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Radiative Processes in Astrophysics
Final exam, Spring 2018

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1. (10 points) Hyperfine Emission from Neutral Hydrogen

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level (singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the *excitation temperature*, T_{ex}, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{ex}} . \tag{1}$$

Here $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state to excited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{ex} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21$ cm transition as the *spin temperature*. But use of the term "excitation temperature" is general to any line transition; it is simply a measure of how excited an atom is.

(a) Define $T_* = h\nu/k$ and compute its value.

(b) Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\phi(\nu)$ (the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and T_*/T_{ex}. *Do not forget the correction for stimulated emission.*

(c) Write down the volume emissivity, j_ν (units of $\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}$), for this transition. Use whatever quantities defined above that you need.

(d) Write down the source function, S_ν, for this transition.

(e) Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.

Does your answer depend on T_{ex}?

If someone gives you a spectrum of the 21 cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities can you infer from the spectrum?

(f) Write down the optical depth of the cloud. Does your answer depend on T_{ex}?

(g) How large would L have to be for the cloud to be marginally optically thick? Use a gas density of $n = 1\,\mathrm{cm^{-3}}$, a gas temperature of $T = 100\,\mathrm{K}$, and an excitation temperature $T_{ex} = T$. Assume the line is only thermally broadened.

Solution:

(a) $T_* = \frac{h\nu}{k} = -T_{ex} \ln\left(\frac{n_1}{n_0}\frac{g_0}{g_1}\right)$, For $\lambda = 21$ cm, $T_* = \frac{h\nu}{k} = \frac{hc}{k\lambda} \simeq 0.064$ K

(b) Using the connection between α_ν and Einstein coeff we obtain

$$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu)\left(n_0 B_{01} - n_1 B_{10}\right)$$

$$\begin{cases} g_0 B_{01} = g_1 B_{10} \\ A_{10} = \frac{2h\nu^3}{c^2} B_{10} \end{cases}$$

Combine this, and use $\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-T_*/T_{ex}}$ we obtain

$$\alpha_\nu = \frac{1}{4\pi}\phi(\nu) A_{10} \frac{\lambda^2}{2} n_1 \left(e^{\frac{T_*}{T_{ex}}} - 1\right) = \frac{1}{4\pi}\phi(\nu) A_{10} \frac{n_1\lambda^2}{2}\left(e^{\frac{T_*}{T_{ex}}} - 1\right) = \frac{1}{4\pi}\phi(\nu) A_{10} \frac{\lambda^2}{2} \cdot 3n_0 \left(1 - e^{-T_*/T_{ex}}\right)$$

(c) Use the relation between j_ν and Einstein Coeff we obtain

$$j_\nu = \frac{h\nu}{4\pi}\phi(\nu)\, n_1 A_{10} = \frac{h\nu}{4\pi}\phi(\nu) A_{10} \cdot n_0 \frac{g_1}{g_0} e^{-T_*/T_{ex}}$$

$$= \frac{h\nu}{4\pi}\phi(\nu) A_{10} \cdot n_0 \cdot 3 e^{-T_*/T_{ex}}$$

(d) The Source function $S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{2hc}{\lambda^3}\frac{1}{e^{T_*/T_{ex}} - 1}$

(e) The Solution of transfering function $\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu$ is

$$I_\nu = I_\nu(0)\, e^{-\tau} + S_\nu\left(1 - e^{-\tau}\right)$$

$$\simeq \tau S_\nu$$

$$= \alpha_\nu \cdot L \cdot \frac{j_\nu}{\alpha_\nu} = j_\nu \cdot L = \frac{h\nu}{4\pi}\phi(\nu) A_{10} \cdot n_0 \cdot 3 \cdot e^{-T_*/T_{ex}} \cdot L$$

That depends on T_{ex}.

We can infer $n_0 e^{-T_*/T_{ex}}$ from the Spectrum, that is, the combination of State occupation n_0 and n_1

If thermal equilibrium is assumed, the temperature can be derived.

(f) $\tau = \alpha_\nu L = \frac{\phi(\nu)}{4\pi} A_{10} \frac{\lambda^2}{2} \cdot 3n_0 \left(1 - e^{-T_*/T_{ex}}\right) L$ that depends on T_{ex}.

(g) The optical thick condition is $\tau \gg 1$. So $\frac{\phi(\nu_0)}{4\pi} A_{10} \frac{\lambda^2}{2} \cdot 3n_0 \left(1 - e^{-T_*/T_{ex}}\right) L \gg 1$

where $\phi(\nu_0) = \frac{1}{\sqrt{\pi}\, \nu \sqrt{\frac{2kT}{m_a c^2}}}$

So we find $L \gg \left[\frac{\phi(\nu_0)}{4\pi} A_{10} \frac{\lambda^2}{2} 3n_0 \left(1 - e^{-T_*/T_{ex}}\right)\right]^{-1} \approx \left[\frac{\phi(\nu_0)}{4\pi} A_{10} \frac{\lambda^2}{2} 3n_0 \frac{T_*}{T_{ex}}\right]$

it depends on A_{10}, I don't know A_{10} ?

3 2. (10 points) Galactic Synchrotron Emission

The *brightness temperature* of synchrotron emission in the Galactic plane is measured to be

$$T_b = 250 \left(\frac{\nu}{480\text{MHz}} \right)^{-2.8} \text{K}, \quad (1)$$

valid for 8 GHz > ν > 480 MHz. This emission arises from cosmic ray electrons gyrating in the Galactic magnetic field of strength $B \sim 3\mu$G. Take the size of the emitting region to be ~10 kpc.

(Aside from being a nuisance foreground for CMB researchers, Galactic synchrotron emission is thought to probe the supernova rate in the Galaxy, since cosmic ray electrons and protons at the relevant energies are thought to be accelerated in supernova shock waves. In addition, a famous "FIR-radio correlation" is observed to exist for normal spiral galaxies; the radio traces electrons accelerated by supernovae, while the far-infrared (FIR) emission traces warm dust grains. Where there are supernovae, there is active star formation; where there is star formation, there are dust grains warmed by the ISRF. Or so goes the traditional interpretation of the FIR-radio correlation.)

(a) Provide an approximate expression for the differential energy spectrum of cosmic ray electrons. Express in units of particles m^{-2} sr^{-1} s^{-1} Gev^{-1}. Indicate the approximate range of energies (E_{min} and E_{max}) for which your expression is valid. You can solve this in three steps.

- Estimate the number density of electrons, $n_e(\gamma_{min})$, responsible for emission at $\nu=\nu_{min}$=480 MHz.
- Calculate the desired differential energy spectrum at electron energy E_{min} using $n_e(\gamma_{min})$.
- Calculate the slope of the differential energy spectrum.

(b) Estimate the gyro-radii of cosmic ray electrons at E_{min} and E_{max}. Compare to the size of the (baryonic disk of the) Galaxy, 10 kpc. You should be able to understand why cosmic ray astronomy is so difficult.

Solution: (a) • For Synchrotron Radiation $\omega = \frac{eB}{\gamma mc}$ => $\gamma = \frac{eB}{2\pi \nu m_e c}$ and $d(\gamma mc^2) = dE = \frac{eBc^2}{2\pi\nu^2 c}\Delta\nu$

$E = \frac{eBc^2}{2\pi\nu c}$ => $\begin{cases} E_{min} = \frac{eBc^2}{2\pi\nu_{max}c} \\ E_{max} = \frac{eBc^2}{2\pi\nu_{min}c} \end{cases}$

The radiation Power for a single electron is $\langle P \rangle = \frac{4}{3}\cdot\frac{8}{3}\pi r_0^2 c \gamma^2 \frac{1}{8\pi} B^2$

The total radiation $\frac{dW}{dt} = \langle P \rangle n_e \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = 4\pi\cdot\Delta\nu\cdot\pi\left(\frac{d}{2}\right)^2\cdot T_b\cdot\frac{2\nu^2 k}{c^2}$ ⟹ $n_e(\gamma_{min}) \simeq \frac{27\Delta\nu T_b \frac{2k}{c^3}\nu^4 m_e^2 \pi^3}{\left(\frac{d}{2}\right) r_0^2 e^2 B^4}$

where $r_0 = \frac{e^2}{m_e c^2}$

4.

• The energy spectrum at E_{min} is

$$\frac{dN}{dA\,dt\cdot d\Omega\cdot dE} = \frac{n_e(\gamma_{min})\,\frac{4}{3}\pi\left(\frac{d}{2}\right)^3}{\pi\left(\frac{d}{2}\right)^2\cdot\frac{d}{c}\cdot 4\pi\times\frac{eBc^2}{2\pi\nu^2 c}\Delta\nu}$$

✗

• If the slope is $-p$, or $\frac{dN}{dA\,dt\,d\Omega\,dE} \propto \gamma^{-p}$, Then the radiation $I_\nu \propto \nu^{-\frac{(p-1)}{2}}$

From question we know $\frac{p-1}{2} = 2.8$. So $p = 6.6$

✗

(b) The gyro-radii is $r = \frac{v}{\omega_B} = \frac{\beta\gamma m c^2}{eB} \simeq \frac{\gamma m_e c^2}{eB}$

Using $\gamma = \frac{eB}{2\pi \nu m_e c}$ we know $\begin{cases} \gamma_{min} = \frac{eB}{2\pi \nu_{max} m_e c} \\ \gamma_{max} = \frac{eB}{2\pi \nu_{min} m_e c} \end{cases}$

So the radii $\begin{cases} r_{min} = \frac{c^2}{2\pi \nu_{max} c} = 3.26\times 10^{-17}\ pc \\ r_{max} = \frac{c^2}{2\pi \nu_{min} c} = [illegible].9 \times 10^{-18}\ pc \end{cases} \ll 10\ kpc$

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3. (10 points) Faraday Rotation

Consider the propagation of light through a magnetized plasma. The magnetic field is uniform $\vec{B_0} = B_0\hat{z}$. Light travels parallel to $\hat{z}$.

An electron in the plasma feels a force from the electromagnetic wave, and a force from the externally imposed B-field. Its equation of motion reads

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B_0} \tag{1}$$

where the electric field $\vec{E}$ can be decomposed into right-circularly-polarized (RCP) and left-circularly-polarized (LCP) waves:

$$\vec{E} = E_0(\hat{x} \mp i\hat{y})e^{i(k_{\mp}z-\omega t)} \tag{2}$$

where it is understood that the real part should be taken. The upper sign (-) corresponds to RCP waves, while the lower sign (+) corresponds to LCP waves.

In the equation of motion above, we have neglected the Lorentz force from the wave's B-field, since it is small (by v/c) compared to the force from the wave's E-field.

(a) Prove that the solution of the equation of motion reads

$$\vec{v} = \frac{-ie}{m(\omega \pm \omega_{cyc})}\vec{E} \tag{3}$$

where $\omega_{cyc} = eB_0/mc$. This is the hardest part of the derivation for the dispersion relation of RCP and LCP waves. But it is fairly straightforward.

(b) Rybicki & Lightman Problem 8.3

8.3—The signal from a pulsed, polarized source is measured to have an arrival time delay that varies with frequency as $dt_p/d\omega = 1.1 \times 10^{-5}$ s^2, and a Faraday rotation that varies with frequency as $d\Delta\theta/d\omega = 1.9 \times 10^{-4}$ s. The measurements are made around the frequency $\omega = 10^8$ s^{-1}, and the source is at unknown distance from the earth. Find the mean magnetic field, $\langle B_{\parallel} \rangle$, in the interstellar space between the earth and the source:

$$\langle B_{\parallel} \rangle \equiv \frac{\int nB_{\parallel}\,ds}{\int n\,ds}.$$

Solution: (a)

Assume $\vec{v} = V_0(\hat{x} \mp i\hat{y})\, e^{i(k_{\mp}z-\omega t)}$. Then from $m\dot{\vec{v}} = -i\omega m\vec{v} = -e\vec{E} - \frac{e}{c}\vec{v}\times\vec{B_0}$, $\vec{B_0} = B_0\hat{e}_z$

we obtain $-im\omega v_0 = -eE_0 \pm \frac{e}{c}v_0 B_0 i$ $\Rightarrow$ $v_0 = \frac{-iE_0 e}{m(\omega \pm \omega_{cyc})}$ $\Rightarrow$ $\vec{v} = \frac{-ie\vec{E}}{m(\omega \pm \omega_{cyc})}$, $\omega_{cyc} = \frac{eB_0}{mc}$ ✓

(b): The rotation angle $\Delta\theta = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d nB_{\parallel}\,ds$, so $\frac{d\Delta\theta}{d\omega} = \frac{+2}{\omega^3}\frac{2\pi e^3}{m^2c^2}\int_0^d nB_{\parallel}\,ds$

$$\Rightarrow \int_0^d nB_{\parallel}\,ds = \frac{d\Delta\theta/d\omega}{\frac{+2}{\omega^3}\frac{2\pi e^3}{m^2c^2}} = \cdots$$

~~At another hand, in plasma the $n = \frac{c}{v_p} = \sqrt{1-\omega_p^2/\omega^2} = \sqrt{1-\frac{4\pi n e^2}{m_e \omega^2}}$~~

At another hand. phase difference $kd = \frac{\omega}{c} n d = \frac{\omega}{c}\int_0^d n\, ds$

So the time difference

$$\Delta t_p = \frac{\Delta(kd)}{\omega} \doteq \frac{1}{c}\frac{\Delta\omega}{\omega}\int_0^d n\,ds$$

$$\Rightarrow \frac{dt_p}{d\omega} \simeq \frac{\int_0^d n\,ds}{\omega c}$$

$$\Rightarrow \int_0^d n\,ds \simeq dt_p/d\omega \cdot \omega c = \cdots$$

Finally

$$\langle B_{\parallel}\rangle = \frac{\int_0^d n B_{\parallel}\,ds}{\int_0^d n\,ds} = \frac{\dfrac{d\,\Delta\theta/d\omega}{\frac{2}{\omega^3}\,\frac{2\pi e^3}{m^2c^2}}}{\omega c \times dt_p/d\omega} = \cdots$$

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4. (5 points) Practice with j_ν, α_ν, S_ν, B_ν, I_ν.

(a) A plane-parallel slab of uniformly dense gas is known to be in LTE (local thermodynamic equilibrium) at a uniform temperature T. Its thickness normal to its surface is s. Its absorption coefficient is $\alpha_{\nu,\mathrm{gas}}$. Write down the specific intensity, I_ν, viewed normal to the slab, in terms of the variables given.

(b) The same slab is now filled uniformly with non-emissive dust having absorption coefficient $\alpha_{\nu,\mathrm{dust}}$. The dust is non-emissive, so its emissivity $j_{\nu,\mathrm{dust}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given so far.

(c) The slab of gas and dust is further mixed with a third component: an emissive, non-absorptive uniform medium having emissivity $j_{\nu,\mathrm{med}}$ and absorption coefficient $\alpha_{\nu,\mathrm{med}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given.[1]

Solution:

(a) The solution for transfer function $\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$ is

$$I_\nu = I_\nu(0)\, e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

In this case $\tau_\nu = \alpha_{\nu\,\mathrm{gas}} \cdot s$. $S_\nu = B_\nu$ = Planck function, so we have

$$I_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \left(1 - e^{-\alpha_{\nu\,\mathrm{gas}} \cdot s}\right)$$ ✓

(b) Now $S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_{\nu\,\mathrm{gas}}}{\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}} = \frac{B_\nu \cdot \alpha_{\nu\,\mathrm{gas}}}{\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}}$

So the specific intensity

$$I_\nu = \frac{B_\nu \cdot \alpha_{\nu\,\mathrm{gas}}}{\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}} \left(1 - e^{-s\cdot(\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}})}\right)$$ ✓

(c) Now $S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_{\nu\,\mathrm{gas}} + j_{\nu\,\mathrm{med}}}{\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}}$, $\tau_\nu = s \cdot \left(\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}\right)$, so we have

$$I_\nu = \frac{\alpha_{\nu\,\mathrm{gas}} B_\nu + j_{\nu\,\mathrm{med}}}{\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}}} \left(1 - e^{-s\cdot(\alpha_{\nu\,\mathrm{gas}} + \alpha_{\nu\,\mathrm{dust}})}\right)$$ ✓

[1] A physical realization of this problem might be an HII region surrounding an ionizing O star. The material in LTE would be the fully ionized plasma, emitting thermal bremsstrahlung radiation. The dust would be dust. The emissive, non-absorptive medium would be the same ionized plasma emitting recombination (line) radiation. For the assumptions stated in the problem to be valid, we would have to evaluate ν at, say, an optical recombination line like Hα.

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5. (5 points) Pulsar Dispersion Measure

A radio astronomer notes that pulses from a certain pulsar observed at a radio frequency of $\nu = 2$ GHz arrive slightly ahead of the pulse train observed at $\nu = 1$ GHz. The lead time is 1 s.

(a) Use the dispersion relation derived in class for a cold, ionized plasma, and the information above, to derive the column density of electrons along the line of sight to this pulsar. Express in standard pulsar-community units of cm^{-3} pc.

The dispersion relation (as derived in class) is given by

$$\left(\frac{ck}{\omega}\right)^2 = 1 + \frac{4\pi n e^2}{m_e(\omega_o^2 - \omega^2)}.$$

For a plasma, $\omega_o^2 = 0$, so the dispersion relation becomes

$$\left(\frac{ck}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

where

$$\omega_p^2 = \frac{4\pi n e^2}{m_e}.$$

(b) For an assumed density of electrons in the interstellar medium of $0.03\,\text{cm}^{-3}$, calculate the distance to this pulsar. Does this seem reasonable?

(c) Calculate the optical depth to Thomson scattering along the line-of-sight to this pulsar.

Solution: (a) The group velocity is $v_g = \frac{\partial \omega}{\partial k} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

So the time difference

$$\Delta t = \frac{\Delta d}{d v_g} = \frac{\Delta d}{c} \cdot \frac{\frac{\omega_p^2}{\omega^2}\frac{d\omega}{\omega}}{\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{3/2}} \overset{\omega \gg \omega_p}{\simeq} \frac{\Delta d}{c}\frac{\omega_p^2}{\omega^2}\frac{d\omega}{\omega}$$

So we can solve it and obtain

$$n\Delta d \simeq \frac{m_e \cdot \Delta t \cdot c \cdot \omega^2 \frac{\omega}{d\omega}}{4\pi e^2} \simeq \frac{511\,\text{keV} \left(\frac{1}{3.26}\text{pc}/\text{yr}\right)^{-1} 1\,\text{sec}\; 4\pi^2 \times 1\text{GHz}^2 \times 1}{4\pi \times 1.44\,\text{nm}\cdot\text{eV}} \doteq 1.2 \times 10^2 \cdot \text{cm}^{-3}\,\text{pc}$$

(b) If $n = 0.03\,\text{cm}^{-3}$, then $\Delta d = \frac{n\Delta d}{n} = \frac{1.2 \times 10^2\,\text{cm}^{-3}\text{pc}}{0.03\,\text{cm}^{-3}} = 4$ kpc. < Size of Milky Way

Quite reasonable!

(c) $\tau = \underbrace{\sigma}_{\sigma_T} \cdot n \times \Delta d \simeq \frac{8\pi}{3} r_0^2 \cdot n\Delta d = \frac{8\pi}{3}(0.28 \times 10^{-12}\,\text{cm})^2 \cdot 1.2 \times 10^2\,\text{cm}^{-3}\text{pc} = 7.5 \times 10^{-5} \ll 1$

Optical thin!