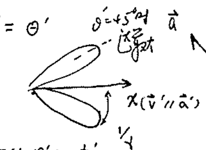
$\rightarrow dt_A = \gamma(1-\beta\mu)dt'$ the time interval of the radiation as received by stationary observer in K.

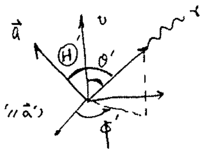
received power 用这个 P_r

而 $P = \frac{dW}{dt}$, so $\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\left(\gamma^2 a_\parallel^2 + a_\perp^2\right)}{(1-\beta\mu)^4}\sin^2\Theta'$.

Case 1. acceleration $\vec{a}' \parallel \vec{v}$ velocity, $\Theta' = \theta'$

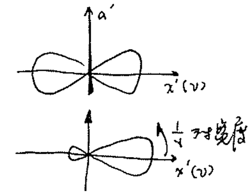
$$\frac{dP_\parallel}{d\Omega} = a_\parallel^2 \frac{\sin^2\theta'}{(1-\beta\mu')^4} \times \text{const.}$$





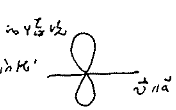
Case 2. acceleration $\perp$ velocity, $\cos\Theta' = \sin\theta'\cos\phi'$

$$\frac{dP}{d\Omega} = \frac{1}{(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right] \times \text{Const.}$$



不考虑静止时的分布

in K'



$\beta\mu \sim 1$

时分母最大值

$\Delta\mu = \frac{1}{\gamma}$

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \cong \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4$$

width $\theta \sim \frac{1}{\gamma}$, peaked at 0.

小测验：第一章～第三章，可带一张A4纸双面；作业 R&L 5.2；网络学堂有作业答案

（考：电子和离子的库仑作用～辐射）

自由的电子在经过的场中加速(减速)的辐射 ～ Free～Free emission

经典时用偶极辐射描述；

Bremsstrahlung (free-free) 韧致辐射

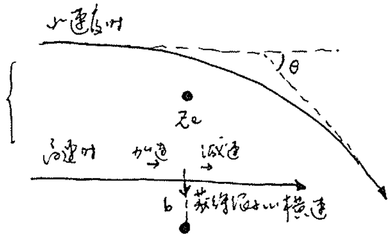
经典的偶极辐射 $\vec{d}=\sum q_i\vec{r}_i \propto \sum_i m_i\vec{r}_i \propto \vec{r}_c$

if like-ions　　质心是没有加速度的

No bremsstrahlung for like particle (e.g. e^-e^-, pp)

e～ion Bremsstrahlung (e.g. p-e，p的加速度很小，$m_e \ll m_p$，电子可看作 as moving in the fixed Coulomb field of p)

- Emission from single-speed electrons

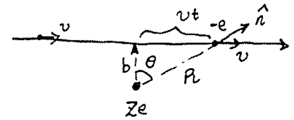


$$\vec{d}=-e\vec{R}$$
$$\ddot{\vec{d}}=-e\dot{\vec{v}}$$

Fourier transformation $\frac{d}{dt}\to -i\omega$

$$-\omega^2\tilde{d}(\omega)=\frac{-e}{2\pi}\int_{-\infty}^{\infty}\dot{\vec{v}}e^{i\omega t}dt$$

讨论高速情况



Collision time (碰撞时间，受有效库仑力影响的特征时间) $\sim \tau=\frac{b}{v}$

for $\omega\tau\gg 1$ ($b\gg\frac{v}{\omega}$), $e^{i\omega t}$ oscillates rapidly for $|t|\lesssim\tau$

$$\int\dot{\vec{v}}e^{i\omega t}dt\simeq 0$$

for $\omega\tau\ll 1$, $e^{i\omega t}\simeq 1$

$$\int\dot{\vec{v}}e^{i\omega t}dt\simeq\int\dot{\vec{v}}dt=\Delta\vec{v}$$

Combine these

$$\tilde{d}(\omega)=\begin{cases}\frac{e}{2\pi\omega^2}\Delta\vec{v}, & b\ll\frac{v}{\omega}\ (\lesssim)\\ 0, & \text{otherwise}\end{cases}$$

In dipole approximation, power spectrum

$$\frac{dW}{d\omega}=\frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2=\begin{cases}\frac{2e^2}{3\pi c^3}|\Delta\vec{v}|^2, & b\lesssim\frac{v}{\omega}\\ 0, & \text{otherwise}\end{cases}$$

From Coulomb's law $\dot{\vec{v}}=-\frac{Ze^2}{mR^2}\hat{n}$

$$\dot{\vec{v}}_\perp=-\frac{Ze^2}{mR^3}\vec{b}$$

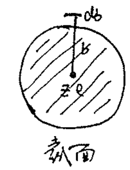
$$|\Delta\vec{v}|=\int_{-\infty}^{\infty}\frac{Ze^2b}{m(b^2+v^2t^2)^{3/2}}dt=\frac{Ze^2b}{mvb^2}\underbrace{\int_{-\infty}^{\infty}\frac{d\left(\frac{vt}{b}\right)}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}}}_{=2}=\frac{2Ze^2}{mvb}$$

Small-Scale

small-angle scattering regime

$$\frac{dW(b)}{d\omega}=\begin{cases}8Z^2e^6/3\pi c^3m^2v^2b^2, & b\lesssim\frac{v}{\omega}\\ 0, & \text{otherwise}\end{cases}$$

- For a flux of electrons $\frac{dN_e}{dA\,dt}=n_ev$, incident on 1 ion



$$\frac{dW}{d\omega\,dV\,dt}=\int_{b_{min}}^{\infty}\frac{dW}{d\omega}\underbrace{n_e\cdot v\cdot 2\pi b\,db}_{\frac{dN_e}{dt}}\times\underbrace{n_i}_{\frac{dN_i}{dV}}=\int_{b_{min}}^{\infty}n_en_i\cdot 2\pi v\frac{dW(b)}{d\omega}b\,db$$

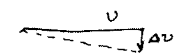
$\frac{dN_i}{dV}$ ～ 单位体积的离子数目

$$=n_en_i\cdot 2\pi v\int_{b_{min}}^{b_{max}}\frac{8Z^2e^6}{3\pi c^3m^2v^2}\frac{db}{b}=\frac{16e^6}{3c^3m^2v}n_en_iZ^2\ln\frac{b_{max}}{b_{min}}$$

b_{max} 的计算十分容易 $b_{max}\simeq\frac{v}{\omega}$

b_{min} 的决定因素有：

① straight-line approximation $\Delta v\ll v$ $(\lesssim)$



$$\Leftrightarrow b\gtrsim\frac{2Ze^2}{mv^2},\quad b_{min}=\frac{2Ze^2}{mv^2}$$

书里面多3-4$\left(\frac{2}{\pi}\right)^2$因子，$b_{min}^{(1)}=\frac{2Ze^2}{mv^2}\left(\frac{2}{\pi}\right)$?

② Uncertainty principle $\Delta x\Delta p\gtrsim\hbar/2$

$\Delta x\sim b$, $\Delta p\sim mv$

If $b_{min}^{(1)}\gg b_{min}^{(2)}$　　$\Leftrightarrow b_{min}^{(2)}=\frac{\hbar}{mv\cdot 2}$

Classical treatment is valid, $b_{min}=b_{min}^{(1)}$

$$\frac{2Ze^2}{mv^2}\gg\frac{\hbar/2}{mv}\Leftrightarrow v\ll\frac{2Ze^2}{\hbar/2}$$

$$\Leftrightarrow\frac{1}{2}mv^2\ll Z^2\times 4\frac{me^4}{2\hbar^2/4}=4Z^2R_y$$

where $R_y=\frac{me^4}{2\hbar^2}\approx 13.6eV$ 是氢原子的电离能

是氢原子的基态能量

Rydberg energy for Hydrogen

书里面多3.14 4倍，即$4Z^2R_y$

Alternatively, $\frac{1}{2}mv^2\gg Z^2R_y$

$b_{min}=b_{min}^{(2)}$

Quantum limit

In general

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16\pi e^6}{3\sqrt{3}c^3m^2 v} n_e n_i Z^2 g_{ff}(v,\omega)$$

where Gaunt factor $g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\frac{b_{max}}{b_{min}}$

- Thermal Bremsstrahlung emission

Average the single-speed expression over a thermal distribution of speeds.

对单一速度是否分布的
如热平衡下是 Maxwell 分布

$\begin{cases}\text{Thermal: Maxwell distribution} \\ \text{Non-thermal: some distribution of speeds.}\end{cases}$

Maxwell probability $dp \propto e^{-E/kT} d^3v \propto v^2 e^{-\frac{mv^2}{2kT}} dv$

$$\frac{dW(T,\omega)}{d\omega\, dV\, dt} = \frac{\int_{v_{min}}^{\infty} \frac{dW(v,\omega)}{d\omega\, dV\, dt} v^2 e^{-\frac{mv^2}{2kT}} dv}{\int_0^\infty v^2 e^{-mv^2/2kT} dv}$$

归一化常数

辐射产生的光子能量应 $h\nu \le \frac{1}{2}mv^2$

$\Rightarrow v_{min} = \sqrt{\frac{2h\nu}{m}}$ 最低速度

引入无量纲参数 $x = \frac{v}{v_{min}}$

$\frac{mv^2}{2kT} = \frac{mv_{min}^2}{2kT}x^2$

$$= \frac{\frac{16\pi e^6 n_e n_i Z^2}{3\sqrt{3}c^3m^2}}{\frac{\sqrt{\pi}}{4}\left(\frac{m}{2kT}\right)^{3/2}} \int_{v_{min}}^{\infty} g_{ff}(v,\omega)\, v e^{-\frac{mv^2}{2kT}} dv$$

$$= \frac{2h\nu}{m} e^{-h\nu/kT} (\quad) \int_1^\infty g_{ff}(v,\omega)\, e^{-\frac{h\nu}{kT}(x-1)} x\, dx$$

$$= \frac{kT}{m} e^{-h\nu/kT} (\quad)\ \bar{g}_{ff}$$

$$= \frac{32\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-\frac{h\nu}{kT}} \bar{g}_{ff}$$

where $\bar{g}_{ff} = \frac{2h\nu}{kT}\int_1^\infty g_{ff}(v,\omega)\, e^{-\frac{h\nu}{kT}(x-1)} x\, dx$ is velocity averaged Gaunt factor, typically of order $1 \sim O(1)$ for $\frac{h\nu}{kT} \sim 1$

define $\varepsilon_\nu^{ff} = \frac{dW}{d\nu\, dV\, dt} = 2\pi \frac{dW}{d\omega\, dV\, dt} = 6.8\times10^{-38} Z^2 n_e n_i T^{-1/2} e^{-\frac{h\nu}{kT}} \bar{g}_{ff}$ $\quad \mathrm{erg\cdot s^{-1} cm^{-3} Hz^{-1}}$

$(\omega = 2\pi\nu)$

$d\omega = 2\pi\, d\nu$

Gaunt factor $\bar{g}_{ff} \sim \begin{cases} O(1) & \text{for } \frac{h\nu}{kT}\sim 1 \\ 1\sim 5 & \text{for } 10^{-4} < \frac{h\nu}{kT} < 1 \end{cases}$

Total power per volume

$$\frac{dW}{dt\, dV} = \int \frac{dW}{dt\, dV\, d\nu} d\nu = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{32\pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B$$

$$= 1.4\times10^{-27}\, T^{1/2} n_e n_i Z^2 \bar{g}_B$$

where $\bar{g}_B = \int \bar{g}_{ff}\, e^{-\frac{h\nu}{kT}} d\left(\frac{h\nu}{kT}\right)$

frequency averaged of the velocity averaged Gaunt factor $\approx 1.1 - 1.5$

相当于一个没有方向的辐射系数

- Thermal bremsstrahlung

Free-Free Absorption (是 bremsstrahlung 的反过程)

free-free emission $4\pi j_\nu^{ff} = \frac{dW}{dt\, dV\, d\nu}$

这里没有角分布.

free-free absorption $\alpha_\nu^{ff} = \frac{j_\nu^{ff}}{B_\nu(T)}$, 代入 Planck func 的 $B_\nu(T)$ 得到 $\alpha_\nu^{ff} \approx 3.7\times10^{8}\, T^{-1/2} Z^2 n_e n_i \nu^{-3} (1-e^{-h\nu/kT})\bar{g}_{ff}$

for $h\nu \gg kT$, $\alpha_\nu^{ff} \propto \nu^{-3}$

absorption 不重要，发射为主

for $h\nu \ll kT$, $\alpha_\nu^{ff} = 0.018\, T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff} \propto \nu^{-2}$

低频吸收更强，发射也存在

Relativistic Bremsstrahlung (free-free) = Compton scattering of the virtual quanta of the ion's electrostatic field as seen in electron's frame.

思路：通过 Lorentz 变换，要到静止参考系.

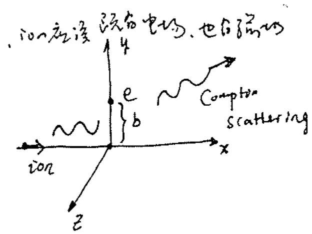
"method of virtual quanta" 虚光子方法

这在电子坐标系下，ion产生的静电场，也会辐射

electro static field → radiation

被Compton散射 (速度高, $v_{ion} \sim c$ 不是 Thomson 散射)

(实际，不是平面波)



得到辐射功率谱

$$\frac{dW'}{dA'd\omega'} = c\left|\tilde{E}(\omega')\right|^2 = \frac{q^2}{\pi^2 b^2 c}\left(\frac{b\omega'}{\gamma c}\right)^2 K_1^2\left(\frac{b\omega'}{\gamma c}\right)$$

$q^2 = Z^2e^2$

这里 $\tilde{E}(\omega) = \frac{1}{2\pi}\int E_2(t) e^{i\omega t} dt = \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}(\gamma^2v^2t^2+b^2)^{-3/2} e^{i\omega t} dt$, $\gamma = (1-\beta^2)^{-1/2}$

$$= \frac{q}{\pi b v} K_1\left(\frac{b\omega}{\gamma v}\right)\frac{b\omega}{\gamma v}$$

$K_1(x)$ modified Bessel function of the order 1

所有的 prime 都表示在电子坐标系下

In the long-wavelength limit $\hbar\omega' \lesssim m_e c^2$, 这里是 Thomson 散射

$$\frac{dW'}{d\omega'} = \frac{dW'}{dA'd\omega'}\int d\sigma_T = \frac{dW'}{dA'd\omega'}\sigma_T$$

Thomson 散射截面

$\frac{dW'}{d\omega'} = \frac{dW}{d\omega}$ $\quad b' = b$, $\omega = \gamma\omega'(1+\beta\cos\theta')$ → in average about 0.6: $\omega = \gamma\omega'$

能量，不变 $\quad$ 垂直速度方向的长度，不变

$$\sigma_T = \frac{8}{3}\pi\left(\frac{e^2}{mc^2}\right)^2$$

In lab frame

$$\frac{dW}{d\omega} = \frac{8 z^2 e^6}{3\pi b^2 c^5 m^2}\left(\frac{b\omega}{\gamma^2 c}\right)^2 K_1^2\left(\frac{b\omega}{\gamma^2 c}\right)$$

书里面公式 (5.23) 有typo. 少了平方

实验室坐标系下的功率谱

If $v \to c$, and add γ, from NR to Relativistic. formulae

NR: Non-Relativistic.

for a flux of electrons

$$\frac{dW}{d\omega\, dt\, dV} = \int_{b_{min}}^{\infty} \frac{dW(b)}{d\omega} \cdot 2\pi b\, db\; n_e v \cdot n_i \sim \frac{16 z^2 e^6 n_e n_i}{3 c^4 m^2} \ln\left(\frac{0.68\gamma^2 c}{\omega b_{min}}\right)$$

其中 ln 的来源是 $\int_{x=\frac{\omega b_{min}}{\gamma^2 c}}^{\infty} K_1^2(x)\cdot x\,dx$

$= x K_0(x) K_1(x) - \frac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right) \xrightarrow{\hbar\omega \ll \gamma m c^2} \ln\left(\frac{0.68\gamma^2 c}{\omega b_{min}}\right)$

In the high-frequency, $\hbar\omega' \gtrsim mc^2$ (高频极限)

$$\frac{dW'}{d\omega'} = \frac{dW'}{d\omega' dA'}\,\sigma$$

$\sigma \to$ Compton Scattering cross-section

Klein-Nishina cross-section

$$\frac{dW}{dt\,dV} \approx 1.4\times10^{-27}\, T^{1/2} z^2 n_e n_i\, \bar{g}_B \underbrace{(1+4.4\times10^{-10}\,T)}_{\text{relativistic Correction}}$$

对ω积分

total power per volume

relativistic Correction (Novikov + Thorne 1973)

0409/2018 week 7 GB211

Synchrotron Radiation 同步辐射

低速, 使用 Larmor 公式, 谱是单色的

高速 (相对论性),
- 谱不是单色的 (beam effect) *1
- 单粒子的谱辐射功率 $\propto F(\frac{\omega}{\omega_c})$ *2
- 多粒子辐射功率 $\propto \omega^{-s}$ *3

Particles accelerated by B-field can radiate

- Non-relativistic particles: $\omega = \omega_g$, cyclotron radiation

  非相对论加速辐射 $\Delta t \to \infty$ $\Delta\omega \to 0$

- Ultra-relativistic particles: spectrum of ω, synchrotron radiation

  相对论加速辐射 $\Delta t_A \sim \frac{1}{\omega_c}$, $\omega_c \cong \frac{3}{2}\gamma^3 \omega_B \sin\alpha$

  A: arrival

  ω_c: critical frequency

  α: pitch angle, $\vec{v}$ 与 $\vec{B}$ 的夹角

介于两者之间的情况不好讨论, 略去

前者只了解, 这里讨论后者

Consider a particle, m, q, in a B-field

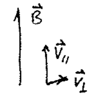
$$a^\mu = \frac{q}{m_0 c} F^\mu{}_\nu u^\nu$$

$$\begin{cases} \frac{dW}{dt} = q\vec{E}\cdot\vec{v} = 0 & ① \\ \frac{d\vec{p}}{dt} = q(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) = \frac{q}{c}\vec{v}\times\vec{B} & ② \end{cases}$$

From ① $\frac{d}{dt}(\gamma m c^2) = 0$, γ is constant $\Rightarrow$ v is constant

From ② $LHS = \frac{d}{dt}(\gamma m\vec{v}) = \gamma m\frac{d\vec{v}}{dt}$ $\Rightarrow$ $\frac{d\vec{v}}{dt} = \frac{q}{\gamma m c}\vec{v}\times\vec{B}$

将速度按 $\parallel \vec{B}$ 和 $\perp \vec{B}$ 分解 (w.r.t $\vec{B}$): $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$



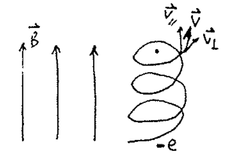
所以 $\frac{d\vec{v}_\parallel}{dt} = 0 \Rightarrow \vec{v}_\parallel$ is constant

$$\frac{d\vec{v}_\perp}{dt} = \frac{q}{\gamma m c}\vec{v}_\perp\times\vec{B} \Rightarrow \dot{\vec{v}}_\perp \perp \vec{v}_\perp$$

$|\vec{v}_\perp| = $ constant $|\vec{a}_\perp| = $ constant

所以运动是 helical orbit

回旋频率 (gyration frequency) $\omega_B = \frac{qB}{\gamma m c}$ ✓ (因为 $a_\perp = \left|\frac{d\vec{v}_\perp}{dt}\right| = v_\perp \omega_B$ ✓) ($a_\perp$ w.r.t. $\vec{B}$, $= a$)



Total Emission 只一部分的公式

$$P = \frac{2q^2\gamma^4}{3c^3}\left(a_\perp^2 + \gamma^2 a_\parallel^2\right) = \frac{2q^2\gamma^4}{3c^3}\left(\frac{qB}{\gamma m c}\right)^2 \beta_\perp^2 c^2 \quad \beta_\perp = \frac{v_\perp}{c}$$

$a_\perp$: 加速; $a_\parallel = 0$ w.r.t $\vec{v}$; $\beta_\perp$: 与磁场垂直的速度

$$= \frac{2}{3} r_0^2 c\, \gamma^2 B^2 \beta_\perp^2 , \quad r_0 = \frac{q^2}{mc^2} \text{ ✓ is classical radius}$$

(见 Thomson 散射)

对于多粒子系统, 可是有分布的. 需求平均

速度 $\vec{v}$ 的方向均匀分布

但 $\vec{v}$ 的大小都一样 $\langle\beta_\perp^2\rangle = \int(\beta\sin\theta)^2\frac{d\Omega}{4\pi} = \frac{2}{3}\beta^2$ ($d\Omega = \sin\theta\, d\theta\, d\phi$)

$$\langle P\rangle = \frac{2}{3}r_0^2 c\gamma^2 B^2 \frac{2}{3}\beta^2 = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B$$

σ_T 是 Thomson scattering cross section $\frac{8}{3}\pi r_0^2$

$U_B = \frac{1}{8\pi}B^2$ is magnetic field energy density.

辐射的角分布

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{(1-\beta\mu)^4}\sin^2\Theta'$$

($a_\parallel$, $a_\perp$ w.r.t. $\vec{v}$)

在 $\vec{a}\perp\vec{v}$ 的情况下, 将 $\vec{a}, \vec{v}$ 分别放在 x, z 方向

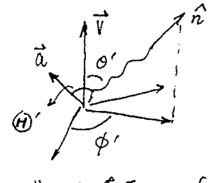
$$\cos\Theta' = \frac{\vec{a}'\cdot\hat{n}'}{|\vec{a}'|} = \hat{e}_x\cdot\hat{n}' = \sin\theta'\cos\phi'$$

而 $\sin^2\Theta' = 1 - \sin^2\theta'\cos^2\phi'$

而易证明 $\phi = \phi'$

利用 $\mu = \frac{\mu'+\beta}{1+\beta\mu'} \xrightarrow{\beta\to-\beta} \mu' = \frac{\mu-\beta}{1-\beta\mu}$ ($\mu = \cos\theta$)

所以 $\sin^2\theta' = 1-\mu'^2 = 1-\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2}$



" ' " 指在 rest frame

Θ': angle between acceleration and the direction of emission

$$\sin^2\Theta' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}$$

最后结果是 $\frac{dP}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{(1-\beta\mu)^4}\left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right)$

在速度非常大时，
$\mu = \cos\theta \doteq 1 - \frac{\theta^2}{2}$
$\beta = (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2\gamma^2}$
$1-\beta\mu \simeq \frac{1+\gamma^2\theta^2}{2\gamma^2}$

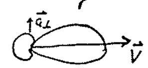
当 $\theta = 0$ 时，$1-\beta\mu$ 最小
当 $\theta = \frac{1}{\gamma}$ 时，$1-\beta\mu$ 已经很大
所以辐射集中在 $\frac{1}{\gamma}$ 范围的立体角内（相对于速度方向）

此时 $\frac{dP_\perp}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{16\gamma^8}{(1+\gamma^2\theta^2)^4}\left(1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right) = \frac{4q^2 a_\perp^2\gamma^8}{\pi c^3}\cdot\frac{1-2\gamma^2\theta^2\cos 2\phi + \gamma^4\theta^4}{(1+\gamma^2\theta^2)^6}$

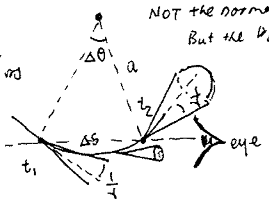
θ 很大时 $\propto \frac{\gamma^4\theta^4}{(\gamma^2\theta^2)^6} \simeq \frac{1}{(\gamma\theta)^8}$

$\phi = 90°$ 时 $\propto \frac{(1+\gamma^2\theta^2)^2}{(1+\gamma^2\theta^2)^6} = \frac{1}{(1+\gamma^2\theta^2)^4}$

The factor is sharply peaked at $\theta = 0$
with an angular scale $\frac{1}{\gamma}$
This is the beam effect along $\vec{v}$



观察的频谱：在观察者的



NOT the normal plane to B
But the 瞬时曲率圆平面

a 是 radius of the Curvature of the path

a' 是 $\perp\vec{B}$ 平面的投影运动的半径
$a' = \frac{v\sin\alpha}{\omega_B}$

$\Delta\theta \doteq 2\frac{1}{\gamma}$ 可以见到辐射的转角

$\Delta s = a\Delta\theta = \frac{2a}{\gamma}$ 路径长

$a = \frac{\Delta s}{\Delta\theta} = \frac{v\Delta t}{|\Delta v|/v} = \frac{v^2}{|\Delta v|} = \frac{v^2}{\frac{q}{\gamma mc}vB\sin\alpha} = \frac{v}{\omega_B\sin\alpha}$

$\gamma m\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$ $\qquad \omega_B = \frac{qB}{\gamma mc}$

α 是投掷角，即 $\vec{v}$ 与 $\vec{B}$ 的夹角

可观测时间 $\Delta t^A = \Delta t_2^A - \Delta t_1^A$ arrival time . 由测不准关系 $\Delta t^A\Delta\omega \gtrsim 1$ 即 $\Delta\omega \gtrsim \frac{1}{\Delta t^A}$

$\Delta t = \frac{\Delta s}{v} = \frac{2}{\gamma\omega_B\sin\alpha} \simeq \Delta t^A$ 所以 $\Delta\omega \gtrsim \frac{\gamma\omega_B\sin\alpha}{2} \gg \frac{\omega_B}{2\pi}$

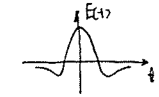
在经典的回旋辐射时，是接近单色的；但由于 beam effect，观测 Δt^A 有限，$\Delta\omega$ 有下限

更准确的估计 $\Delta t^A = \Delta t - \frac{\Delta s}{c} = \Delta t\left(1-\frac{v}{c}\right) = \frac{2}{\gamma\omega_B\sin\alpha}\left(1-\frac{v}{c}\right)$

利用 $1-\left(\frac{v}{c}\right)^2 = \gamma^{-2}$，$\frac{v}{c} \simeq (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2\gamma^2}$，$1-\frac{v}{c} \simeq \frac{1}{2\gamma^2}$ for $\gamma \gg 1$

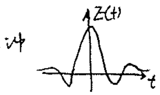
则 $\Delta t^A \simeq \frac{1}{\gamma^3\omega_B\sin\alpha}$

所以有 $\Delta t^A = \Delta t\left(1-\frac{v}{c}\right) = \frac{1}{\gamma^3\omega_B\sin\alpha}$



$\frac{\Delta t^A}{\Delta t_{gyro}} = \frac{1/\gamma^3\omega_B\sin\alpha}{2\pi/\omega_B} \sim \frac{1}{\gamma^3} \ll 1$ 所以接收到的是一个一个脉冲

回旋时间



定义 Critical frequency $\omega_c = \frac{3}{2}\gamma^3\omega_B\sin\alpha$，则 $\Delta t^A = \frac{3}{2}\omega_c^{-1}$

在非相对论时 $\Delta t^A \to \infty$ 线谱

根据测不准关系 $\Delta t^A\Delta\omega \sim 1$，辐射展宽 extend of the spectrum $\Delta\omega \sim \frac{2}{3}\omega_c$

总的辐射功率

由于 $\Delta\theta \sim \frac{1}{\gamma}$，不妨假定 $E(t) = F(\gamma\theta)$
即 $\gamma\theta$ 唯一决定了 $E(t)$

由几何关系
$\theta \simeq \frac{s}{a}$, $t = \frac{s}{v}\left(1-\frac{v}{c}\right)$
$\gamma\theta = \frac{\gamma s}{a} = \frac{\gamma}{a}vt/\left(1-\frac{v}{c}\right) \simeq \frac{\gamma vt}{\frac{3}{2}\frac{\gamma^3 v}{\omega_c}\times\frac{1}{2\gamma^2}} = \frac{4}{3}\omega_c t$

所以 $E(t) \propto g(\omega_c t)$，再做其 Fourier Transform

$\tilde{E}(\omega) \propto \int_{-\infty}^{\infty} g(\omega_c t)e^{i\omega t}dt$，若令 $\xi = \omega_c t$ 则 $\tilde{E}(\omega) = \int_{-\infty}^{\infty} g(\xi)e^{i\frac{\omega}{\omega_c}\xi}d\xi$

最后 $\tilde{E}(\omega)$ 应是 $f\left(\frac{\omega}{\omega_c}\right)$，即都是 $\frac{\omega}{\omega_c}$ 的函数；功率谱又因为 $\tilde{P}_\Omega(\omega) = \frac{dW}{d\omega d\Omega} \propto |\tilde{E}(\omega)|^2$.

$\tilde{P}(\omega) = \frac{dW}{d\omega dt} = \frac{1}{T}\int\frac{dW}{d\omega d\Omega}d\Omega \propto |\tilde{E}(\omega)|^2 = C_1 F\left(\frac{\omega}{\omega_c}\right)$

independent of ω $\qquad$ some constant $\qquad$ dimensionless

同步辐射总功率 $P = \int_0^\infty\tilde{P}(\omega)d\omega = C_1\omega_c\int_0^\infty F(x)dx$，$x = \frac{\omega}{\omega_c}$

另一方面的总的辐射功率应是 $P = 2q^4B^2\gamma^2\beta^2\sin\alpha/3m^2c^3$，所以 $C_1\int_0^\infty F(x)dx = \frac{P}{\omega_c} = \frac{4q^3B\beta^2\sin\alpha}{9mc^2} \simeq \frac{4q^3B\sin\alpha}{9mc^2}$ $\quad \to \beta \simeq 1$

最终 $\tilde{P}(\omega) = \frac{\sqrt{3}q^3B\sin\alpha}{2\pi mc^2}F\left(\frac{\omega}{\omega_c}\right)$

ω_c only dependent on γ

Special Cases

$\tilde{P}(\omega) \propto \omega^{-s}$
power law，经常见到
这里的 s 称为 spectral index

对于多粒子系统，若速度给出，distribution of velocity

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma, \quad \gamma_1<\gamma<\gamma_2$$

总辐射功率 $P_{tot}(\omega) = C\int_{\gamma_1}^{\gamma_2}\tilde{P}(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2}F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma \propto -\int_{x_1}^{x_2}F(x)\,\omega_c^{-\frac{p}{2}+\frac{1}{2}}\frac{dx}{x} \propto \omega^{-\frac{(p-1)}{2}}\int_{x_2}^{x_1}F(x)\,x^{\frac{p-3}{2}}dx$

$x=\frac{\omega}{\omega_c}$，$\omega_c \propto \gamma^2$

$= \frac{3\gamma^2 qB\sin\alpha}{2mc} \propto \gamma^2$

$x \propto \gamma^{-2}$

$\frac{dx}{x} \simeq -2\frac{d\gamma}{\gamma}$

若 $\omega_c(\gamma_1) \ll \omega \ll \omega_c(\gamma_2)$，可令 $x_2\to 0$，$x_1\to\infty$，则

$$P_{tot}(\omega) \propto \omega^{-\frac{(p-1)}{2}}$$

所以，在极端相对论情况下，若 velocity 分布以 p 为 power law index，则 $P_{tot}(\omega)$ 以 $\frac{p-1}{2}$ 为 power law index.

0423 | 2018　　week 9　　6B211　　hw　R&L 6.1, 7.1

物理学实验和辐射机制

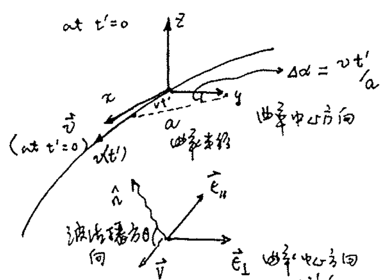
推导同步辐射和总辐射功率的定量表达式

Spectrum of Synchrotron Radiation

沿方向 $\hat{n}$ 的功率谱

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \hat{n}\times(\hat{n}\times\vec{\beta})\exp i\omega\left(t'-\frac{\hat{n}\cdot\vec{r}(t')}{c}\right)dt'\right|^2$$

retarded time



其中 $t' - \frac{\hat{n}\cdot\vec{r}(t')}{c} = t' - \frac{a}{c}\cos\theta\sin\Delta\alpha$

$\vec{\beta}(t') = (\cos\Delta\alpha, \sin\Delta\alpha, 0)$, $|\vec{\beta}|$

$\hat{n} = (\cos\theta, 0, \sin\theta)$ 不失一般性，假设在 x-z 平面

$\hat{\epsilon}_{\perp} = (0,1,0)$

$\hat{\epsilon}_{\parallel} = \hat{n}\times\hat{\epsilon}_{\perp} = (-\sin\theta, 0, \cos\theta)$

只需考虑小的 $\Delta\alpha$：由于 beam effect，也只需考虑小的 θ

$$\simeq t' - \frac{a}{c}\left(1-\frac{\theta^2}{2}\right)\left(\frac{vt'}{a}-\frac{1}{6}\left(\frac{vt'}{a}\right)^3\right)$$

$$= t'\left(1-\frac{v}{c}\right) + \frac{v}{c}\frac{\theta^2}{2}t'$$

而已推导曲率半径 $a = \frac{3}{2}\gamma^3\frac{v}{\omega_c} \simeq \frac{3\gamma^3 c}{2\omega_c}$

$$\simeq 2\gamma^{-2}t' + \frac{\theta^2}{2}t' = (2\gamma^2)^{-1}\left((1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)$$

是一个很大的量，指数积分为0（为什么？）
只有 $\gamma\theta \lesssim 1$ 时，积分不为零
另外还需 $\frac{\gamma t'}{a} \lesssim 1$

而 $\hat{n}\times(\hat{n}\times\vec{\beta}) = (\hat{n}\cdot\vec{\beta})\hat{n}-\vec{\beta} = -\underset{\text{小量}}{\hat{\epsilon}_{\perp}\sin\Delta\alpha} + \hat{\epsilon}_{\parallel}\cos\Delta\alpha\underset{\text{小量}}{\sin\theta}$，所以 $\hat{n}\times(\hat{n}\times\vec{\beta})$ 很小。

功率谱可以写为 $\parallel$ 和 $\perp$ 方向的贡献之和

$$\frac{dW}{d\omega d\Omega} = \frac{dW_{\parallel}}{d\omega d\Omega} + \frac{dW_{\perp}}{d\omega d\Omega}$$

注意 $v\simeq c$

其中，along $\hat{\epsilon}_{\perp}$，$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int\frac{t'c\,dt'}{a}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta^2\gamma t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]\right|^2$

$\theta_\gamma = 1+\gamma^2\theta^2$

along $\hat{\epsilon}_{\parallel}$，$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left|\int dt'\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]\right|^2$

所以 $\frac{dW_{\perp}}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2\left|\int_{-\infty}^{\infty}y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$，let $y = \gamma\frac{ct'}{a\theta_\gamma}$

$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{a^2\theta_\gamma^2}{\gamma^4 c}\right)^2\left|\int_{-\infty}^{\infty}\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2$，$\eta \equiv \frac{\omega a\theta_\gamma^3}{3c\gamma^3}$

or $\gamma^2 c$

because cancelled by γ in y

$$\theta\simeq 0,\quad \eta\simeq\eta(\theta=0) = \frac{\omega}{2\omega_c}$$

所以 $\frac{dW}{d\omega d\Omega}$ depends on $\frac{\omega}{\omega_c}$ and $\gamma\theta$

积分后得到

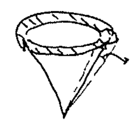
$$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{q^2\omega^2}{3\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{2/3}^2(\eta)$$

→ modified Bessel function

$$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2 K_{1/3}^2(\eta)$$

由于粒子做螺旋线运动，速度扫过一个立体角，半锥角 pitch angle

$$d\Omega = 2\pi\sin\alpha\,d\theta$$



$\theta\sim\frac{1}{\gamma}$

总的辐射谱

$$\frac{dW}{d\omega} = \int\frac{dW}{d\omega d\Omega}d\Omega = \int\left(\frac{dW_{\perp}}{d\omega d\Omega}+\frac{dW_{\parallel}}{d\omega d\Omega}\right)d\Omega = \frac{dW_{\perp}}{d\omega}+\frac{dW_{\parallel}}{d\omega}$$

$$\frac{dW_{\perp}}{d\omega} = \frac{2q^2\omega^2a^2\sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^4 K_{2/3}^2(\eta)\,d\theta = \frac{\sqrt{3}q^2\gamma\sin\alpha}{2c}(F(x)-G(x))$$

$$\frac{dW_{\parallel}}{d\omega} = \sim\int_{-\infty}^{\infty}\theta_\gamma^2\theta^2K_{1/3}^2(\eta)\,d\theta = \sim(F(x)-G(x))$$

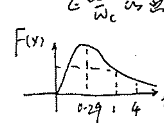
$\frac{dW}{d\omega} = \sim\times 2F(x)$

or Ω?

其中 $F(x) = x\int_x^{\infty}K_{5/3}(\xi)d\xi$，$x=\frac{\omega}{\omega_c}$

是 $\frac{\omega}{\omega_c}$ 的函数

$$\begin{cases}\frac{4}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} & x\ll 1\\ \left(\frac{\pi}{2}\right)^{1/2}e^{-x}x^{1/2} & x\gg 1\end{cases} \simeq F(x)$$



$\omega/\omega_c<1$ 有显著贡献

在时间 $\frac{2\pi}{\omega_B}=T$ 内，平均功率

$$\frac{dW}{T\,d\Omega} = P(\omega) = \frac{\sqrt{3}\,q^3 B\sin\alpha}{2\pi mc^2} F\left(\frac{\omega}{\omega_c}\right)$$

or ω?

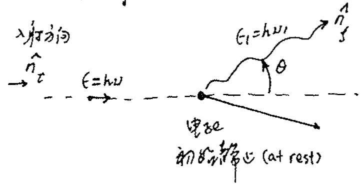
Thomson 散射是弹性散射，只是相对论性情况的近似。

如果入射光的能量很高，那么电子会被冲击，光的频率也会变化；

$h\nu \simeq m_e c^2$

另外，如果入射光能量很大，散射截面的公式也要改变 } 量子效应

Compton Scattering



4-momentum of γ

$P^{\mu}_{\gamma i} = \frac{\epsilon}{c}(1, \hat{n}_i)$ $\quad P^{\mu}_{\gamma f} = \frac{\epsilon_1}{c}(1, \hat{n}_f)$

4-momentum of electron

$P^{\mu}_{e i} = (mc, \vec{0})$ $\quad P^{\mu}_{e f} = \left(\frac{E}{c}, \vec{p}\right)$

metric $(-1, 1, 1, 1)$

运动学

Conservation of momentum

$$P^{\mu}_{ei} + P^{\mu}_{\gamma i} = P^{\mu}_{ef} + P^{\mu}_{\gamma f}$$

$$\Rightarrow \left|P^{\mu}_{ef}\right|^2 = \left|P^{\mu}_{ei} + P^{\mu}_{\gamma i} - P^{\mu}_{\gamma f}\right|^2$$

$\left|P^{\mu}_{ef}\right|^2 = -\left(\frac{E}{c}\right)^2 + p^2 = -(mc)^2$

$\left|P^{\mu}_{ei} + P^{\mu}_{\gamma i} - P^{\mu}_{\gamma f}\right|^2 = \left|\left(\frac{\epsilon}{c} - \frac{\epsilon_1}{c} + mc,\ \frac{\epsilon}{c}\hat{n}_i - \frac{\epsilon_1}{c}\hat{n}_f\right)\right|^2 = -\left(\frac{\epsilon}{c} - \frac{\epsilon_1}{c} + mc\right)^2 + \left|\left(\frac{\epsilon}{c}\hat{n}_i - \frac{\epsilon_1}{c}\hat{n}_f\right)\right|^2$

$$\Rightarrow -m^2c^4 = -(\epsilon - \epsilon_1 + mc^2)^2 + \left|\epsilon\hat{n}_i - \epsilon_1\hat{n}_f\right|^2$$

$\left|\epsilon\hat{n}_i - \epsilon_1\hat{n}_f\right|^2 = \epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1 \underbrace{\hat{n}_i\cdot\hat{n}_f}_{\cos\theta}$

$$= -(\epsilon-\epsilon_1)^2 - m^2c^4 - 2mc^2(\epsilon-\epsilon_1) + \epsilon^2 + \epsilon_1^2 - 2\epsilon\epsilon_1\cos\theta$$

$$\Rightarrow \epsilon\epsilon_1(1-\cos\theta) = mc^2(\epsilon - \epsilon_1)$$

$$\Rightarrow \epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1-\cos\theta)}$$ 此即散射角和散射能量的关系

$\epsilon = \frac{hc}{\lambda}$

$$\Rightarrow \lambda_1 = \lambda\left(1 + \left(\frac{\lambda}{\lambda_c}\right)^{-1}(1-\cos\theta)\right)$$

定义 $\frac{\epsilon}{mc^2} = \frac{h/mc}{\lambda} = \frac{\lambda_c}{\lambda}$，$\lambda_c = \frac{h}{mc}$ 为 Compton wavelength（是电子的特征波长）

当 $\lambda < \lambda_c$ 即 $\epsilon > mc^2$ 时，Compton 效应明显

电子的质量很小，$\lambda_c \simeq 0.024\,\text{Å}$，波长较长，实验容易观察到。

另外，λ_c 只与粒子质量有关。

散射前后频率变化

$\Delta\lambda = \lambda_1 - \lambda = \lambda_c(1-\cos\theta)$ 只与散射粒子的质量有关，和散射角有关（最重要的运动学结论）。

$\Delta\lambda$: loss of energy；λ_c: $\frac{h}{mc}$

反弹时 $\Delta\lambda = 2\lambda_c$；不改方向时 $\Delta\lambda = 0$

对于长波 $\frac{\lambda_1-\lambda}{\lambda}$ 很小，就回到 Thomson 散射。

之前的 Thomson 散射截面 $\sigma_T = \frac{8}{3}\pi r_0^2$，微分散射截面 $\frac{d\sigma}{d\Omega} \cong \frac{1}{2}r_0^2(1+\cos^2\theta)$

Note:

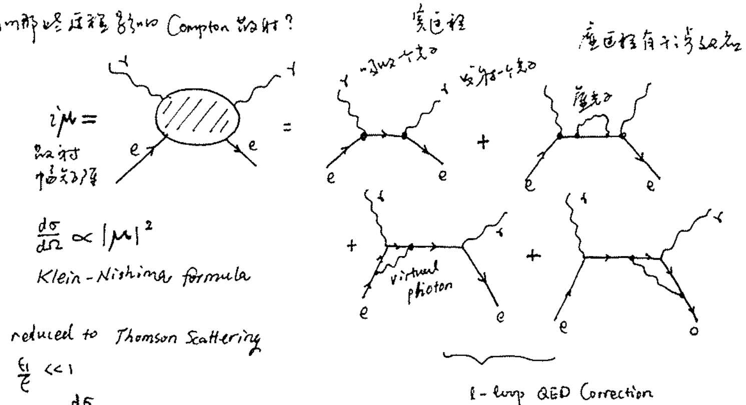
$\iint_{\text{all}} \cos^2\theta\, d\Omega = \frac{4}{3}\pi$

$r_0 = \frac{e^2}{mc^2}$

现在，加入 QED 效应，对 Compton 散射

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}r_0^2\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon_1}{\epsilon} - \sin^2\theta\right) \Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{KN} = \frac{1}{2}r_0^2\left(\frac{\epsilon_1}{\epsilon} + \left(\frac{\epsilon_1}{\epsilon}\right)^3 - \left(\frac{\epsilon_1}{\epsilon}\right)^2\sin^2\theta\right)$$

可以有哪些过程贡献 Compton 散射？



$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2$

Klein-Nishina formula

$\epsilon_1 \sim \epsilon$, reduced to Thomson Scattering

$\epsilon \gg \epsilon_1$, $\frac{\epsilon_1}{\epsilon} \ll 1$

$$\left(\frac{d\sigma}{d\Omega}\right)_{KN} \ll \frac{1}{2}r_0^2(1+\cos^2\theta)$$

能量很高时，散射截面反而变小

Compton Scattering becomes less efficient

定义 $x = \frac{h\nu}{mc^2} = \frac{\lambda_c}{\lambda} \gg 1$，可以展开得到截面积。

$$\sigma = \sigma_T \cdot \frac{3}{4}\left(\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right\} + \frac{1}{2x}\ln(1+2x) - \frac{1+3x}{(1+2x)^2}\right)$$

$$\begin{cases} \sigma_T\left(1 - 2x + \frac{26}{5}x^2 + \cdots\right) & \text{for } x \ll 1 \ (\text{non-relativistic}) \\ \frac{3}{8}\sigma_T x^{-1}\left(\ln 2x + \frac{1}{2}\right) \propto \frac{\ln 2x}{x} & \text{for } x \gg 1 \ (\text{ultra-relativistic}) \end{cases}$$

电子处于加热状态时，可以有很高的能量；可以与CMB发生逆 Compton 散射

$T_e \gtrsim 10^4 K \simeq 1eV$ $\qquad$ $T_{CMB} \simeq 3K(1+6) \sim 2\times10^{-3}eV \ll T_e$

$\underbrace{1+6}_{\text{reionization redshift}}$

电子打光子 { 电子损失能量 / 光子获得能量

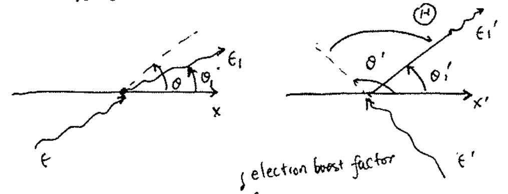
在电子的坐标系下，仍然是光子损失能量（电子坐标系下，光子有很高的能量）

下面定性估计光子获得的能量

Inverse Compton Scattering 低能光子和高能电子散射

In the rest frame of electron, $h\nu \ll mc^2$ non-relativistic $\quad \sigma = \sigma_T$ ignore QED correction

Let K be the lab frame, K' be the electron rest frame $\qquad$ 看起来是 Compton 散射，事实上仍是 Thomson 散射（宣读者）



Doppler shift $\quad \epsilon' = \epsilon\gamma(1-\beta\cos\theta)$ $\quad$ γ electron boost factor

$\epsilon_1 = \epsilon_1'\gamma(1+\beta\cos\theta_1')$

$$\epsilon_1' = \frac{\epsilon'}{1+\frac{\epsilon'}{mc^2}(1-\cos\Theta)} \simeq \epsilon'\left(1-\frac{\epsilon'}{mc^2}(1-\cos\Theta)\right)$$

for Thomson Scattering

By definition

$$\cos\Theta = \hat{n}'_{\gamma i}\cdot\hat{n}'_{\gamma f} = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')\cdot(\sin\theta_1'\cos\phi_1', \sin\theta_1'\sin\phi_1', \cos\theta_1')$$

$$= \sin\theta'\sin\theta_1'\cos(\phi'-\phi_1') + \cos\theta'\cos\theta_1'$$

In case of relativistic electrons $\qquad \epsilon' \ll mc^2$（忽略前后光子能量）

$\gamma^2 - 1 \gg \frac{h\nu}{mc^2}$ $\qquad \epsilon' : \epsilon_1' = 1+\frac{\epsilon'}{mc^2}(1-\cos\Theta) \simeq 1$

$E_{\gamma i} : E'_{\gamma i} : E_{\gamma f}$ $\qquad \epsilon_1' : \epsilon_1 = \frac{1}{\gamma(\beta\cos\theta_1'+1)} \sim \frac{1}{\gamma}$

$\epsilon : \epsilon' = \frac{1}{\gamma(1-\beta\cos\theta)} \overset{\theta=\frac{\pi}{2}}{\simeq} \frac{1}{\gamma}$

（boost 使能获得的能量提升）

所以 $E_{\gamma i} : E'_{\gamma i} : E_{\gamma f} = 1 : \gamma : \gamma^2$，能量获得了两次提升，完全由于 Doppler 效应

所以，光子能量提升是 γ^2 倍。

低能光子 low energy photon ϵ $\longrightarrow$ high energy photon $\gamma^2\epsilon$

$m_e c^2 \simeq 500\,keV$, $\epsilon = 100\,keV \xrightarrow{\gamma^2}$ 远大于 500keV，此时不是很有效 就不是 Inverse

γ^2 提升只有在非相对论情况下适用

对高能光子不适用

（$\gamma\epsilon \ll m_ec^2$，光子获得的能量显著

$\gamma\epsilon \sim m_ec^2$，光子获得的能量不显著）

Review

$\epsilon : \epsilon' : \epsilon_1' : \epsilon_1 \simeq 1 : \gamma : \gamma : \gamma^2$

ϵ: photon energy before scattering in K; ϵ': before in K'; ϵ_1': after in K'; ϵ_1: after in K

两次 boost 获得能量

0507/2018 $\quad$ week 11 $\quad$ 6B211 $\quad$ 物理辐射机制 $\qquad$ HM R&L 7.4, 8.2

05/07 7:20-9:45 六教 6B210

Inverse Compton Power for Single Scattering

Want to know the average emitted power for isotropic distribution of photon scattering off an isotropic distribution of electrons.

Let $n(p)$ = photon phase space distribution function

$\underbrace{n\,d^3p\,d^3x}_{\text{invariant}}$ = photon numbers (invariant)

所以 $n = n'$, distribution is invariant

$n\,d^3p$ = photon number density in $[\epsilon, \epsilon+d\epsilon]$

所以 $\epsilon = \gamma\epsilon'$, $d^3p = \gamma d^3p'$ $\Rightarrow$ $\frac{n\,d^3p}{\epsilon} = \frac{n'd^3p'}{\epsilon'}$ Lorentz invariant

The total emitted power in K' ← rest frame of electron

假定电子是静止的

光子场在 K 下是各向同性分布的

$$\frac{dE_1}{dt} = \frac{dE_1'}{dt'} = c\sigma_T\int \epsilon_1'\,n'd^3p'$$

$c\sigma_T$：单位时间的散射截面；ϵ_1'：单个光子的能量，emitted photon energy in K'；$n'd^3p'$：单位体积的数目

$$= c\sigma_T\int \underset{\epsilon'^2}{\epsilon_1'^2}\frac{n'd^3p'}{\underset{\epsilon'}{\epsilon_1'}} = c\sigma_T\int \underset{\epsilon'^2}{\epsilon_1'^2}\frac{n\,d^3p}{\underset{\epsilon'}{\epsilon_1}}$$

$$= c\sigma_T\int[\gamma(1-\beta\cos\theta)\underset{\epsilon}{\epsilon_1}]^2\frac{n\,d^3p}{\underset{\epsilon}{\epsilon_1}}$$

$$= c\sigma_T\gamma^2\int(1-\beta\cos\theta)^2\epsilon\,n\,d^3p$$

Suppose $\underbrace{\gamma^2-1}_{\text{光子获得能量}} \gg \underbrace{\frac{\epsilon}{mc^2}}_{\text{K系电子损失能量}}$

即在 K' 系的能量交换 ≪ K 系的能量交换

即能量交换主要通过 Boost 产生

i.e. $\epsilon_1' = \epsilon'$

also $\frac{dE_1'}{dt'} = \frac{dE_1}{dt}$ （γ factor 消掉了）

For isotropic photon distribution $\langle(1-\beta\cos\theta)^2\rangle = 1+\frac{1}{3}\beta^2$, $\langle\cos\theta\rangle = 0$, $\langle\cos^2\theta\rangle = \frac{1}{3}$ （这是对角度作平均）

defines photon energy density (initial) $\quad u_{ph} = \int\epsilon\,n\,d^3p$

$$\frac{dE_1}{dt}=\frac{dE_1'}{dt'} = c\sigma_T\gamma^2\left(1+\frac{1}{3}\beta^2\right)u_{ph}$$

The rate of decrease of initial photon energy in K

$$\frac{dE_1}{dt} = -c\sigma_T\int \epsilon n\, d^3p = -c\sigma_T u_{ph}$$

逆的 Compton 散射的功率

$$P_{Compton} = \frac{dE_{rad}}{dt} = c\sigma_T u_{ph}\left[\overset{\text{总散射, 获得}}{\gamma^2\left(1+\frac{1}{3}\beta^2\right)} \underset{\text{原入射, 丢失}}{-1}\right]$$

$\downarrow\ \gamma^2-1 = \beta^2\gamma^2$

$$= \frac{4}{3}c\sigma_T u_{ph}\gamma^2\beta^2$$

注: 这里假设了 Energy Transfer in K' is neglected.

如果不忽略, 则为

$$P_{Compton} = \frac{4}{3}c\sigma_T u_{ph}\gamma^2\beta^2\left(1-\frac{63}{10}\frac{\gamma\langle\epsilon^2\rangle}{mc^2\langle\epsilon\rangle}\right)$$

光子能量

表示K'下光子的能量损失 $\sim \frac{\langle\epsilon\rangle}{mc^2}$ 为修正

1阶修正

如果电子的速度有分布

- Power law distribution of electron γ (UltraRelativistic)

$N(\gamma)d\gamma$ = electron number density in $[\gamma, \gamma+d\gamma]$

$$P_{tot} = \int P_{Compton}(\gamma)\, N(\gamma)\, d\gamma \qquad \text{(total power per volume)}$$

if $N(\gamma) = \begin{cases} C\gamma^{-p}, & \gamma_{min}<\gamma<\gamma_{max} \\ 0, & \text{otherwise}\end{cases}$

$$P_{tot} \xrightarrow{\beta\simeq 1} \cong \int \gamma^{2-p}\, d\gamma \propto \gamma^{3-p}\Big|_{\gamma_{min}}^{\gamma_{max}} = \frac{4}{3}\sigma_T c u_{ph} C(3-p)\left(\gamma_{max}^{3-p} - \gamma_{min}^{3-p}\right)$$

- Thermal distribution (of non-rel electrons)

$\gamma \simeq 1$, $\langle\beta^2\rangle = \langle v^2\rangle/c^2 = 3kT/mc^2$

$$P_{tot} = n_e\left\langle P_{Compton}\right\rangle = n_e\frac{4}{3}\sigma_T c\frac{3kT}{mc^2}u_{ph} = \left(\frac{4kT}{mc^2}\right)c\sigma_T n_e u_{ph}$$

Energy Transfer for repeated Scatterings in a finite thermal medium

CMB的谱不是完美的黑体谱

其 spectral distortion 是由高温 ($>10^4$ K) 的电子的逆Compton 散射获得能量造成

由 Y parameter 描述

define the y parameter

y parameter = whether a photon will significantly change its energy during repeated scattering

$$= \left(\begin{array}{c}\text{average fractional}\\ \text{energy change per scattering}\end{array}\right)\times\left(\begin{array}{c}\text{mean number}\\ \text{of scatterings}\end{array}\right)$$

If $y \gtrsim 1$, total energy of photon and spectrum will be significantly changed

$y \ll 1$, Not much changed

In non-rel ($\gamma\epsilon \ll mc^2$)

在K'系下光子能量变化不显著

In K' $\quad \epsilon_1' = \dfrac{\epsilon'}{1+\frac{\epsilon'}{mc^2}(1-\cos\Theta)} \simeq \epsilon'\left(1-\frac{\epsilon'}{mc^2}(1-\cos\Theta)\right)$

$$\frac{\Delta\epsilon'}{\epsilon'} = \frac{\epsilon_1'-\epsilon'}{\epsilon'} = -\frac{\epsilon'}{mc^2}(1-\cos\Theta)$$

K'系下的散射角

后面假定散射是各向同性的

average over angle $\quad \dfrac{\Delta\epsilon'}{\epsilon'} = -\dfrac{\epsilon'}{mc^2}$ (in K')

In lab frame

$$\frac{\Delta\epsilon}{\epsilon} = \underbrace{-\frac{\epsilon}{mc^2}}_{\text{损失的能量}} + \alpha\frac{kT}{mc^2}$$

和电子的动能有关

we guess, at least to the lowest order

光子获得能量 不变

因为原子与电子的相互作用, 若光子的分布

$$N(\epsilon) = K\epsilon^2 e^{-\epsilon/kT} \quad \text{(Non-degenerate ultra-rel particles)}$$

Planck 分布

相当于化学势为0的 Bose 分布

$\langle\Delta\epsilon\rangle = 0$ must be held

计算 $\quad \langle\epsilon\rangle = \int\epsilon\frac{dN}{d\epsilon}d\epsilon \Big/ \int\frac{dN}{d\epsilon}d\epsilon = 3kT$

$$\langle\epsilon^2\rangle = \int\epsilon^2\frac{dN}{d\epsilon}d\epsilon \Big/ \int\frac{dN}{d\epsilon}d\epsilon = 12(kT)^2$$

$$\langle\Delta\epsilon\rangle = -\frac{\langle\epsilon^2\rangle}{mc^2} + \alpha\frac{kT}{mc^2}\langle\epsilon\rangle = \left(\frac{kT}{mc}\right)^2(-12+3\alpha) \Rightarrow \alpha = 4$$

所以对非相对论性的电子

electrons in thermal equilibrium

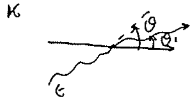
$$(\Delta\epsilon)_{NR} = \frac{\epsilon}{mc^2}(4kT-\epsilon)$$

- High temperature $4kT > \epsilon$, photons gain energy from electrons. (Inverse Compton Scattering)

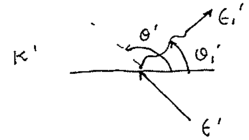
Low temperature $4kT < \epsilon$, photon lose energy (出射散射, 光子损失能量)

In ultra-relativistic limit ($\gamma \gg 1$) (but $\gamma\epsilon \ll mc^2$)

K



K'



$$\epsilon' = \epsilon\gamma(1-\beta\cos\theta)$$

$$\epsilon_1 = \epsilon_1'\gamma(1+\beta\cos\theta_1')$$

in K' $\epsilon' \simeq \epsilon_1'$ (because $\gamma\epsilon \ll mc^2$)

θ_1'各向同性的散射 also $\langle\ \rangle$后为0

$$\epsilon_1 = \epsilon\gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta_1') = \epsilon\gamma^2(1-\beta\cos\theta+\beta\cos\theta_1'-\beta^2\cos\theta\cos\theta_1')$$

对于光子各向同性的 $\langle\ \rangle$后为0

$\langle\ \rangle$后为 $-1/3$
$\simeq -\langle\cos^2\theta\rangle$
$\theta_1' \simeq 180° - \theta$?

$$(\Delta\epsilon)_R = \epsilon_1 - \epsilon = \epsilon\left(\gamma^2(1+\tfrac{1}{3}\beta^2) - 1\right)$$

$\gamma^2 - 1 = \gamma^2\beta^2$

$$= \frac{4}{3}\epsilon\gamma^2\beta^2$$

$$\overset{\beta=1}{\simeq} \frac{4}{3}\epsilon\gamma^2$$

$$\langle\gamma^2\rangle = \frac{\langle E^2\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2,$$

$$(\Delta\epsilon)_R = 16\epsilon\left(\frac{kT}{mc^2}\right)^2$$

有散射时，不能用直线情况下的光的传播方程
(辐射转移)

结论是

mean number of scattering $\simeq \text{Max}(\tau_{es}, \tau_{es}^2)$

主要为 τ_{es}^2
光薄时 τ_{es}

, τ_{es}是电子散射光深
es $\simeq$ electron scattering

$$\tau_{es} \simeq \underbrace{\rho\,\kappa_{es}}_{[cm^{-1}]}\,R$$

κ_{es} = electron scattering opacity $= \sigma_T/m_p = 0.40\,cm^2\cdot g^{-1}$

R = size of finite medium

$$(y\text{ parameter})_{NR} \cong \frac{4kT}{mc^2}\text{Max}(\tau_{es}, \tau_{es}^2) \qquad (\text{if } 4kT \gg \epsilon)$$

$$(y\text{ parameter})_{\text{Relativistic}} \cong 16\left(\frac{kT}{mc^2}\right)^2\text{Max}(\tau_{es}, \tau_{es}^2)$$

要研究相空间分布 $n(p)$ 的演变，需要从第一性原理出发(考虑Boltzman方程)
(或 $n(\omega)$)

Kompaneets Equation $\sim$ Boltzmann方程的具体应用

(Repeated Scatterings by non-relativistic electrons)

let $n(\omega)$ = photon phase space density
↑ assume isotropic
$n(\vec{p})$

Reaction Equation for scattering $\gamma + e \rightleftharpoons \gamma + e$
$\omega\ \vec{p} \quad \omega_1\ \vec{p}_1$

let $f_e(\vec{p})$ = phase space density of electrons

Boltzmann equation for $n(\omega)$ due to scatterings

Similar to that in Saha equation
"+" because Bose-Einstein enhancement

$$\frac{\partial n(\omega)}{\partial t} = c\int d^3p\int\frac{d\sigma}{d\Omega}d\Omega\left[f_e(\vec{p}_1)n(\omega_1)(1+n(\omega)) - f_e(\vec{p})n(\omega)(1+n(\omega_1))\right]$$

electron photon scattering cross-sec

Scattering into ω by photons of ω_1

为什么没有电子的 Pauli exclusion?

对于角动量守恒，满足守恒的结果，所以只有 $\int d^3p$ 有积 $\int d\Omega$

Boltzmann Eq. can be expanded to 2nd order $\Rightarrow$ Fokker-Planck eq.

non relativistic thermal distribution of electrons

For non-relativistic electron in thermal equilibrium

Kompaneets equation (1957)

$$f_e(E) = n_e(2\pi mkT)^{-3/2}e^{-E/kT}$$

$E = \frac{p^2}{2m}$, n_e是电子的numberdensity
(electron number per unit volume)

在Maxwell分布里是
$e^{-\frac{1}{2}\frac{mv^2}{kT}}$
现在在是更一般的Boltzmann分布

Define energy transfer of electrons

$$\Delta = \frac{\hbar(\omega_1-\omega)}{kT} \ll 1$$

非相对论性电子在K下不会显著改变ω
这是一个小量，可以用来展开Boltzmann eq.

$$n(\omega_1) = n(\omega) + (\omega_1-\omega)\frac{\partial n(\omega)}{\partial\omega} + \frac{1}{2}(\omega_1-\omega)^2\frac{\partial^2 n(\omega)}{\partial\omega^2} + \cdots \qquad (*1)$$

$$f_e(E_1) = f_e(E) + (E_1-E)\frac{\partial f_e}{\partial E} + \frac{1}{2}(E_1-E)^2\frac{\partial^2 f_e}{\partial E^2} + \cdots \qquad (*2)$$

Note: $f_e(E)$ 不用 $e^{-E/kT}$ 形式 仅适合 $E \gg kT$

For electron，由能量守恒 $E_1 - E = -(\omega_1-\omega)\hbar$，又 $f_e(E) \propto e^{-E/kT}$，$\frac{\partial f_e}{\partial E} = -\frac{1}{kT}f_e$

所以将 $\frac{\partial f_e}{\partial E}$ 代入 (*2) 有

$$(E_1-E)\frac{\partial f_e}{\partial E} = -f_e(E)(-1)\frac{\hbar(\omega_1-\omega)}{kT} = f_e(E)\Delta$$