Homework 1st

Radiative Process in Astrophysics

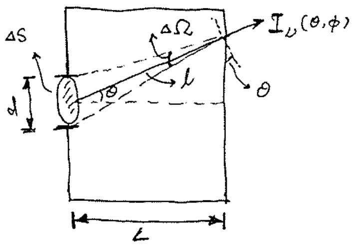
陈洋遥
2017311337
物研172班，物理系天体中心

5/5

1.1. A "pinhole camera"

The energy flux at "film plane" can be given by

$$F_\nu(\theta,\phi) = \frac{dE}{dA\,dt\,d\nu} = I_\nu(\theta,\phi)\cos\theta\,\Delta\Omega \qquad (*1)$$



where the solid angle $\Delta\Omega$ spaned by the pinhole with respect to the "film plane" is

$$\Delta\Omega = \frac{\Delta S\cos\theta}{l^2} \qquad (*2)$$

Use geometry relation, it is easy to show

$$\Delta S = \pi\left(\frac{d}{2}\right)^2 \qquad (*3)$$

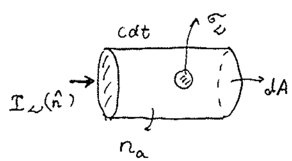
$$l = \frac{L}{\cos\theta} \qquad (*4)$$

Combine (*1)(*2)(*3)(*4), we can get

$$F_\nu(\theta,\phi) = I_\nu(\theta,\phi)\cos^4\theta\cdot\frac{\pi}{4(L/d)^2}$$

1.2. Photoionization

From the definition of $I_\nu(\hat{n})$, we know the energy transfering per unit frequency per unit solid angle is



$$\frac{dE}{d\nu\,d\Omega} = I_\nu(\hat{n})\,dA\,dt \qquad (*1)$$

So that, the number of photon then can cause ionization is

$$\frac{dN_\gamma}{d\nu\,d\Omega} = \frac{dE/(d\nu\,d\Omega)}{h\nu} \qquad (*2)$$

The fraction of photon that will be absorbed is

$$frac = \frac{d\sigma_\nu}{dA} = \frac{dV\cdot n_a\cdot\sigma_\nu}{dA} \qquad (*3)$$

So the number of photo-ionization per unit volume per unit time is

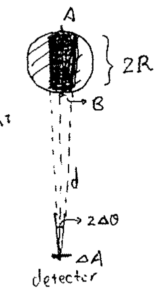
$$res = \int_{\nu_0}^{\infty} d\nu \int_{all} d\Omega\,\frac{dN_\gamma}{d\nu\,d\Omega}\cdot frac\cdot\frac{1}{dt\cdot dV} \qquad (*4)$$

Combine (*1 ~ *4), also use the fact $4\pi J_\nu = \int_{all} I_\nu(\hat{n})\,d\Omega$, and $u_\nu = \frac{4\pi J_\nu}{c}$, we obtain

$$res = 4\pi n_a \int_{\nu_0}^{\infty}\frac{J_\nu\sigma_\nu}{h\nu}\,d\nu = c\,n_a\int_{\nu_0}^{\infty}\frac{\sigma_\nu u_\nu}{h\nu}\,d\nu$$

1.3 X-ray clouds

a. When the source is completely resolved, only the light generated by the black region can be observed, so similar to the radiation transfer, the emission coefficient can be define as



$$j = \frac{dN}{dt\,dV\,d\Omega} = \frac{\Gamma}{4\pi} \qquad (*1)$$

Use radiation transfer function, we have

$$\frac{dI}{ds} = j \qquad (*2)$$

Integrate (*2) from A to B, we have

$$I = j\cdot 2R = \frac{\Gamma R}{2\pi}$$

b. When the source is completely unresolved, it can be viewed as a source-point. The particle it emit at dt is

$$dN = \Gamma\cdot dt\cdot\frac{4}{3}\pi R^3 \qquad (*3)$$

The received number is

$$dN_{rec} = dN\cdot\frac{\Delta A}{4\pi d^2} \qquad (*4)$$

So the Intensity is

$$I = \frac{dN_{rec}}{dt\cdot\Delta A}\,\frac{1}{\pi\Delta\theta^2} \qquad (*5)$$

Combine (*3 ~ *5) we obtain

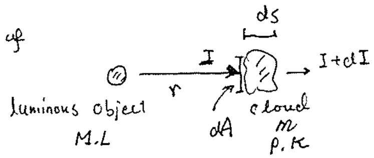
$$I = \frac{\Gamma R^3}{3\pi d^2\Delta\theta^2}$$

2/2

1.4 Radiation Pressure by a Nearby Object.

Reference : Solutions in the textbook. 1·3

a. For the cloud, after the absorption, the change of intensity is



$$dI = -\alpha \, ds \cdot I$$

The Pressure is

$$\text{Pressure} = \dot{P} \;(\text{momentum change}) = \int_{all} \frac{|dI|}{c} \cos^2\theta \, d\Omega = \int_{all} \frac{\alpha \, ds \cdot I}{c} d\Omega = \frac{\alpha \, ds}{c} F \quad (*1)$$

($\theta \approx 0$)

where $F = \frac{L}{4\pi r^2}$.

The Condition for ejection is

$$\frac{G \cdot ds \, dA \cdot \rho \cdot M}{r^2} < \text{Pressure} \cdot dA \quad (*2)$$

Combine (*1) (*2) and $\kappa = \frac{\alpha}{\rho}$ we obtain

$$\frac{M}{L} < \frac{\kappa}{4\pi c G}$$ ✓

b. From a we can calculate the force acting on the cloud

$$f = \text{Pressure} \times dA - \frac{G \cdot ds \, dA \, \rho \cdot M}{r^2}$$

$$= \frac{GMm}{r^2} \left(\frac{\kappa L}{4\pi c G M} - 1 \right)$$

where m is mass of the cloud. We can re-write the force f as the gradient of a potential function. That is

$$f = -\frac{d\phi(r)}{dr}$$

where the potential $\phi(r) = \frac{GMm}{r}\left(\frac{\kappa L}{4\pi c G M} - 1\right)$. Now use energy conservation we can calculate the speed v at $r = \infty$ by

$$\phi(R) = \frac{1}{2} m v^2$$

This immediately gives $v = \frac{2GM}{R}\left(\frac{\kappa L}{4\pi c G M} - 1\right)$. ✓

c. This require $L < \frac{4\pi c G M}{\kappa}$, when the κ achieve minimum $\kappa_{min} = \frac{\sigma_T}{m_H}$, the L can have maximum value

$$L_{max} = \frac{4\pi c G M}{\kappa_{min}} = \frac{4\pi c G M \cdot m_H}{\sigma_T}$$ ✓

Substitute it with $c = 3.0 \times 10^8$ m/s, $m_H = 1.674 \times 10^{-27}$, $M = 1.989 \times 10^{30}$ kg, $G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$.

we obtain $L_{max} = 1.259 \times 10^{38}$ erg·s⁻¹. $M/M_\odot$

Radiative Process In Astrophysics.

Hm 2nd, week 3rd

陈洋遥

2017311337

7/7

1.7 Einstein Coefficients 2/2

a. From the relation between Einstein Coefficients we know (text book 1.72a~b)

$$g_1 B_{12} = g_2 A_{21} \frac{c^2}{2h\nu^3} \qquad (*1)$$

Now use the 'equilibrium condition', we also have (text book 1.71, where B_{21} is ignored)

$$\bar{J} = B_\nu = \frac{A_{21}}{\frac{g_1 B_{12}}{g_2} e^{\frac{h\nu}{kT}}} \qquad (*2)$$

Combine (*1)(*2) we immediately obtain Wien's law

$$\bar{J} = B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT}}$$

b. The Fermions forbid two particle at same energy state, so the input neutrino will weaken the spontaneous emission, cause a 'negative' stimulated emisson. Similar to text book 1.71:

$$\bar{J} = \frac{A_{21}/B_{21}}{\frac{g_1 B_{12}}{g_2 B_{21}} e^{\frac{h\nu}{kT}} - 1} \qquad (*3)$$

But now we have $B_{21} < 0$. Compare with the radiation law of Fermion

$$\bar{J} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} + 1} \qquad (*4)$$

We realize the relation between A and Bs by Combine (*3) and (*4)

$$g_1 B_{12} = -g_2 B_{21}$$

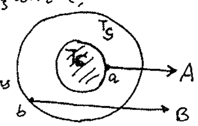
$$A_{21} = -\frac{2h\nu^3}{c^2} B_{21} \qquad (B_{21} < 0)$$

✓ Ok. but write $B_{21} \to -B_{21}$.

1.9 A spherical object surrounded by a shell. 1/1

For Ray 'A', it starts at 'a' because this object is black, so it starts with T_c;

For Ray 'B', it starts at 'b', so it starts with $T = 0$;



For simplicity, we use the Radiation trasfer function at Rayleigh-Jeans scale, so $\frac{dT}{d\tau_\nu} = -T + T_s$, and

$$T = T_{(0)} e^{-\tau_\nu} + T_s (1 - e^{-\tau_\nu})$$

For Ray A, at ν_0, $\tau_\nu \to \infty$, $T = T_s$

at ν_1, $\tau_\nu \to 0$, $T = T_c$

For Ray B, at ν_0, $\tau_\nu \to \infty$, $T = T_s$

at ν_1, $\tau_\nu \to 0$, $T = 0$

So we have the Table at right side;

T	ν_0	ν_1
A	T_s	T_c
B	T_s	0

a. When $T_s < T_c$, along A, ν_1 is brighter;

along B, ν_0 is brighter.

b. when $T_s > T_c$, along A or B, both ν_0 are brighter. ✓

2.1 Average of oscillated field 1/1

Let $A(t) = Re(\mathcal{A} e^{-i\omega t})$, $B(t) = Re(\mathcal{B} e^{-i\omega t})$, $Re\mathcal{A} = R_1$, $Im\mathcal{A} = I_1$, $Re\mathcal{B} = R_2$, $Im\mathcal{B} = I_2$, $T = \frac{2\pi}{\omega}$.

So we can write the average in time period:

$$\langle A(t)B(t)\rangle_T = \frac{1}{T}\int_0^T dt \cdot Re(A(t)) \cdot Re(B(t))$$

$$= \frac{1}{T}\int_0^T dt \cdot (R_1 \cos\omega t + I_1 \sin\omega t)(R_2 \cos\omega t + I_2 \sin\omega t)$$

$$= \frac{1}{T}\int_0^T dt \cdot \left(\frac{1}{2}R_1R_2 + \frac{1}{2}I_1I_2\right) \qquad \uparrow \int_0^T \cos 2\omega t \cdot dt = \int_0^T \sin 2\omega t \cdot dt = 0$$

$$= \frac{1}{2}(R_1R_2 + I_1I_2) = Re\left[\frac{1}{2}(R_1 - iI_1)(R_2 + iI_2)\right] = Re\left(\frac{1}{2}\mathcal{A}^*\mathcal{B}\right)$$

$$= Re\left[\frac{1}{2}(R_1 + iI_1)(R_2 - iI_2)\right] = Re\left(\frac{1}{2}\mathcal{A}\mathcal{B}^*\right)$$ ✓

2/2

2.2 Electromagnetic field in Conducting medium.

In conducting medium, we have $-\frac{\partial\rho}{\partial t} = \nabla\cdot\vec{j} = \nabla\cdot(\sigma\vec{E}) = (\nabla\cdot\vec{E})\sigma = \frac{4\pi\rho}{\varepsilon}\sigma$

(↑ charge conservation; ↑ 1st Maxwell eq.)

Solve this we have $\rho = e^{-\frac{4\pi\sigma}{\varepsilon}t} \to 0$ (t足够大).

So in the conducting medium we can let $\rho \to 0$, and the Maxwell equations now become:

$$\begin{cases} \nabla\cdot\vec{E} = 0 & \nabla\cdot\vec{H} = 0 \\ \nabla\times\vec{E} = -\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t} & \nabla\times\vec{H} = \frac{4\pi\vec{j}}{c} + \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t} \end{cases}$$

By taking derivative '$\nabla\times$' of $\nabla\times\vec{E}$ and $\nabla\times\vec{H}$, we can obtain

$$\frac{\partial^2\vec{F}}{\partial t^2} + \frac{4\pi\sigma}{\varepsilon}\frac{\partial\vec{F}}{\partial t} - \frac{c^2}{\mu\varepsilon}\nabla^2\vec{F} = 0 \qquad (*1)$$

where $\vec{F}$ can either be $\vec{H}$ or $\vec{E}$. Solve this equation by assuming the solution has form

$$\vec{F} = \tilde{F}\hat{a}\, e^{i(\tilde{k}\hat{n}\cdot\vec{x} - \omega t)} \qquad (*2)$$

Combine (*1) and (*2) give

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\mu\varepsilon\left(1 + \frac{4\pi\sigma i}{\omega\varepsilon}\right) \qquad (*3)$$ ✓

a. From eq (*3) we immediately know $m^2 = \mu\varepsilon(1 + \frac{4\pi\sigma i}{\omega\varepsilon})$ if $\tilde{k}^2 = \frac{\omega^2}{c^2}m^2$.

b. The flux of energy $\langle S\rangle = \frac{c}{4\pi}\cdot\frac{1}{2}\langle E_0^* H_0\rangle = \frac{c}{8\pi}\left\langle \tilde{E}^* e^{-(Im\tilde{k})\hat{n}\cdot\vec{x}}\, \tilde{H} e^{-(Im\tilde{k})\hat{n}\cdot\vec{x}}\right\rangle = e^{-2Im\tilde{k}\hat{n}\cdot\vec{x}}\langle S\rangle_0$

By definition of α_ν, we should have $\langle S \rangle = e^{-\alpha_\nu \hat{n}\cdot\vec{x}} \langle S \rangle_0$. Compared with above eq we obtain

$$\alpha_\nu = 2\,\mathrm{Im}(\tilde{k}) = 2\,\frac{\omega}{c}\,\mathrm{Im}(n).$$

2.4 Displacement Current

If there is no $\frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ term, the Maxwell's equation of $\nabla\times\vec{H}$ becomes

$$\nabla\times\vec{H} = \frac{4\pi}{c}\vec{j} \qquad (*1)$$

And combine $\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$, $\nabla\cdot(\nabla\times\vec{H}) = 0$, we have

$$\begin{cases} \nabla\cdot\vec{j} = 0 \\ \frac{\partial\rho}{\partial t} = 0 \end{cases}$$

This seems only to allow "static field", which is contradictory with our experience.

Also, by taking '$\nabla\times$' of $(*1)$, we have

$$\nabla\times(\nabla\times\vec{H}) = \frac{4\pi}{c}\nabla\times\vec{j}$$

In linear medium, where $\vec{H} = \vec{B}/\mu$, $\vec{j} = 0$, where we have

$$\nabla^2\vec{H} = 0$$

This is not a "wave equation", so the "wave solution" does not exist in this case.

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hm3 week4

陈洋遥

2017311337

6.5/7

3.1 1.5/2

Pulsar.

(a) The relation between magnetic field and magnetic dipole m is

$$B_0 = \frac{2m}{R^3} \qquad (*1)$$

不是很理解这里的B_0是什么意思

While for a rotated pulsar, the change of dipole is

$$|\ddot{\vec{m}}| = |\vec{\omega}\times(\vec{\omega}\times\vec{m})| = \omega^2 \sin\alpha \cdot m \qquad (*2)$$

For the radiation, analog to that of electric dipole, we have

$$\frac{dW}{dt} = \frac{2|\ddot{\vec{m}}|^2}{3c^3} \qquad (*3)$$

Combine (*1 ~ 3) we obtain

$$\frac{dW}{dt} = \omega^4 B_0^2 R^6 \sin^2\alpha / 6c^3 \qquad (*4)$$

✓

(b) The inertia of a sphere is $I = \frac{2}{5}MR^2$, So the rotation energy is

$$E = \frac{1}{2} I \omega^2 \qquad (*5)$$

By definition, the Power

$$\frac{dW}{dt} = -\frac{dE}{dt} \qquad (*6)$$

The slow down time-scale thus is obtained by combination of (*4 ~ 6):

$$\tau = \frac{\omega}{-\dot{\omega}} = \frac{12c^3 M}{5\omega^2 R^4 B_0^2 \sin^2\alpha}$$

✓

(c) Substitute these values into τ above we obtain

$$\tau(\omega = 10^4) \doteq 41.1 \text{ yr}$$

$$\tau(\omega = 10^3) \doteq 4110 \text{ yr}$$

$$\tau(\omega = 10^2) \doteq 411000 \text{ yr}$$

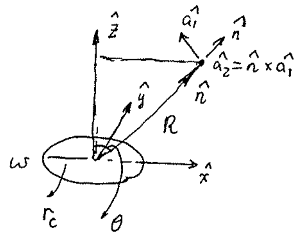
P = ?

✓

4/4

3.2

Radiation from non-relativistic circular motion.



(a) The Circular motion, radius r_c, angular speed ω, can be described by

$$\vec{r} = r_c(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}) \qquad (*1)$$

Electric field at direction $\hat{n}$ thus should be

$$\vec{E} = \frac{-e}{Rc^2}\hat{n}\times(\hat{n}\times\ddot{\vec{r}}) \qquad (*2)$$

where the ortho-normal basis $\hat{n} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$, $\hat{a}_1 = -\cos\theta\,\hat{x} + \sin\theta\,\hat{z}$. $\hat{a}_2 = \hat{n}\times\hat{a}_1$

Combine (*1, 2) we have

$$\vec{E} = \frac{r_c e}{Rc^2}\omega^2(\cos\theta\cos\omega t\,\hat{a}_1 + \sin\omega t\,\hat{a}_2) \qquad (*3)$$

The Power emitted per unit solid angle is

$$\left\langle \frac{dW}{dt\,d\Omega} \right\rangle = \frac{c}{4\pi}\langle \vec{E}\cdot\vec{E}\rangle R^2 = \frac{(1+\cos^2\theta)e^2 r_c^2\omega^4}{8\pi c^3} \qquad (*4)$$

✓

Total Power is

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{2}{3}\frac{e^2 r_c^2 \omega^4}{c^3} \qquad (*5)$$

(b) from (*3) we can see

$$\left(\frac{E_x}{\cos\theta}\right)^2 + \left(\frac{E_y}{1}\right)^2 = \left[\frac{r_c e}{Rc^2}\omega^2\right]^2$$

it is obviously an elliptical equation. So the wave is elliptically polarized. ✓

(c) Since $\left(\frac{dW}{dt}\right) = \frac{2}{3}\frac{e^2 r_c^2 \omega^4}{c^3}$, only the single angular frequency ω is allowed. ✓

(d) For the circular motion in B field, we have

$$\frac{mv^2}{r_c} = e\,v B/c$$

So it is easily to show $\omega_B = \frac{eB}{mc}$. Substitute this into (*5) we obtain

$$P = \frac{2}{3} r_0^2 c \left(\frac{v}{c}\right)^2 B^2$$

✓

where $v_\perp = \omega_B r_c$, $r_0 = \frac{e^2}{mc^2}$

(e) For circular polarized radiation $\vec{E} = \hat{x}E_x\cos\omega t + \hat{y}E_y\sin\omega t$, where $E_x = E_0$ while $E_y = E_0 = E_x$.

In this case the incident flux

$$|\langle \vec{S}\rangle| = \frac{c}{4\pi}\langle \vec{E}\cdot\vec{E}\rangle = \frac{c}{4\pi}E^2 \qquad (*6)$$

The Scatter flux per unit solid angle is (*4), where now $\frac{eE_0}{m} = \omega_B^2 r_c$. Substitute this into (*4), where we find

$$\frac{d\sigma}{dt\,d\Omega} = \frac{(1+\cos^2\theta)}{8\pi c^3}e^2\left(\frac{eE_0}{m}\right)^2 \qquad (*7)$$

From the definition of $d\sigma$, we have

$$\frac{d\sigma}{d\Omega} = \frac{dW}{dt\,d\Omega}\frac{1}{\langle S\rangle} = \frac{1}{2}(1+\cos^2\theta)r_0^2$$

✓

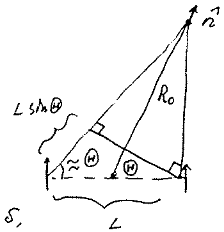
Integral by total solid angle we have total Cross Section

$$\sigma = \int_{all}\frac{d\sigma}{d\Omega}d\Omega = \frac{8\pi}{3}r_0^2$$

✓

3.3 Two dipoles.

(a) From the picture shown in the right, it is easily to show when $R_0 \gg L$, the difference of retarded time is



$$|t^{(1)} - t^{(2)}| = \frac{L\sin\Theta}{c}$$

So the phase difference of two dipoles should be $\frac{L\sin\Theta}{c}\omega = \delta$,

That means $\vec{d}_1(t^{(1)}) = \vec{d}_1 \cos(\omega t^{(1)})$, $\vec{d}_2(t^{(2)}) = \vec{d}_2 \cos[(t^{(1)} + \frac{\delta}{\omega})\omega] = \vec{d}_2 \cos(\omega t^{(1)} + \delta)$

So the total $\vec{E}$ field is

$$\vec{E}(t) = [\ddot{\vec{d}}_1(t^{(1)}) + \ddot{\vec{d}}_2(t^{(2)})] \frac{\sin\Theta}{c^2 R_0}$$

$$= \left(\vec{d}_1 \cos(\omega t^{(1)}) + \vec{d}_2 \cos(\omega t^{(1)} + \delta)\right) \frac{\omega^2 \sin\Theta}{c^2 R_0}$$

The flux per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{4\pi} R_0^2 \left\langle |\vec{E}(t)|^2 \right\rangle = \frac{c}{4\pi} \frac{\omega^4 \sin^2\Theta}{c^4} \left(\frac{1}{2} d_1^2 + \frac{1}{2} d_2^2 + d_1 d_2 \cos\delta\right)$$

$$= \frac{\omega^4 \sin^2\Theta}{8\pi c^3} (d_1^2 + d_2^2 + 2 d_1 d_2 \cos\delta)$$

(b) when $L \ll \lambda$, $\frac{L\sin\Theta}{c}\omega = \frac{L\sin\Theta \cdot 2\pi}{\lambda} \ll 1$, so

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 \sin^2\Theta}{8\pi c^3} (d_1 + d_2)^2$$

which is similar to a single dipole $d_1 + d_2$.

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5/5

4.3 2/2

a. The Transformation of acceleration

We already know the transformation of velocity

$$\begin{cases} \frac{dt'}{dt} = \frac{1}{\gamma\sigma} & (*0) \\ u_x = \frac{1}{\sigma}(v+u_x') & (*1) \\ u_y = \frac{u_y'}{\gamma\sigma} & (*2) \\ u_z = \frac{u_z'}{\gamma\sigma} & (*3) \end{cases}$$

, where $\sigma = 1+\frac{vu_x'}{c^2}$

Take differentiation of (*1) we have

$$a_x = \frac{du_x}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\left[\frac{1}{\sigma}(v+u_x')\right]$$

Use the defintion $\frac{du_x'}{dt'} = a_x'$ and $\frac{d\sigma}{dt'} = \frac{v}{c^2}a_x'$ and (*0) we obtain

$$a_x = \frac{1}{\gamma^3\sigma^3}a_x'$$

Take differentiation of (*2) we have

$$a_y = \frac{du_y}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\frac{u_y'}{\gamma\sigma}$$

Similarly use $\frac{du_y'}{dt'} = a_y'$, $\frac{d\sigma}{dt'} = \frac{v}{c^2}a_x'$ and (*0) we obtain

$$a_y = \frac{a_y'}{\gamma^2\sigma^2} - \frac{1}{\gamma^2\sigma^3}\frac{v}{c^2}a_x' u_y'$$

Similarly we can derive a_z, it is

$$a_z = \frac{a_z'}{\gamma^2\sigma^2} - \frac{1}{\gamma^2\sigma^3}\frac{v}{c^2}\cdot u_z' a_x'$$

✓

b. To be simple, just let x-axis to be the direction of particle moving, and $u_x' = u_y' = u_z' = 0$, $\sigma = 1$

That means

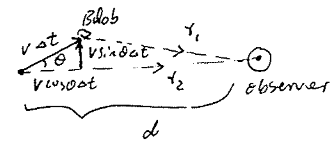
$$\begin{cases} a_x = \frac{a_x'}{\gamma^3} \\ a_y = \frac{a_y'}{\gamma^2} \\ a_z = \frac{a_z'}{\gamma^2} \end{cases} \Rightarrow \begin{cases} a_x' = \gamma^3 a_x \\ a_{y或z}' = \gamma^2 a_{y或z} \end{cases} \Rightarrow \begin{cases} a_\parallel' = \gamma^3 a_\parallel \\ a_\perp' = \gamma^2 a_\perp \end{cases}$$

✓

4.7 Blob's transverse motion 2/2

a. See the right figure, the distance of light-ray r_1 and r_2 are $d_1 = d - v\cos\theta\,\Delta t$

$d_2 = d$



So the time interval of receiving the r_1 and r_2 is

$$t_1 - t_2 = \left(\Delta t + \frac{d_1}{c}\right) - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

That means, the transverse velocity is

$$v_{app} = \frac{v\sin\theta\Delta t}{t_1 - t_2} = \frac{v\sin\theta}{1-\frac{v}{c}\cos\theta}$$

✓

b. It is easy to show, when $v \sim c$, the v_{app} approaches to

$$v_{app} \approx \frac{v\sin\theta}{1-\cos\theta} \approx c\frac{\sin\theta}{1-\cos\theta}$$

if $\sin\theta + \cos\theta = \sqrt{2}\sin(\theta+\phi) > 1$, then $v_{app} \approx c\frac{\sin\theta}{1-\cos\theta} > c$

To find the maximum, we simply take derivative

$$\frac{dv_{app}}{d\theta} = \frac{v\left(\cos\theta - \frac{v}{c}\right)}{\left(1-\frac{v}{c}\cos\theta\right)^2}$$

When $\cos\theta = \frac{v}{c}$, or $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{1}{\gamma}$, the v_{app} obtain maximum

$$v_{app}^{max} = \frac{v\cdot\frac{1}{\gamma}}{1-\frac{v}{c}\cdot\frac{v}{c}} = v\cdot\frac{1}{\gamma}\gamma^2 = v\gamma$$

✓

4.14 1/1

The force $F_\perp$ and $F_\parallel$ should NOT be the "force" in the particle's rest frame K', so from $F^\mu = \frac{dU^\mu}{d\tau}m_0 = \frac{m_0}{\gamma}\frac{du^\mu}{dt}$

$$a_\parallel' = \frac{F_\parallel'}{m} = \frac{F_\parallel}{m},\quad a_\perp' = \frac{F_\perp'}{m} = \frac{\gamma F_\perp}{m}$$

From textbook eq (4.92), it is easy to show the power

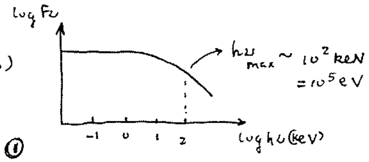
$$P = \frac{2e^2}{3c^3}\left(a_\parallel'^2 + a_\perp'^2\right) = \frac{2e^2}{3c^3m^2}\left(F_\parallel^2 + \gamma^2F_\perp^2\right)$$

✓

陈泽道 天体物理中的辐射机制. 04/09/2018 2/3

Ex 5.2

For the free-free emission, we have formula from (5.15b)



$$\varepsilon^{ff} = \frac{dW}{dt\,dV} = 1.4\times10^{-27}\,T^{1/2}\,n_e n_i Z^2 \bar{g}_B \quad \cdots ①$$

In this case the number density $n_e = n_i = \frac{\rho}{m_H}$, $\bar{g}_B \simeq 1.2$.

Besides, the critical temperature T decide the frequency ν_{max} above which the radiation decrease sharply, that is

$$h\nu_{max} = kT \quad \cdots ②$$

In this case, from the figure, we guess $h\nu_{max} \simeq 10^5\,eV$, and use $1\,eV \sim 10^4\,K$ we have $T \simeq 10^9 K$.

The radiation received by the observer F, should be due to the free-free emission, that is

$$\varepsilon^{ff} = \frac{dW}{dt\,dV} = \frac{F\cdot 4\pi L^2}{\frac{4}{3}\pi R^3} \quad \cdots ③$$

Again the hydrostatic state requies the potential energy $V \sim$ kenetic energy T, that is

$$\frac{3}{2}kT \times 2_{\text{electron + proton}} = \frac{GMm_H}{R} \quad \cdots ④$$

Combine ②④, we obtain

$$R \simeq 5\times10^8\left(\frac{M}{M_\odot}\right)\,cm \quad \cdots ⑤$$

✓

Combine ①②③⑤, we have

$$\rho \simeq 4\times10^{-26} L F^{1/2}\left(\frac{M}{M_\odot}\right)^{-3/2}$$

For $F = 10^{-8}\,erg\cdot cm^{-2}\cdot s^{-1}$, $L = 10\,pc$, we have $\rho \simeq 1.2\times10^{-7}\,g\cdot cm^{-3}\left(\frac{M}{M_\odot}\right)^{-3/2}$ ✓

For the absorption of free-free process and scattering, let their mass-absorption-coeffs to be κ^{ff} and κ^{es}, the κ^{ff} can be evaluated by $\frac{\varepsilon^{ff}}{\rho}$, while κ^{es} can be evaluated by ... (? I don't know, just see the answer ...), so

$$\frac{\kappa^{ff}}{\kappa^{es}} \simeq 10^{-15}\left(\frac{M}{M_\odot}\right)^{-3/2}$$

If $M/M_\odot \gtrsim 10^{-10}$, $\kappa^{ff}/\kappa^{es} \lesssim 1$, in this case

$$\kappa_* \simeq \sqrt{\kappa^{ff}\kappa^{es}} \simeq 10^{-8}\left(\frac{M}{M_\odot}\right)^{-3/4}\,cm^2\cdot g^{-1}$$

↖ (why?)

The optical depth

$$\tau \simeq \kappa_*\rho\Delta R \simeq \kappa_*\rho R \simeq 6\times10^{-7}\left(\frac{M}{M_\odot}\right)^{-5/4}$$

For $\frac{M}{M_\odot} \gtrsim 10^{-4}$ (10^{-5}), $\tau \lesssim 6\times10^{-2} \ll 1$, In this case the optical-thin condition is satisfied.

Hm Chapter 6 & 7

陳泽道
物研 172
2017311337

(415)

6.1 Energy loss of Synchrotron Radiation

2/2

The Emiting Power of Synchrotro radiation is $P = \frac{2}{3} r_0^2 c \gamma^2 B^2 \cdot \sin^2\alpha \frac{v^2}{c^2}$

Here r_0 is the classical radius of electron, $r_0 = \frac{e^2}{mc^2}$, $\gamma^2 = \frac{1}{1-v^2/c^2}$. The energy change rate is

$$-\frac{d}{dt}(\gamma mc^2) = P \qquad \langle 1 \rangle$$

Simplifying of eq <1> gives

$$A\,dt = \frac{d\gamma}{1-\gamma^2}, \quad \text{where } A = \frac{2}{3}\frac{e^4 B_\perp^2}{m^3 c^5}, \quad B_\perp = B\sin\alpha$$

For $\gamma \gg 1$, $1-\gamma^2 \simeq -\gamma^2$, we have integral

$$\gamma = \gamma_0 (1 + A\gamma_0 t)^{-1}. \qquad \langle 2 \rangle$$

✓

When electron lose half of its energy, $\gamma/\gamma_0 = 1/2$. From e.q. <2> it is easy to show

$$t_{1/2} = (A\gamma_0)^{-1} = \frac{5.1\times10^8}{B_\perp^2}\gamma_0^{-1}.$$

✓

Eqs (6.1) don't consider the radiation field (the emitted B and E fields) of electron. If taken this into consideration. the decrease of γ will be reproduced naturally.

7.1 Inverse Compton Scattering of Hot Opaque gas

2/3

a. For the case $\tau_{es} \gg 1$, we may know $N_{scat} = \tau_{es}^2$ (eq 1.90a)

We also know the energy gain from each single Compton Scattering for photon with energy ϵ is

$$\Delta\epsilon = \epsilon\frac{4kT - \epsilon}{mc^2} \qquad \text{(eq. 7.36)}$$

For the case $\epsilon_i \ll 4kT$, $\Delta\epsilon \simeq \epsilon\frac{4kT}{mc^2}$. That means after one scattering, the energy become $\epsilon_1 = \epsilon_i\left(\frac{4kT}{mc^2}+1\right)$, after two scattering $\epsilon_2 = \epsilon_i\left(\frac{4kT}{mc^2}+1\right)^2$, ..., after N_{scat} Scattering

$$\epsilon_f = \epsilon_i\left(\frac{4kT}{mc^2}+1\right)^{N_{scat}} \underset{4kT \ll mc^2}{=} \epsilon_i\left(1 + N_{scat}\frac{4kT}{mc^2}\right)$$

$$= \epsilon_i \exp\left(\frac{4kT}{mc^2}\tau_{es}^2\right) \simeq \epsilon_i\left(1 + \frac{4kT}{mc^2}\tau_{es}^2\right) \qquad \langle 3 \rangle$$

b. If $\epsilon_f = 4kT$, then $\Delta E \simeq 0$. the Compton process should be less efficient.

Set $\epsilon_f = 4kT$ in eq. <3>, we have

$$\tau_{crit} = \tau_{es} = \left[\frac{mc^2}{4kT}\left(\frac{4kT}{\epsilon_i} - 1\right)\right]^{1/2}$$

✓

c. For fixed medium (fixed T. m. τ_{es}), Since $\epsilon_f = \epsilon_i\left(1 + \frac{4kT}{mc^2}\tau_{es}^2\right)$. the parameter should be

$$P = \frac{4kT}{mc^2}\tau_{es}^2.$$

✓

Chapter 7 and Chapter 8

张泽逵 2017311337
物研172

(5/5)

7.4 Derive Eqs (7.53) to (7.55) for the Kompaneets equation.

3/3

① Derive (7.53) $\Delta = \frac{x\vec{p}\cdot(\vec{n}_1-\vec{n})}{mc} + O\left(\frac{kT}{mc^2}\right)$

Suppose the four-momentum before scattering for electron is $P = \left(\frac{E}{c}, \vec{p}\right)$, for photon is $P_\gamma = \left(\frac{\epsilon}{c}, \frac{\epsilon}{c}\vec{n}\right)$. After scattering, the four-momentum for electron is $P_1 = \left(\frac{E_1}{c}, \vec{p}_1\right)$. for photon is $P_{\gamma_1} = \left(\frac{\epsilon_1}{c}, \frac{\epsilon_1}{c}\vec{n}_1\right)$. Use the Conservation of four-momentum, we obtain

$$P + P_\gamma = P_1 + P_{\gamma_1}$$

$$\Leftrightarrow \quad P_1 = P + P_\gamma - P_{\gamma_1}$$

The square of above equation is

$$P_1^2 = (P + P_\gamma - P_{\gamma_1})^2$$

Use $P_1^2 = P^2$, $P_\gamma^2 = P_{\gamma_1}^2 = 0$, we Obtain

$$PP_\gamma - PP_{\gamma_1} - P_\gamma P_{\gamma_1} = 0$$

Substiture definition of $P, P_\gamma, P_{\gamma_1}$ into above, it is easy to show

$$\frac{\epsilon}{c}\vec{p}\cdot\vec{n} - \frac{\epsilon_1}{c}\vec{p}\cdot\vec{n}_1 = \frac{\epsilon}{c}\frac{E}{c} - \frac{\epsilon_1}{c}\frac{E}{c} + \frac{\epsilon\epsilon_1}{c\,c}(\vec{n}\cdot\vec{n}_1 - 1)$$

$$\simeq \frac{(\epsilon-\epsilon_1)}{c}\vec{p}\cdot\vec{n} + \frac{\epsilon}{c}\vec{p}\cdot(\vec{n}-\vec{n}_1) \qquad \simeq 0$$

Drop higher order of above equation, remaining

$$(\epsilon_1 - \epsilon)\left(\frac{E}{c} - \vec{p}\cdot\vec{n}\right) = \epsilon\vec{p}\cdot(\vec{n}_1 - \vec{n})$$

Because $\frac{1}{\frac{E}{c}-\vec{p}\cdot\vec{n}} \overset{\text{non-rel}}{\simeq} \frac{1}{m_e c} + O\left(\frac{|\vec{p}|}{E/c}\right) \simeq \frac{1}{mc} + O\left(\frac{kT}{mc^2}\right)$, we have

$$\epsilon_1 - \epsilon = \frac{\epsilon\vec{p}\cdot(\vec{n}_1-\vec{n})}{mc} + O\left(\frac{kT}{mc^2}\right)$$

$$\Leftrightarrow \quad \Delta x = \frac{x\vec{p}\cdot(\vec{n}_1-\vec{n})}{mc} + O\left(\frac{kT}{mc^2}\right)$$

✓

② Derive (7.54) $I_2 = 2x^2 n_e \sigma_T \left(\frac{kT}{mc^2}\right) + O\left(\frac{kT}{mc^2}\right)^2$

From definition $I_2 = \int d^3p \frac{d\sigma}{d\Omega} d\Omega \cdot f_e \Delta^2$

Here $\Delta^2 = \left[\frac{x\vec{p}\cdot(\vec{n}_1-\vec{n})}{mc}\right]^2 = \frac{x^2 p^2 (\vec{n}_1-\vec{n})^2}{m^2c^2}\cos^2\beta$, $d^3p = p^2\sin\beta\cdot dp\,d\beta\,d\phi$, $f_e = n_e(2\pi mkT)^{-3/2} e^{-E/kT}$

β是 $\vec{p}$ 与 $\vec{n}_1-\vec{n}$ 的夹角 $\qquad \frac{d\sigma}{d\Omega} = \frac{1}{2}r_0^2(1+\cos^2\Theta)$

The Integration Can be written as

$$I_2 = \left(\frac{x}{mc}\right)^2 \underbrace{\int d^3p\, f_e\cdot\cos^2\beta\cdot p^2}_{= n_e mkT} \cdot \underbrace{\int \frac{d\sigma}{d\Omega} d\Omega\, [\vec{n}_1-\vec{n}]^2}_{= \frac{16}{3}\pi r_0^2 = 2\sigma_T} + O\text{ (higher order)}$$

$$= 2\sigma_T n_e x^2\left(\frac{kT}{mc^2}\right) + O\text{(higher order)}$$

✓

③ Derive (7.55a) $\frac{\partial n}{\partial t} = -\frac{1}{x^2}\frac{\partial}{\partial x}(x^2 j(x))$

From $\int \frac{\partial n}{\partial t}x^2 dx = 0$, the only possible way is to let $\frac{\partial n}{\partial t}x^2$ behave like a 'divergence' term. or

$$\frac{\partial n}{\partial t}x^2 = \frac{\partial}{\partial x}\left(\text{some function}\right) = -\frac{\partial}{\partial x}(x^2 j(x))$$

$$\Leftrightarrow \quad \frac{\partial n}{\partial t} = -\frac{1}{x^2}\frac{\partial}{\partial x}(x^2 j(x))$$

④ Derive (7.55b) $j = g(x)\left[n' + h(n,x)\right]$

Using (7.55a) we can show

$$\frac{\partial n}{\partial t} = -\left(\frac{2j}{x} + j'\right) \qquad (*1)$$

If we use (7.52), we can See $\frac{\partial n}{\partial t} = n'' C_2(x) + n' C_1(n,x) + C_0(n,x)$, which means $\frac{\partial n''}{\partial x^2}$ only depends on x. We can say that $j = j(n, n', x)$ because $\frac{\partial n}{\partial t}$ should not exceed 2 order.

From (*1) we write

$$\frac{\partial n}{\partial t} = -\left(\frac{2j(n,n',x)}{x} + \frac{\partial j}{\partial x} + \frac{\partial j}{\partial n}n' + \frac{\partial j}{\partial n'}n''\right)$$

$$\Rightarrow \quad -\frac{\partial j}{\partial n'} = C_2(x)$$

$\Rightarrow$ j is linear function of $n'(x)$, so $j(n,n',x) = g(x)n' + g(x)h(n,x)$

✓

8.2 The wave packet $\psi(r,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kr-\omega(k)t)} dk$, satisfy $A(k) e^{-i\omega(k)t} = \mathcal{F}(\psi(r,t))$

2/2

where $\mathcal{F}$ denotes Fourier transformation. Also, $r\psi(r,t)$ can be expressed as

$$r\psi(r,t) = \int_{-\infty}^{\infty} r\, A(k)\, e^{i(kr-\omega(k)t)} dk$$

$$= \int_{-\infty}^{\infty} A(k) \frac{1}{i}\frac{\partial}{\partial k} e^{i(kr)} e^{-i(\omega(k)t)} dk$$

$$= \int_{-\infty}^{\infty} e^{ikr}\; i\frac{\partial}{\partial k} A(k) e^{-i\omega(k)t}\; dk$$

$$\Leftrightarrow \quad i\frac{\partial}{\partial k} A(k) e^{-i\omega(k)t} = \mathcal{F}(r\cdot\psi(r,t))$$

Using Parseval identity, we obtain

$$\int |\psi(r,t)|^2 dr = \frac{1}{2\pi}\int \mathcal{F}(\psi)\, \mathcal{F}^*(\psi)\, dk = \frac{1}{2\pi}\int |A(k)|^2 dk$$

independent of time

$$\int r|\psi|^2 dr = \frac{1}{2\pi}\int \mathcal{F}(r\psi)\, \mathcal{F}^*(\psi)\, dk = \frac{1}{2\pi}\int \left(|A|^2 t \frac{\partial\omega}{\partial k} + A^* i \frac{\partial}{\partial k} A \right) dk$$

So we can express $\frac{d}{dt}\langle r(t)\rangle$ as

$$\frac{d}{dt}\langle r(t)\rangle = \frac{d}{dt} \frac{\int \left(|A|^2 t \frac{\partial\omega}{\partial k} + A^* i \frac{\partial}{\partial k} A \right) dk}{\int |A|^2 dk} = \frac{\int \frac{\partial\omega}{\partial k} |A|^2 dk}{\int |A|^2 dk}$$

$$= \left\langle \frac{\partial\omega}{\partial k} \right\rangle$$

✓

张泽遥 2017311337 物研172 辐射机制 7/7

10.4 Derive $\sigma_{bf}(\hbar\omega \gg R_y) \simeq \dfrac{(2\alpha)^{9/2}\pi Z^5 c^{7/2}}{3a_0^{3/2}\omega^{7/2}}$ for bound-free process, using non-rel Born approximation.

2/2

Solution: From text book we already have

$$\frac{d\sigma_{bf}}{d\Omega} = \frac{\alpha v V}{2\pi\omega}\left|\langle f | e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}\cdot\nabla | i\rangle\right|^2 \qquad (10.52)$$

For the lowest order Born approximation, $e^{i\vec{k}\cdot\vec{r}} \simeq 1$. then for the initial state (n=0) $\left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Z\frac{r}{a_0}}$

And the free state $\frac{1}{V^{1/2}} e^{i\vec{q}\cdot\vec{r}}$, the matrix element

$$\langle f | e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}\cdot\nabla | i\rangle \simeq \langle f | \vec{\epsilon}\cdot\nabla | i\rangle \xlongequal{\text{Hermitian of } \nabla} \langle i | \vec{\epsilon}\cdot\nabla | f\rangle$$

$$= \int d^3r \cdot \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Z\frac{r}{a_0}}\ \vec{\epsilon}\cdot\nabla \frac{1}{V^{1/2}} e^{i\vec{q}\cdot\vec{r}}$$

$$= (\vec{\epsilon}\cdot i\vec{q})\left[\frac{Z^3}{\pi a_0^3 V}\right]^{1/2} \underbrace{\int e^{-\frac{Z}{a_0}r} e^{i\vec{q}\cdot\vec{r}} d^3r}$$

$$= \int_0^\infty e^{-\frac{Z}{a_0}r} r^2 dr \cdot 2\pi \int_{-1}^{1} d\mu \cdot e^{iqr\mu}$$

$$= \int_0^\infty \frac{4\pi}{q} r\, e^{-\frac{Z}{a_0}r} \sin qr\, dr$$

$$= \frac{8\pi Z}{a_0}\left[\left(\frac{Z}{a_0}\right)^2 + q^2\right]^{-2}$$

$$= i\vec{q}\cdot\vec{\epsilon}\left[\frac{Z^3}{\pi a_0^3 V}\right]^{1/2} \frac{8\pi Z}{a_0}\left[\left(\frac{Z}{a_0}\right)^2 + q^2\right]^{-2}$$

So from (10.52) we obtain

$$\frac{d\sigma_{bf}}{d\Omega} = \frac{32\alpha}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{\hbar q (\vec{q}\cdot\vec{\epsilon})^2}{\left[\left(\frac{Z}{a_0}\right)^2 + q^2\right]^4} \xlongequal[\Rightarrow\ q \gg \frac{Z}{a_0}]{\text{for } \hbar\omega \gg R_y,\ \hbar q \gg P_e(n=0) = \frac{\hbar}{a_0} \simeq \frac{\hbar Z}{a_0}} \frac{32\alpha}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{\hbar(\vec{q}\cdot\vec{\epsilon})^2}{q^7}$$

Now by integrate over all Ω we obtain

$$\sigma_{bf} = \int \frac{d\sigma_{bf}}{d\Omega} d\Omega = \frac{32\alpha}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{\hbar}{q^7}\int q^2\cos^2\theta\, d\Omega = \frac{32\alpha\hbar}{m\omega}\frac{1}{q^5}\left(\frac{Z}{a_0}\right)^5 \frac{4}{3}\pi.$$

$\downarrow\ \hbar\omega \simeq \frac{\hbar^2 q^2}{2m},\quad a_0 = \frac{\hbar^2}{me^2},\quad \alpha = \frac{e^2}{\hbar c}$

$$= \frac{(2\alpha)^{9/2}\pi c^{7/2} Z^5}{3a_0^{3/2}\omega^{7/2}}$$

✓

10.5 For radiation emitted from optic thin material, the absorption (α) can be neglected. So the spectrum is fully determined by line profile $\phi(\nu)$. For natural broadening $\phi_N(\nu) = \frac{1}{4\pi^2}\frac{\gamma}{(\nu-\nu_0)^2+(\frac{\gamma}{4\pi})^2}$, and for Doppler broadening $\phi_D(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} e^{-\frac{(\nu-\nu_0)^2}{(\Delta\nu_D)^2}}$. The width of them is $\Delta\nu_N = \frac{\gamma}{4\pi}$, $\Delta\nu_D = \nu_0\sqrt{\frac{2kT}{m_H c^2}}$.

From text book we know

$$g_1 f_{1n} = \frac{2^9 n^5 (n-1)^{2n-4}}{3(n+1)^{2n+4}} \qquad (10.46)$$

$$g_u A_{ul} = \frac{-8\pi^2 e^2 \nu_{ul}^2}{mc^3} g_u f_{ul} = \frac{8\pi^2 e^2 \nu_{ul}^2}{mc^3} g_l f_{lu} \qquad (10.34)$$

$$\Rightarrow \gamma = A_{21} = \nu_{21}^2 \frac{8\pi^2 e^2}{mc^3}\frac{2^{14}}{3^9}\frac{g_1}{g_2}$$

$$\text{where } 2\pi\hbar\nu_{21} = \frac{3}{8}\frac{e^2}{a_0} = \frac{3}{8}\frac{me^4}{\hbar^2}$$

For higher temperature $T \gg T_c$, doppler broadening dominates, thus

$$\Delta\nu = \Delta\nu_D \propto \sqrt{T}$$

For lower temperature $T \ll T_c$, natural broadening dominates, therefore

$$\Delta\nu = \Delta\nu_N \propto \gamma \quad \text{NOT dependent on temperature.}$$

Now we want to find critical temperature T_c, by setting

$$\Delta\nu_D = \frac{\gamma}{4\pi}.$$

$$\Rightarrow \frac{2kT_c}{m_H c^2}\nu_{21}^2 = \frac{1}{16\pi^2}\gamma^2 = \frac{1}{16\pi^2}\left(\nu_{21}^2\frac{8\pi^2 e^2}{mc^3}\frac{2^{14}}{3^9}\frac{g_1}{g_2}\right)^2$$

$$\Rightarrow kT_c = \left(\frac{e^2}{\hbar c}\right)^6 m_H c^2 \frac{2^{21}}{3^{18}}$$

$$\Rightarrow T_c \simeq 8.5\times10^{-3}\,K.$$

10.6 The transient probability $w_{fi} \propto |d_{fi}|^2 = \frac{1}{3}\left(|d(x)_{fi}|^2 + |d(y)_{fi}|^2 + |d(z)_{fi}|^2\right)$

For single atom, $\vec{d} \propto \vec{r}$. So we have

$$w_{fi} \propto |\langle f|\vec{r}|i\rangle|^2$$

The matrix element

$$\langle f|\vec{r}|i\rangle = \int r^2 R_f(r) R_i(r)\vec{r}\, Y^*_{l_f m_f}(\theta,\phi) Y_{l_i m_i}(\theta,\phi)\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int r^3 R_f(r) R_i(r) dr \int Y^*_{l_f m_f}(\theta,\phi) Y_{l_i m_i}(\theta,\phi)\sin\theta\, d\theta\, d\phi \times \hat{r}$$

For its z component

$$\langle f|\vec{r}|i\rangle_z \propto \int Y^*_{l_f m_f}(\theta,\phi) Y_{l_i m_i}(\theta,\phi)\cos\theta\sin\theta\, d\theta\, d\phi$$

$$\propto \int P_{l_f}^{m_f}(\theta) P_{l_i}^{m_i}(\theta)\,\mu\, d\mu \int e^{i(m_i-m_f)\phi} d\phi$$

$$\mu P_l^m = \frac{1}{2l+1}\left[(l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m\right]$$

$$= 0 \quad \text{unless} \quad m_i = m_f, \quad l_f - l_i = \pm1$$

For its x-y component

$$\langle f|\vec{r}|i\rangle_x \pm i\langle f|\vec{r}|i\rangle_y \propto \int Y^*_{l_f m_f}(\theta,\phi) Y_{l_i m_i}(\theta,\phi)\sin\theta\, e^{\pm i\phi}\sin\theta\, d\theta\, d\phi$$

$$\propto \int P_{l_f}^{m_f}(\theta) P_{l_i}^{m_i}(\theta)\sqrt{1-\mu^2}\, d\mu \int e^{i(m_i-m_f\pm1)\phi} d\phi$$

$$\sqrt{1-\mu^2}\, P_l^{m-1} = \frac{1}{2l+1}\left[P_{l-1}^m - P_{l+1}^m\right]$$

$$= 0 \quad \text{unless} \quad l_f - l_i = \pm1, \quad m_i - m_f = \pm1$$

Add them together, we obtain the simple selection rule:

$$\Delta l = \pm1$$

$$\Delta m = 0, \pm1$$