物理宇宙学　　茅奕(sh?)　　02/28/2018

Physical Cosmology　　6教 6B301

现代宇宙学
- 宇宙的膨胀
- 热历史和相变
- 物质分布、运动、演化及其在CMB/LSS上的特征
- 暴胀理论

e.g. SLOAN & WMAP movie

{ 传统天文学 — 看图说话
{ 现代天文学 — 天体物理与各种model的定量刻画

教材　S. Weinberg, Cosmology — 对初学者比较友好
　　　S. Dodelson, Modern Cosmology } 泛用　推荐英文版

Grades: Homework 30% (9 sets) — 一周后交
Quizzes 30%
Final Exam 40% } closed form

Lecture 1　General Picture of the expanding Universe

中国古代宇宙观
- 浑天说（晋《浑天仪注》）浑天如鸡子，地如卵中黄
- 《道德经》 道生一，一生二，…
- 《太极》 太极生两仪…
- 有了天道以明人事 — 中国古代天文学是为帝王服务的
  帝王

- 肉眼的宇宙是不动的　要研究宇宙的演化，就需要光学（感谢光速的有限性）
- 宇：空间　宙：时间　四方上下谓之宇，往古今来谓之宙　刘安《淮南子·齐俗》

欧洲宇宙观　托勒密的地心说　哥白尼的日心说

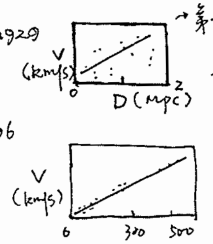
宇宙中心 → 地，日，银河

Multiverse 多重宇宙

Universe ← 哈勃发现河外星系 local group / cluster

- classical evidence of Big Bang Theory (BBT)

{ Hubble 1929
CMB
SNIa 1996
BBN



经过拟合的直线

一大争论！没有系统对应！

→ 第一次暗示非静态宇宙模型

→ 195x年，伽莫夫的热大爆炸理论 Garmov　Hot BBT

→ Friedmann eq.　FLRW metric　1920s

→ 1970s Bell lab, CMB discovery　T~5K　支持Garmov理论

→ 1981 Inflation model/Theory 解决Hot BBT 该理论自洽问题与扰动

→ 1992 COBE Satellite
CMB的Black Body Radiation性质（支持BBT）
CMB的anisotropy 10^{-5}

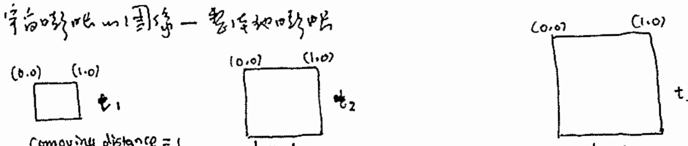
$H_0 \approx 67.4$ km/s/Mpc

$v = H_0 D$　Hubble constant parameter

BBT + inflation = standard model of cosmology

Standard model of particle physics → 丑！
$SU(3) \times SU(2) \times U(1)$

- 宇宙膨胀的图像 — 均匀地膨胀



comoving distance = 1
physical distance = 1

$d_c = 1$
$d_p = 1 \times a(t_2)$ — scale factor $a(t)$，与位置无关

$d_c = 1$
$d_p = a(t_3) > a(t_2) > a(t_1) = 1$

某时刻的距离，总是可以用某一固定时刻的距离(d_c)乘以$a(t)$得到

Hubble Rate $H(t)$:

$$x_{phy} = x_{com}\, a(t)$$

$$\Rightarrow v_{phy} = \frac{d x_{phy}}{dt} = x_{com}\dot{a} \quad \Rightarrow \quad \frac{v_{phy}}{x_{com} a} = \frac{v_{phy}}{x_{phy}} = \frac{\dot{a}}{a} = H(t)$$

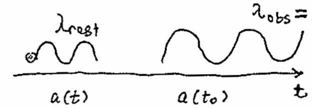
Hubble rate

Redshift z:

$$1+z = \frac{\lambda_{obs}}{\lambda_{rest}} = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}$$

若选取 $a(t_0)=1$

Cosmological Redshift



$\lambda_{obs} = \lambda_{rest}\frac{a(t_0)}{a(t)}$ → 观测时

* 所以宇宙学红移只是几何效应而已，与具体的宇宙学模型无关（$a(t)$演化的）
　运动学效应　动力学

* 可以建立 $z \leftrightarrow a \leftrightarrow t$ 的一一对应关系，都可以做为坐标量
　运动学　动力学

$v \ll c$, $z \simeq \frac{v}{c}$

* why not Doppler redshift?（为什么不用Doppler红移来解释退行，而一定要引入宇宙膨胀？）

红移 z 可以大于1吗？

狭义相对论的Doppler红移公式 $z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1$

当 $v \to c$ 时 $z \to \infty$ 也可以

在膨胀的宇宙中，星系的退行有它不同的；

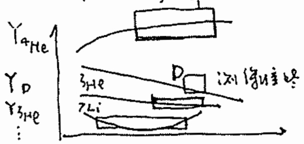
CMB、大爆炸核合成等 证据可以证实宇宙的膨胀。只有Hubble定律是不够的

- Big Bang Nucleosynthesis (BBN) → 专指宇宙冷却过程中的合成

p, n, e, γ, ν, …
$p + n \to D_{氘} + \gamma + \dots$
$D + n \to T_{氚} + \gamma + \dots$
↓
D, T, ^3He, ^4He, ^7Li 等 light element

宇宙的Big Bang → H 合成 (BBN)
但更重的元素是以后SN → 合成更重的元素

light element abundance 可以给出宇宙早期的状态



不同元素的给出的结果是自洽的

- Cosmic Microwave Backgroup (CMB)

$p(H^+)$, e, ^4He ion, …

↓ 温度变低

$p+e \xrightarrow{\text{recombine}} H+\gamma$ 13.6eV

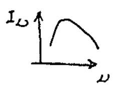
fire wall　opaque

transparent　γ　$z=1100$ 产生CMB

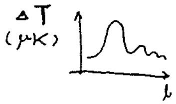
($H+\gamma \to p+e$ 是逆过程)

- 光子与电子是有多个的
  当其中有能量 E 大于 13.6eV，就能电离 H，此时光子与物质才透明了
- 极近黑体辐射，各向同性的 CMB 是 Big Bang 的证据
  早期的热平衡状态
  各向同性存在 Horizon Problem
  需要 Inflation 解决 / 或和平宇宙学原理
  所以，Hot Big Bang 预言 T~3K 的黑体谱
  但不一定是各向同性的

- 1992年，COBE卫星测CMB辐射



Perfect Black-body
$T=2.725K$



ΔT (μK)

第一次测到 density flunctuation
$\delta \sim 10^{-5}$ T

意义：$\Delta T(\Omega_b, \Omega_m, \Omega_\Lambda, H_0, A_s, n, \alpha, \tau, \Omega_K, \Omega_\nu \cdots)$

所以 ΔT 曲线可以约束宇宙学参数

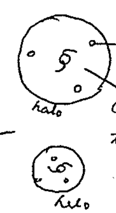
- 现在的 Cosmic Probes：(温度和密度的分布统计)
  CMB (T+p)
  LSS - large scale structure (galaxy survey
  Lyα forest
  weak lensing
  Lyα emitters, Lyman break emitters.
  21cm (radio) intensity mapping

- Cosmic inventory

Dark energy ~70%

matter 30%
- 25% darkmatter，不发光
- 4%　free H. He，在星际间，不发光
- 0.5%　star，发光
- 0.3%　neutrino
- 0.03%　heavy elements

不发光的物质并不一定是暗物质
大部分重子也不发光



star + ISM = galaxy
CGM (Circum-galactic) 不同一个halo
halo
温度较高 有可能发光
IGM
(H. He 离子 自由的，极电离，不发光)
(主要是 z>6，z>3 存在再电离，使中性的 H. He 变为电离的 H. He)

- CMB spectrum distortion
  CMB 偏离黑体谱的小的移动

- CMB的 polarization，不同偏振的功率谱 (fluctuation) {T, EE, B → BICEP2, Ali，与原初引力波有关

● Dark Matter ← 修改引力，可能显得很不自然

evidence
- rotation Curve
- bullet cluster (透镜得质量中心
  X-ray 得可见物质的中心

  lensing　Xray　lensing
- CMB power spectrum
  $\Omega_m \sim 0.3$
  Ω_b 由 BBN 给出为 ~0.05，因而应存在 $\Omega_{dm} \sim 0.25$
- Formation of galaxies at today
  $\delta \sim 10^{-5}$ 的原初涨落，如果在短时间形成非线性结构？
  若只有 baryon，形成引力井的时标太长 (与 c_{sound} 有关)，因而需要一种能更有效的聚集的物质 (Dark matter，速度更慢)

为什么说暗物质是粒子来组成的，而不是不发光重子来组成的？
↳ bullet cluster 是很好的证据，表明存在只有引力作用的物质

● Dark energy ← cosmology constant 是仅理论上允许的

evidence for $\Omega_\Lambda > 0$
- SNIa (1998)
  能反映宇宙的膨胀历史，$d_{c.a.L}(z)$
  即可以约束 Ω_Λ
- CMB (WMAP 2003年)

  θ_1 对 Ω_K、Ω_Λ 敏感
  给出 $\Omega_m \sim 0.3$
  $\Omega_K \lesssim 10^{-2}$
  约束 Ω_Λ
- BAO
  与膨胀历史有关
- 21cm intensity mapping

DE explanation
- Cosmological Constant {fine tuning problem　$\rho_{\text{vac energy from particle}} / \rho_\Lambda \sim 10^{125}$; coincidence problem　$\Omega_m \sim \Omega_\Lambda$
  如何解决真空能？使之接近为0但又不是0？
- Quintessence 暗能物质
- f(R), f(T) gravity
  修改广义相对论

ΛCDM 和宇宙的命运

宇宙会因Λ而使其不断膨胀下去吗？ 例如，如果人类处在 z=1，可能就看不到 Ω_Λ，对宇宙的命运的判断取决于 Ω_K

所以，对过去的回顾并不能保证对未来的预言。

Lecture 2 03/07/2018

Mathematical Framework

广义相对论是经典的引力理论，这里假定 $\hbar=c=k_B=1$

hw: Dodelson, Chapter 2, Ex(1+)2+3 不交

Errata: http://home.fnal.gov/~dodelson/errata.html

Ex 2. (2.9.1) $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

Ex 3. (c). $\frac{d^2x^i}{dt^2} = -\frac{1}{a}\delta^{ij}\frac{\partial\phi}{\partial x^j}$

Review of General Relativity (GR)

line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$

Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$

$\underset{\text{time}}{\mu = 0}, \underset{\text{space}}{1,2,3}$ $\quad i = 1,2,3$

in general situation $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$

1. Transformation $x^\mu \to x'^\mu$

$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}dx^\nu = R^\mu{}_\nu dx^\nu \quad , R^\mu{}_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu}$$

- 线元ds是不变的，因此 $ds'^2 = ds^2 \Rightarrow g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \overset{?}{=} g_{\mu\nu}(x)R^\mu{}_\rho R^\nu{}_\sigma$

Proof: $LHS = ds'^2 = g'_{\rho\sigma}(x')dx'^\rho dx'^\sigma = g'_{\rho\sigma}(x')R^\rho{}_\mu R^\sigma{}_\nu dx^\mu dx^\nu$

$RHS = ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$

注意 $\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x'^\rho}{\partial x^\kappa} = \delta^\mu{}_\kappa$

两边等号均成立，因而可得结论（可代入验证）

线元是不变的，因而保号，$ds^2 = \begin{cases} \text{space-like}, >0 & \Rightarrow \text{可选适当的变换，使 } dt=0 \\ \text{null (along a lightray)} = 0 & \Rightarrow \frac{d\vec{x}}{dt} = \vec{c} \\ \text{time-like}, <0 & \Rightarrow \text{可使 } d\vec{x} = 0 \end{cases}$

- Scalar $S(x)$ if $S'(x') = S(x)$

That means $S(x)$ is unchanged at a physical spacetime point. $x \to x'$

Covariant vector: $v_\mu \frac{\partial x^\mu}{\partial x'^\rho} = v'_\rho = S^\mu{}_\rho v_\mu$

Contravariant vector (e.g. dx^μ): $v'^\mu = R^\mu{}_\nu v^\nu$

$$S^\alpha{}_\mu = (R^{-1})^\alpha{}_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} \quad , \text{因为 } \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\mu{}_\nu$$

$\partial_\mu S \equiv \frac{\partial S}{\partial x^\mu}$ 一定是Covariant vector (Prove yourself)

Tensor
$$T'_{\mu_1\cdots\mu_m}{}^{\overset{\text{按contra变化}}{\nu_1\cdots\nu_n}} = (S^{\alpha_1}_{\mu_1}\cdots S^{\alpha_m}_{\mu_m})(R^{\nu_1}_{\beta_1}\cdots R^{\nu_n}_{\beta_n})T_{\alpha_1\cdots\alpha_m}{}^{\beta_1\cdots\beta_n}$$
按co变化

ex1. $\delta^\mu{}_\nu = \begin{cases} 1, & \mu=\nu \\ 0, & \mu\neq\nu \end{cases}$ is a tensor

$$\underset{\uparrow\ \text{左边分量}}{\delta^\mu{}_\nu}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\sigma} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\sigma} = \underset{\uparrow\ \text{右边分量}}{\delta^\rho{}_\sigma}$$

ex2. $g_{\mu\nu}$ is a tensor

$g^{\mu\lambda}g_{\lambda\nu} \equiv \delta^\mu{}_\nu \Rightarrow g^{\mu\lambda} = (g^{-1})^{\mu\lambda}$

So $g^{\mu\nu}$ is a tensor (Proof yourself)

ex3. If A^μ is a vector, so is $A_\mu \equiv A^\nu g_{\mu\nu}$

If A_μ is a vector, so is $A^\nu \equiv g^{\nu\mu}A_\mu$) the same is true inversely

If $T_\mu{}^\nu$ is a ~~vector~~ tensor, so is $g_{\mu\alpha}g^{\nu\beta}T^\alpha{}_\beta$ is a tensor.

So we can always use $g^{\mu\nu}$ ($g_{\rho\sigma}$) raising and lowering index.

2.

For free motion in Minkowski space $\frac{d^2x^i}{dt^2} = 0 \Rightarrow \frac{d\vec{A}}{dt} = 0$ $\left(\frac{dv^i}{dt} = 0\right)$

But in a general space $\partial_\nu A_\mu = 0 \not\Rightarrow \partial'_\nu A'_\mu = 0$ (so ∂A_μ is changed in different coords ...)

because $\partial'_\nu A'_\mu = S^\alpha{}_\nu \partial_\alpha = \frac{\partial x^\alpha}{\partial x'^\nu}\partial_\alpha\left(\frac{\partial x^\beta}{\partial x'^\mu}A_\beta\right)$ $\left(\frac{\partial x^\beta}{\partial x'^\mu}A_\beta\right)$

$$= \frac{\partial x^\alpha}{\partial x'^\nu}(\partial_\alpha A_\beta)\frac{\partial x^\beta}{\partial x'^\mu} + \frac{\partial x^\alpha}{\partial x'^\nu}A_\beta\frac{\partial^2 x^\beta}{\partial x^\alpha\partial x'^\mu}$$

$A_\beta\frac{\partial^2 x^\beta}{\partial x'^\mu\partial x'^\nu}$ → 多出了一项

We can define Covariant derivative

$$\nabla_\nu A_\mu \equiv \partial_\nu A_\mu - \underset{\text{affine connection 仿射连络}}{\Gamma^\lambda{}_{\nu\mu}}A_\lambda$$

$$\nabla'_\nu A'_\mu = S^\alpha_\nu S^\beta_\mu \partial_\alpha A_\beta + \frac{\partial^2 x^\sigma}{\partial x'^\mu\partial x'^\nu}A_\sigma - \Gamma'^\lambda{}_{\nu\mu}S^\sigma{}_\lambda A_\sigma$$

we want $RHS = S^\alpha_\nu S^\beta_\mu \nabla_\alpha A_\beta$

So $\Gamma'^\lambda{}_{\nu\mu} = R^\lambda{}_\sigma S^\alpha_\nu S^\beta_\mu \Gamma^\sigma{}_{\alpha\beta} + \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\nu\partial x'^\mu}$ 表明 $\Gamma^\lambda{}_{\nu\mu}$ 不是tensor，是为了使 $\nabla_\nu A_\mu$ 是tensor而引入的。

Note: (i) $\Gamma^\mu_{\alpha\beta}$ is "minimal" if it is symmetric in lower indices

Otherwise, $\frac{1}{2}(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) \equiv S^\mu_{\alpha\beta}$ is torsion tensor. (Proof yourself)

所以，$\Gamma^\mu_{\alpha\beta}$ 的反对称部分一定是tensor.

后面常用的 $\Gamma^\mu_{\alpha\beta}$ 都是对称的

(ii) Extension $\nabla_\alpha T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \partial_\alpha T_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} - \Gamma^\beta_{\alpha\mu_1}T_{\beta\mu_2\cdots\mu_m}{}^{\nu_1\cdots\nu_n} - \cdots$

$+ \Gamma^{\nu_1}_{\alpha\lambda}T_{\mu_1\cdots\mu_m}{}^{\lambda\nu_2\cdots\nu_n} + \cdots$

(iii) Formally

$$\nabla_\alpha (T^{\cdot\cdot}_{\cdot\cdot} + U^{\cdot\cdot}_{\cdot\cdot}) = \nabla_\alpha T^{\cdot\cdot}_{\cdot\cdot} + \nabla_\alpha U^{\cdot\cdot}_{\cdot\cdot}$$

$$\nabla_\alpha (T^{\cdot\cdot}_{\cdot\cdot} U^{\cdot\cdot}_{\cdot\cdot}) = (\nabla_\alpha T^{\cdot\cdot}_{\cdot\cdot}) U^{\cdot\cdot}_{\cdot\cdot} + T^{\cdot\cdot}_{\cdot\cdot} \nabla_\alpha U^{\cdot\cdot}_{\cdot\cdot}$$

(iv) $\nabla_\beta \nabla_\alpha A_\mu \neq \nabla_\alpha \nabla_\beta A_\mu$ (非常重要的性质)

Big Result: if ① the connection $\Gamma^\mu_{\alpha\beta}$ is symmetric (w.r.t. $\alpha\beta$)

② $\nabla_\alpha g_{\mu\nu} = 0$ (度规在同胚群意义上，平移后是否不变呢 $\forall \alpha\ \nabla^\alpha \nabla_\alpha g_{\mu\nu} = 0$)

Then there is a unique way to construct the covariant derivative, specifically

$$\Gamma^\beta_{\mu\nu} = \left\{ {\beta \atop \mu\nu} \right\} = \frac{1}{2} g^{\lambda\beta} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right)$$

Christoffel Symbol

Proof: $0 = \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} \cdots$

$0 = \nabla_\mu g_{\lambda\nu} = \partial_\mu g_{\lambda\nu} \cdots$ + 利用 $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$ 得到 (do by yourself)

$0 = \nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} \cdots$

所以，度规就完全决定了 $\Gamma^\beta_{\mu\nu}$!

3. Measure

$$\int d^4x\, \mathcal{L}(x)$$

↑

Lagrange density

$\mathcal{L}$ is covariant

而 $d^4x' = \det R\, d^4x$

并不是不变，因而改为 $\int d^4x (\cdots) \mathcal{L}(x) = \int d^4x \sqrt{g}\, \mathcal{L}(x)$

令 $G_{\mu\nu} = g_{\mu\nu}$，坐标变换下 $G \to G' = R^{-1} G (R^{-1})^T$

令 $\det G = g$. 则 $g' = \frac{g}{(\det R)^2}$，立即有 $\sqrt{g'}\, d^4x' = \sqrt{g}\, d^4x$

Extension of $\nabla_\mu g_{\alpha\beta} = 0$

(i) 如果存在场 A_μ，使 $\nabla_\mu g_{\alpha\beta} = A_\mu g_{\alpha\beta}$ (*1)

Weyl 最早引入此假设，希望统一引力和电磁 Gravity + EM

Conformal transformation $g'_{\alpha\beta}(x) = \lambda(x) g_{\alpha\beta}(x)$ 下，如果 A_μ' 的变化满足 共形变换

gauge transformation (类似于电磁 gauge)

$A_\mu' = A_\mu + \partial_\mu \lambda$，则定义 (*1) 是不变的

因为 $A'_\mu g'_{\alpha\beta} = \nabla_\mu g'_{\alpha\beta} = \nabla_\mu (\lambda g_{\alpha\beta}) = \partial_\mu \lambda g_{\alpha\beta} + \lambda \nabla_\mu g_{\alpha\beta} = (\partial_\mu \lambda) g_{\alpha\beta} + \lambda A_\mu g_{\alpha\beta}$

$\cdots = (A_\mu + \partial_\mu \lambda) g'_{\alpha\beta}$

4. Curvature 曲率

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\beta = - R^\alpha{}_{\beta\mu\nu} A_\alpha$$

如它是个 tensor 的非对称部分

$R^\alpha{}_{\beta\mu\nu}$ 只依赖于几何，与作用的 vector A_α 无关

经计算

$$\nabla_\mu \nabla_\nu A_\beta = \partial_\mu (\nabla_\nu A_\beta) - \Gamma^\sigma_{\mu\nu} \nabla_\sigma A_\beta - \Gamma^\sigma_{\mu\beta} \nabla_\nu A_\sigma$$

$\partial_\mu (\nabla_\nu A_\beta) = \partial_\mu \partial_\nu A_\beta - \underset{①}{\partial_\mu (\Gamma^\sigma_{\nu\beta} A_\sigma)}$ 对称部分 不考虑

$\Gamma^\sigma_{\mu\beta} \nabla_\nu A_\sigma$ gives $- \underset{②}{\Gamma^\sigma_{\mu\beta} \partial_\nu A_\sigma} + \underset{③}{\Gamma^\sigma_{\mu\beta} \Gamma^\rho_{\nu\sigma} A_\rho}$

① + ② $= - \partial_\mu \Gamma^\sigma_{\nu\beta} A_\sigma$ + symmetric $\mu\nu$

最终可以得到

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\beta} - \Gamma^\alpha_{\nu\rho} \Gamma^\rho_{\mu\beta}$$

Properties

$R_{\alpha\beta\gamma\delta} = - R_{\alpha\beta\delta\gamma}$

$R_{\alpha\beta\gamma\delta} = - R_{\beta\alpha\gamma\delta}$

$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$

$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$ (对称性)

(Γ 依赖于坐标系，所以 $R^\alpha{}_{\beta\mu\nu}$ 的表达式依赖于坐标系；但 $R_{\alpha\beta\gamma\delta}$ 的性质不依赖于坐标系，是 tensor 等式，因而可以选择任何坐标系证明)

Proof:

Choose the "geodesic coordinates", so that $\Gamma^\alpha_{\beta\gamma}(x_0) = 0$ at point x_0 (一定存在的)

$$\Rightarrow R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(\partial_\alpha \partial_\delta g_{\beta\gamma} + \partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma} \right)$$

可以在此坐标系下证明 Properties

Properties (Bianchi identity)

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0$$

Proof hint:

To prove this, use Lie bracket identity

$$[\nabla_\alpha [\nabla_\beta \nabla_\gamma]] + [\nabla_\beta [\nabla_\gamma \nabla_\alpha]] + [\nabla_\gamma [\nabla_\alpha \nabla_\beta]] = 0$$

5. 构造 $\mathcal{L}$ 的方法 $S = \int d^4x \sqrt{g}\, \mathcal{L}$

$\mathcal{L}$ must be covariant

- Scalar field

$$S = \int d^4x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \underbrace{\partial_\mu \phi\, \partial_\nu \phi}_{\text{动能部分}} - \underbrace{V(\phi)}_{\text{potential}} \right)$$

- Transverse vector field

Transverse 指 $A_\mu \to A_\mu + \partial_\mu \chi$ 有规范不变性. 得到的是无质量场. 纵分量为0.

In Minkowski space, e.m. potential A_μ, e.m. tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

In GR, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha - \partial_\nu A_\mu + \Gamma^\alpha_{\nu\mu} A_\alpha = \partial_\mu A_\nu - \partial_\nu A_\mu$

仍是相同形式

$$S = -\frac{1}{4}\int d^4x\sqrt{g}\; g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = -\frac{1}{4}\int d^4x\sqrt{g}\,F^{\alpha\beta}F_{\alpha\beta}$$ (弯曲空间的电磁场)

注意到 $F_{\mu\nu}$ 的反称性. $F_{\mu\nu}F^{\nu\mu}$ 是不可约的

- Longitudinal vector field A^{μ}

$$\int d^4x\sqrt{g}\,\underbrace{\left(\nabla_{\mu}A^{\mu}\right)}_{\text{直接contraction 得 scalar}}\left(\nabla_{\nu}A^{\nu}\right)$$ 此项是对A的二阶导数

Symmetric under $A^{\alpha}\to A^{\alpha}+\epsilon^{\alpha\beta\gamma\delta}\partial_{\beta}\Lambda_{\gamma\delta}$

- Gravity

$$\sim\int d^4x\sqrt{g}\,(R+\lambda)$$

R: 对g作二阶导的 Scalar; λ: 常数

the only terms up to 2nd order derivative of $g_{\mu\nu}$

$R = g^{\alpha\gamma}g^{\beta\delta}R_{\alpha\beta\gamma\delta}$

改为 R^2. $\frac{1}{R}$ 则为 $f(R)$ gravity

↓

同样有 $f(T)$ gravity

若 $S=\int d^4x\sqrt{g}\,R$, 代入 Euler-Lagrange eq 可以得到 Einstein Equations

注意: $\dfrac{\delta S}{\delta g^{\alpha\beta}} = \left(R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R\right)\sqrt{g}$

得: $R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta} = \frac{1}{2\kappa}T_{\alpha\beta}$

In Newtonian limit $T_{00}=\rho$, $\dfrac{\partial T_{0i}}{\partial x^i}\to 0$ $\Big\}$ $R_{00}=\frac{1}{4\kappa}\rho$

$$\approx \partial_{\gamma}\Gamma^{\gamma}_{00}-\partial_0\Gamma^{\gamma}_{0\gamma}+o(\Gamma^2) = \frac{1}{2}\nabla^2 g_{00}$$

that is $\nabla^2 g_{00} = \frac{1}{2\kappa}\rho$

Compared with Newtonian gravity

$$\nabla^2\phi = 4\pi G\rho$$

we obtain $g_{00}=1+2\phi$

$(4\kappa)^{-1}=4\pi G \Rightarrow \kappa=\frac{1}{16\pi G}$

Define $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$

则 $G_{\mu\nu}=8\pi G T_{\mu\nu}$ (Einstein Field Equations)

由物质分布决定几何

6. 广义相对论的第二个假设 Geodesic motion of free particle

Given geometry $g_{\mu\nu}$

经典力学 $\frac{d}{dt}\left(\frac{dx^i}{dt}\right)=0 \Rightarrow \frac{d^2x^{u}}{ds^2}=0$

↑ 加入时空间因素

改写到 covariant derivative

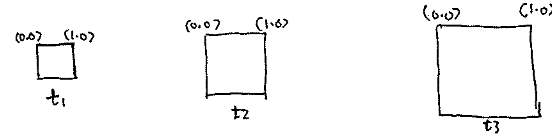
$\frac{D}{Ds}\frac{Dx^{u}}{Ds}=0$ $\quad \frac{Dx^{\mu}}{Ds}=\frac{dx^{u}}{ds}=$ Vector

平移 vector 还要考虑连续 $\frac{D}{Ds}\left(\frac{dx^{\mu}}{ds}\right)=\frac{d^2x^{\mu}}{ds^2}+\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0$ (geodesic equation)

03.14 | 2018 week3 物理宇宙学 6B301

Mathematical Framework

- Space time geometry of the expanding Universe.



	t_1	t_2	t_3
Comoving distance	1	1	1
Physical distance	$a(t_1)$	$a(t_2)$	$a(t_3)$

$ds^2=-c^2dt^2+d\vec{x}^2$ Minkowski space (static flat local)

↓

$ds^2=-c^2dt^2+a^2(t)d\vec{x}^2$ (expanding, flat, global coordinates)

$c=1$ in natural units

↓

$ds^2=-c^2dt^2+a^2(t)ds^2_{3D}$

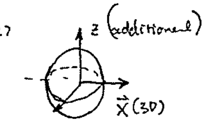
$a(t)\sim$ Robertson-Walker scale factor

$ds_{3D}\sim$ line element in Comoving frame

Require: the Universe is isotropic and homogenous

only allowed (1) a 3D flat Euclidean space, $ds^2_{3D}=d\vec{x}^2$

(2)



a 3D spherical surface in a 4D Euclidean space ← 数学上的概念. 物理上, 并没有额外的维度存在

$ds^2_{3D}=d\vec{x}^2+dz^2$

and $\vec{x}^2+z^2=R^2$

Rescale coordinates $\vec{x}'=R^{-1}\vec{x}$, $z'=R^{-1}z$ so that $ds^2_{3D}=R^2\left(d\vec{x}'^2+dz'^2\right)$ 可以再次吸收到 $a(t)$ 里面

We can absorb R into the definition of a, Now, drop the primes $z'\to z$, $\vec{x}'\to\vec{x}$

$ds^2_{3D}=d\vec{x}^2+dz^2$

and $\vec{x}^2+z^2=1$

$dz'^2+d\vec{x}'^2=$ 4维空间的球面

$z'^2+\vec{x}'^2=1$

Sphere is invariant under SO(4), rotation group in 4D

四维的旋转可以有若干分量，例如：围绕z，3D rotation $\vec{x} \to R_{3D}\vec{x}$

$$\left.\begin{array}{l} \vec{x}^2 \to (R_{3D}\vec{x})^2 = \vec{x}^2 \\ z \text{ unchanged} \end{array}\right\} \Rightarrow \text{isotropic}$$

$$\left.\begin{array}{l} \text{3D translation } \vec{x} \to \vec{x} + \vec{\epsilon} \\ z \text{ unchanged} \end{array}\right\} \text{homogeneous.}$$

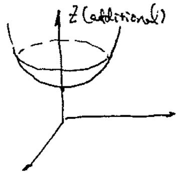
(3) a hyperspherical surface in 4-D pseudo-Euclidean space

$$ds_{3D}^2 = d\vec{x}^2 - dz^2$$

pseudo-Euclidean

$z^2 - \vec{x}^2 = 1$ can also be R, absorbed in $a(t)$

hypersphere



Invariant under SO(3,1) 4D pseudo-rotations

- $$\left.\begin{array}{l} \text{3D rotation } \vec{x} \to R_{3D}\vec{x} \\ \vec{x}^2 = (R_{3D}\vec{x})^2 \\ z \text{ unchanged} \end{array}\right\} \Rightarrow \text{isotropic}$$
- $$\left.\begin{array}{l} \text{3D translation } \vec{x} \to \vec{x} + \vec{\epsilon} \\ \text{(non-flat) } z \text{ unchanged}_{\text{any}} \end{array}\right\} \Rightarrow \text{homogeneous}$$

just like Lorentz transformation, but with z instead of time

(2)(3)

These 2 are maximally symmetric, homogeneous + isotropic metric in space.

$$ds_{3D}^2 = d\vec{x}^2 \pm dz^2$$

$$\downarrow \; z^2 \pm \vec{x}^2 = 1 \quad \Rightarrow \quad \begin{array}{l} 2dz + \vec{x}\cdot d\vec{x} = 0 \\ 2dz = -\vec{x}\cdot d\vec{x} \end{array}$$

$$ds_{3D}^2 = d\vec{x}^2 \pm \frac{(\vec{x}\cdot d\vec{x})^2}{1 \mp \vec{x}^2}$$

$$= d\vec{x}^2 + K\frac{(\vec{x}\cdot d\vec{x})^2}{1 - K\vec{x}^2} \qquad \text{其中 } K = \begin{cases} 1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ 0 & \text{flat (Euclidean)} \end{cases}$$

Space-time metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(d\vec{x}^2 + K\frac{(\vec{x}\cdot d\vec{x})^2}{1 - K\vec{x}^2}\right)$$

上面的形式略去了曲率半径R（吸收进了$a^2(t)$）

若保留，则 $d\vec{x} \to R\,d\vec{x}$

$K \to R^{-2}K$

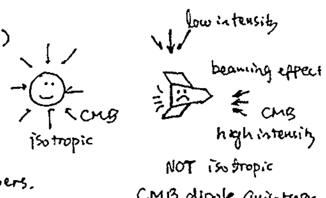
Theorem: FLRW metric is the unique metric (up to coordinate transformation) if the Universe appears spatially homogeneous and isotropic to a set of freely falling observers.

(See Weinberg's book, old version)

Question: what is the frame of reference we use here?

注：宇宙的均匀各向同性是对空间而言的，与参考系有关。

(只是，在一个特殊的参考系下，metric的形式才会显得这么对称.)



The above assumptions are made to a "typical" freely falling observers.

those that move with the average velocity of typical galaxies in the neighborhood.

We can use CMB dipole anisotropy to define a 'CMB' frame. i.e. cosmological frame.

Cosmological frame的存在似乎和牛顿的绝对时空观是一致的！

物理定律总是协变的 / 看起来好像牛顿

在所有参考系下，物理定律是协变的

$\vec{X}$ – quasi-Cartesian coordinate

(但不是Euclidean)

作变换 $\vec{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

则有 $d\vec{x}^2 = dr^2 + r^2 d\Omega^2$, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$\vec{x}\cdot\vec{x} = r^2$, $\frac{\vec{x}\cdot d\vec{x}}{r\cdot dr} = 1$, 则 $ds_{3D}^2 = (r^2 + r^2 d\Omega^2) + K\frac{(r\,dr)^2}{1 - Kr^2} = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2$

最后：

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right)$$

Friedmann – Lemaître – Robertson – Walker (FLRW) metric

($K=0$: 1922; 1927; $K\neq 0$: 1930s)

- Topology of the Universe

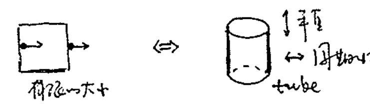
FLRW metric describe the geometry of the Universe, but topology can be different from the simplest.

Simpleast:

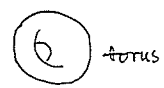
- $K = 1$ — spherical surface in 4D $\Rightarrow$ finite but no boundary
- $K = 0$ — flat $\Rightarrow$ infinite, no boundary
- $K = -1$ — hyperspherical surface in 4D $\Rightarrow$ infinite

在没有暗能量时，K就完全决定了宇宙的命运.

However, in K=0 flat Universe, may have finite space with periodicity.



可以将宇宙沿几个方向卷折起来 warped along some directions



两个方向折叠也可以

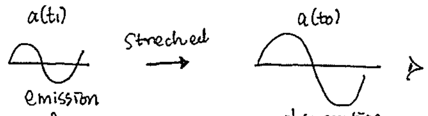
- 可以用 general genus number along $\vec{L}_1, \vec{L}_2, \vec{L}_3$ (are fixed, non-coplanar vectors), identify $\vec{X}$ with $\vec{X} + n_1\vec{L}_1 + n_2\vec{L}_2 + n_3\vec{L}_3$, n_1, n_2, n_3 = integers

  The Volume is $V = a^3\vec{L}_1\cdot(\vec{L}_2 \times \vec{L}_3)$

  All above, K=0, i.e. flat

- 但是，没有这种迹象（CMB没有观察到有两边完全关联），所以 $|\vec{L}_i| \geq 10^{10}$ ly.

- Cosmological Redshift



emission λ_{em} $\nu_{em} = \frac{1}{\delta t_1}$ observation λ_{obs} $\nu_{obs} = \frac{1}{\delta t_0}$

取决于 $\frac{\lambda_{em}}{a(t_1)} = \frac{\lambda_{obs}}{a(t_0)} \rightarrow 1+z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{1}{a(t_1)}$

数学上 $\frac{c\delta t_1}{a(t_1)} = \frac{c\delta t_0}{a(t_0)}$ (见下)，也可以导出 $\frac{\lambda_{em}}{\lambda_{obs}} = \frac{\nu_{obs}}{\nu_{em}} = \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)}$

对光来讲. $0 = ds^2 = -c\,dt^2 + a^2(t)\frac{dr^2}{1-Kr^2}$ $(d\theta = d\phi = 0)$

$$\int_{t_1}^{t_0}\frac{c\,dt}{a(t)} = \int_{r(t_1)}^{r(t_0)}\frac{dr}{\sqrt{1-Kr^2}}$$

$$\delta(LHS) = \frac{c\delta t_1}{a(t_1)} - \frac{c\delta t_0}{a(t_0)} = \delta(RHS) = 0$$

由于 $t \leftrightarrow a(t) \leftrightarrow z(a)$，所以，可以将 z 看作时间变量

cosmological redshift as a time variant.

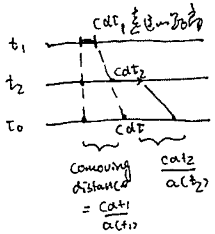
$t_0 = 13.7\ \text{Gyr}$ $a = \frac{1}{1100}$ $z = 1100$

$t = 0$ $a = 1$ $z = 0$

后面还可将 τ 作为时间变量.

- Distance

1. Comoving distance between a distant emitter and us (observer)

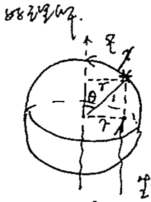


因而, comoving distance $\chi = \int_{t(a)}^{t_0}\frac{c\,dt}{a(t)} = \int_r^0 -\frac{dr'}{\sqrt{1-Kr'^2}} = \int_0^r\frac{dr'}{\sqrt{1-Kr'^2}} = \begin{cases}\sin^{-1} r & K=+1\\ \sinh^{-1} r & K=-1\\ r. & K=0\end{cases}$

Comoving (line-of-sight) distance, L.O.S. distance

反过来. $\underbrace{r = S_k(\chi)}_{\text{Comoving (transverse) distance}} = \begin{cases}\sin\chi, & K=+1\\ \sinh\chi, & K=-1\\ \chi. & K=0.\end{cases}$ χ 可以在已知 $a(t)$ 的情况下计算出. 由此可以得到其他量

说明
- 对 flat=0 的情况, 很容易理解.
- 对 K=1 的情况



polar angle $\theta = \sin^{-1} r = \chi$

坐标距离 $r \neq$ 观测到的 comoving distance χ

- 对 K=-1 的情况, 不可以直观

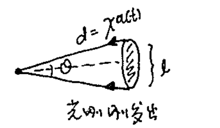
$$\frac{da}{a} = -\frac{dz}{1+z} = -a\,dz$$

因为 $t \leftrightarrow a \leftrightarrow z$, 因为 $\frac{c\,dt}{a} = \frac{c}{a}\frac{da}{\dot a} = \frac{c}{a^2}\frac{da}{H(a)}$, 所以

$$\chi = \int_{t(a)}^{t_0}\frac{c\,dt'}{a(t')} = \int_a^1\frac{c\,da'}{a'^2H(a')} = \int_0^z\frac{c\,dz'}{H(z')}$$

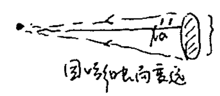
use friedmann equation. $H(z')$ is known, So we obtain $\chi(z)$

2. Angular Diameter Distance



define $d_A = l/\theta$ (l → physical size, θ → observed angular size)

由于宇宙的膨胀, θ会变小, 但观测到的θ由发出时的 d_{phy} 和 l_{phy} 决定

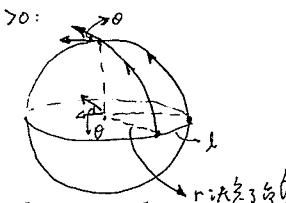


K=0 :

$$d_A = \frac{l}{\theta_{obs}} = \frac{l}{l/a\chi} = a\chi$$

← 发射时 ↑ 发射时

K>0 :



$$d_A = \frac{l}{\theta_{obs}} = \frac{l}{l/ar} = ar = aS_K(\chi)$$

数学上. $ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2d\Omega^2\right]$ line element

对于这个物体, $dr = 0.\ d\phi = 0,\ dt = 0$

则 $ds^2 = a^2(t)r^2d\theta^2 = dl^2$ 表示物体的长度^2

所以角直径距离 $d_A = \frac{dl}{d\theta} = ar$

3. Luminosity Distance

In flat, static space

flux $F = \frac{L}{4\pi d^2}$

So in general space, we define $F_{obs} = \frac{L_{intrisic}}{4\pi d_L^2}$ (d_L ← luminosity distance), $L = \frac{h\nu\Delta N}{\Delta t}$ (hν ← 光子能量, ΔN ← 光子数目)

接收的 time delay

$$\delta t_{em} = \delta t_{obs}\,a(t)$$

光子也红移 $\nu_{em} = \frac{\nu_{obs}}{a(t)}$

所以 $L_{em} = \frac{L_{obs}}{a^2(t)}$, 因而 $F_{obs} = \frac{L_{obs}}{4\pi r^2} = \frac{L_{em}}{1/a^2 \cdot 4\pi r^2}$

为什么是 r ?.
观测到的, 也是在切平面, 按投影距离分布的量
另外, 观测到时 a=1, 这和 d_A 考虑的时刻不同

故定义如下 $d_L = \frac{r}{a(t)} = \frac{S_k(\chi)}{a(t)}$

4. Conformal time

$\eta = \int_0^t\frac{c\,dt}{a(t)}\cdot\frac{1}{c} = \int_0^t\frac{dt}{a(t)}$, 所以 $d\eta = \frac{dt}{a}$. 可以将 FLRW metric 写为 $ds^2 = a^2(t)\left(-d\eta^2 + d\Sigma_{3D}^2\right)$

0321/2018 week4 物理宇宙学 6B301

HW 2
Dodelson Chapter 2
Ex 6, 8.
替换 Ex 8 a) Eq. (2.97)
$\Gamma^i{}_{jk} = \frac{1}{2} g^{il}(\cdots - g_{jk,l})$

Mathematical Framework - II

FLRW metric $g_{\mu\nu} \xrightarrow{GR}$ Friedmann eq.

RW metric:

$$ds^2 = -dt^2 + a^2(t)\left(d\vec{x}^2 + \frac{K(\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2}\right) \quad (x_i dx^i x_j dx^j)$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

in a quasi-Cartesian coordinates
(比 r, θ, ϕ 略去(更一点))

where $g_{00} = -1$, $g_{0i} = 0$, $g_{ij} = a^2(t)\left(\delta_{ij} + \frac{K x_i x_j}{1-K\vec{x}^2}\right) = a^2(t)\tilde{g}_{ij}$ ← 与时间无关

and define $\tilde{g}_{ij} = \delta_{ij} + \frac{K x_i x_j}{1-K\vec{x}^2}$ → 并不是完全的对角化

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} -1 & \\ & a^{-2}(\tilde{g}_{ij}^{-1}) \end{pmatrix}$$

Connection & Ricci tensor:

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}\right)$$

非零项有

$$\Gamma^0{}_{00} = \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0$$

$$\Gamma^0{}_{ij} = \frac{1}{2}(\overset{=0}{g_{0i,j}} + \overset{=0}{g_{0j,i}} - g_{ij,0}) = a\dot{a}\tilde{g}_{ij} \qquad A_{,\alpha} = \frac{\partial A}{\partial x^\alpha}$$

$$\Gamma^i{}_{0j} = \frac{1}{2} g^{il}(\underset{0}{g_{l0,j}} + g_{lj,0} - \underset{0}{g_{0j,l}}) = \frac{\dot{a}}{a}\delta_{ij}$$

$$\Gamma^i{}_{jl} = \frac{1}{2}\tilde{g}^{im}(\tilde{g}_{jm,l} + \tilde{g}_{lm,j} - \tilde{g}_{jl,m}) \equiv \tilde{\Gamma}^i{}_{jl} = K\tilde{g}_{jl}x^i$$

平直空间为0. 对于…空间 …, 以及 $\Gamma^0{}_{ij}$

$\tilde{g}_{ij}$ and $\tilde{\Gamma}^i{}_{jl}$ are purely spatial metric and connection, respectively.

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha}$$

可以用 Mathematica 计算
但对于张量计算

$$R_{00} = \underset{0}{\Gamma^\alpha{}_{00,\alpha}} - \Gamma^\alpha{}_{0\alpha,0} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{00} - \Gamma^\alpha{}_{\beta 0}\Gamma^\beta{}_{0\alpha}$$

$$= -\frac{\partial}{\partial t}\Gamma^i{}_{0i} + \Gamma^i{}_{0j}\Gamma^j{}_{0i} = 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + 3\left(\frac{\dot{a}}{a}\right)^2 = 3\frac{\ddot{a}}{a}$$

$$R_{ij} = \tilde{R}_{ij} - 2\dot{a}^2\tilde{g}_{ij} - a\ddot{a}\tilde{g}_{ij} = -[2K + 2\dot{a}^2 + a\ddot{a}]\tilde{g}_{ij}$$

$\tilde{R}_{ij}$ are purely spatial Ricci tensor.

$\tilde{R}_{ij} = -2K\tilde{g}_{ij}$ (在 $\vec{x}=0$ 处计算 $\tilde{R}_{ij} = -2K\delta_{ij}$ (不协变) $\to -2K\tilde{g}_{ij}$ (协变))

Einstein Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \iff R_{\mu\nu} = -8\pi G S_{\mu\nu}$$

(S应该没有负号)

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda$$

In flat space ($K=0$), $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & pa^2 & & 0 \\ & 0 & pa^2 & \\ & & & pa^2 \end{pmatrix}$ (perfect, isotropic fluid)

ρ: density; p: pressure; $a^2 p\delta_{ij}$

In curved space ($K \neq 0$): $T_{00} = \rho$, $T_{i0} = 0$, $T_{ij} = a^2 p\tilde{g}_{ij} = p g_{ij}$, $T^\mu{}_\nu = g^{\mu\alpha}T_{\alpha\nu} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$

$$S_{ij} = T_{ij} - \frac{1}{2}g_{ij}T^\lambda{}_\lambda = T_{ij} - \frac{1}{2}\tilde{g}_{ij}a^2(T^k{}_k + T^0{}_0) = a^2 p\tilde{g}_{ij} - \frac{1}{2}a^2\tilde{g}_{ij}(3p - \rho)$$

$$= \frac{1}{2}(\rho - p)a^2\tilde{g}_{ij}$$

$$S_{00} = T_{00} - \frac{1}{2}g_{00}T^\lambda{}_\lambda = \rho + \frac{1}{2}(3p - \rho) = \frac{1}{2}(\rho + 3p)$$

$$S_{0i} = 0$$

Now substitute the $S_{\mu\nu}$, $R_{\mu\nu}$ into Einstein equation, we have $R_{\mu\nu} = -8\pi G S_{\mu\nu}$

(i,j): $-\frac{2K}{a^2} - \frac{2\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} = -4\pi G(\rho - p)$

$(0,0)$: $\frac{3\ddot{a}}{a} = -4\pi G(\rho + 3p)$

$(i,j)\times 3 + (0,0) \Rightarrow \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G\rho}{3}$ (Friedmann eq.)

(Friedmann) combine (i,j), 消去K $\Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$ (Energy Conservation)

$\iff a^{-3}\frac{d}{dt}(\rho a^3) = -3\frac{\dot{a}}{a}p$

$\iff \frac{d}{dt}(\rho a^3) = -p\frac{d}{dt}(a^3)$

$\overset{\times V_0}{\iff} \frac{d}{dt}(\rho V) = -p\frac{d}{dt}V$ $\quad$ $V_0 \sim$ comoving, $V \sim$ physical

$\iff dE = -p\,dV$ (1st law of Thermodynamics)

Note that for the energy momentum tensor

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

it has a conservation law

$$0 = T^\mu{}_{\nu;\mu} \xrightarrow[\text{space}]{\text{Minkowski}} \partial_\mu T^\mu{}_\nu = 0$$

$$= \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu}T^\mu{}_\alpha$$

For matter (baryon, dark matter)

$p = 0$, $\frac{d}{dt}(\rho_m a^3) = 0 \Rightarrow \rho_m \propto a^{-3}$

non-relativistic particles; 总的质量守恒

相当于总质量守恒, $V \propto a^3$

$\rho_m = \frac{M}{V} \propto a^{-3}$

Note: $p=0$ 条件必须在非相对论条件下才能成立 $p/\rho \ll 1$

$p = nkT$

$\frac{p}{\rho} = \frac{nkT}{nmc^2} = \frac{kT}{mc^2}$

mc^2: 511 keV $\sim 511\times 10^3 \times 10^4$ K $\sim 10^{10}$ K

所以 T 非常高时, 电子不满足非相对论, 要当成辐射

For radiation (relativistic particles)

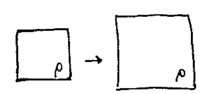
$p = \frac{1}{3}\rho$ (valid for Bosons (e.g. photon) and Fermions (e.g. electron, protons, neutrino))

$$\frac{d\rho_r}{dt} + 4\frac{\dot{a}}{a}\rho_r = 0 \Rightarrow a^{-4}\frac{d}{dt}(\rho_r a^4) = 0 \Rightarrow \rho_r \propto a^{-4}$$

相当于 $\varepsilon \propto a^{-1}$

$\rho_r \propto \frac{\varepsilon}{V} \propto a^{-4}$

For vacuum energy



vacuum energy
体积的增加不改变能量密度

$d(\rho V) = -p\,dV \to$ 立即有 $p = -\rho$ $\xrightarrow{\text{代入 Energy conservation}}$ 立即有 $\dot{\rho}_\Lambda = 0$

($d(\rho V) = \rho\,dV$)

将上面三种形式的能量代入 Friedmann equation $\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G\rho}{3}$

定义 critical density $\rho_{cr} = \frac{3H_0^2}{8\pi G} \approx 1.878\times 10^{-29} h^2\,\mathrm{g/cm^3}$

H_0 → Hubble param. at today $H_0 = 100\,\mathrm{km/s/Mpc}\cdot h$; 而 Planck 常数记为 h_{pl}

, 改写 Friedmann eq 为由 Hubble param $H(t) = \frac{1}{a}\frac{da}{dt}$

表达的形式:

$$\left(\frac{H}{H_0}\right)^2 + \frac{K}{(aH_0)^2} = \frac{\rho}{\rho_{cr}} = \Omega_m a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda$$

默认 $a_0 = 1$, 否则记为 $\frac{a}{a_0}$

曲率的贡献不能忽略, 可以类比 $\rho_{m,R,\Lambda}$

定义曲率的能量密度等效 $\Omega_K = \frac{-K}{(a_0H_0)^2}$ (curvature density)

由于后面把当今曲率的值设定为a. 这里要改写为 $\Omega_K = \frac{-K}{(a_0R)^2H_0^2}$, $K \to KR^{-2}$

形式上已一另定义 $\Omega_m = \frac{\rho_{m,0}}{\rho_{cr}}$, $\Omega_R = \frac{\rho_{R,0}}{\rho_{cr}}$, $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}}$，因 $\rho(t) = \rho_{m,0} a^{-3} + \rho_{R,0} a^{-4} + \rho_\Lambda$，所以各值用"0"的写为

$\rho(t) = \rho_{cr}(\Omega_m a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda)$，就得到上面的宇宙方程结论

由 RW + GR 决定 $\sim$ $$\left(\frac{H}{H_0}\right)^2 = \Omega_m \underbrace{\overbrace{a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda}^{\text{matter density}}}_{\text{Cosmic inventory}} + \Omega_K a^{-2}$$

由曲率决定演化形式

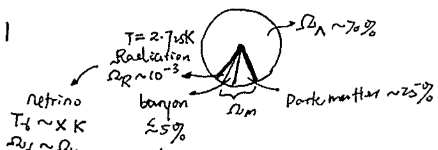
By definition $a(t=t_0)=1$, $H(t=t_0)=H_0$ so we have

$$\Omega_m + \Omega_R + \Omega_\Lambda + \Omega_K = 1$$

Note: 上面的归一关系不是自然的结果，而是合理的定义的结果（恰当选择 H_0, ρ_{cr} 的定义）；

Cosmic Pie

$|\Omega_K| \lesssim 10^{-2}$



Inter-galactic medium
Inter-stellar medium $\Big\}$ invisible ordinary matter (hydrogen) 显然占主导 ~4%

Stars ~0.5%

Helium ~0.25%

neutrino 属于 radiation

Summary: $$\left(\frac{H(t)}{H_0}\right)^2 = \text{Cosmic inventory} = \sum_i \Omega_i a^{-3-3w}$$

其中 w 是状态方程的系数 $P = w\rho$，决定演化方式 $\frac{d\rho}{dt} + \frac{3\dot{a}}{a}(\rho+p) = 0$

$\rho \propto a^{-3-3w}$

$$w = \begin{cases} 0, & \text{matter} \\ 1/3, & \text{radiation} \\ -1, & \text{vacuum energy} \\ <0, & \text{dark energy} \end{cases}$$

$\Big\{$ Radiation 演化快，在早期很主导
Ω_Λ 在宇宙早期无用，现在占主导。Ω_m 在早一些时主导
Ω_K 一直忽略不计

Distance　t-χ-z-a 之间的关系

Define. $H(z) = H_0 E(z)$, where $E(z) = \left[\Omega_\Lambda + \Omega_m (1+z)^3 + \Omega_R (1+z)^4 + \Omega_K (1+z)^2\right]^{1/2}$

$$\approx \left[\Omega_\Lambda + \Omega_m (1+z)^3\right]^{1/2} \quad (\text{for matter dominated stage})$$

Comoving LOS distance $$\chi(z) = \int_t^{t_0} \frac{c\,dt'}{a(t')} = \int_0^z \frac{c\,dz'}{H(z')} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')} = D_H \int_0^z \frac{dz'}{E(z')}$$

where $D_H = \frac{c}{H_0} \approx 3000\ h^{-1}\text{Mpc}$　($c \to 3\times10^5$ km/s; H_0: 100 h km/s/Mpc)

Comoving transverse distance $$S_K(\chi) = \begin{cases} \sin\chi & K=1 \\ \sinh\chi & K=-1 \\ \chi & K=0 \end{cases} = r$$ 投影到三维 (4d→3d)

Angular diameter distance $d_A = a S_K(\chi)$

Luminosity distance $d_L = \frac{1}{a} S_K(\chi)$

Time (Age of Universe)　t-a-z

$$\frac{da}{a} = -\frac{dz}{1+z}, \quad da = a^2(-dz)$$

$$t = \int_0^t dt' = \int_0^a \frac{da'}{\dot{a}'} = -\int_\infty^z \frac{dz'}{(1+z')H(z')} = \int_0^a \frac{da'}{a' H(a')} = \frac{1}{H_0}\int_0^{a = \frac{1}{1+z}} \frac{da'}{a' E(a')}$$

这里有无穷积分，数值上不好用时，用 a 的积分可能更好

注意：这里的 χ 是发出后到我们的距离，但 t 是宇宙开始到某一时候(a/z)的时间。

look-back time: the time between the emission and today

age　look back time
0　$t(z)$　$z=0$　t

$$t_{lb} = \int_t^{t_0} dt' = \int_z^0 \frac{-dz'}{(1+z')H(z')} = \int_0^z \frac{dz'}{(1+z')H(z')} = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')E(z')}$$

少数情况下存在解析解

- Matter dominated Universe $$t = \frac{1}{H_0}\int_0^a \frac{da'}{a'\sqrt{\Omega_m a'^{-3}}} \propto a^{3/2}$$ $$= \frac{1}{H_0\sqrt{\Omega_m}} \frac{2}{3} a^{3/2}$$
  where $\Omega_m = 1$, and at today, $t_0 = \frac{1}{H_0}\frac{2}{3} \simeq 6.52\times10^9\ h^{-1}\text{yr}$
- Radiation dominated Universe $t \propto a^2$ (or $a \propto \sqrt{t}$) 有不同 scaling law
  $t_0 = \frac{1}{2}\frac{1}{H_0}$
- In any case, $t_0 \sim \frac{1}{H_0}$. 所以 $\frac{1}{H_0}$ 经常被称为 Hubble time.

Lecture 5　0328/2018　物理宇宙学　GB301

HW Dodelson chapter 2 Ex 11 (b) $z_{decouple} = 1100$

Cosmological Constant and Vacuum Energy

- Vacuum Energy $p = -\rho_V$
  $$T^\mu{}_\nu = \begin{pmatrix} -\rho_V & \\ & p\,\delta_{ij} \end{pmatrix} = -\rho_V \delta^\mu_\nu$$
  $$T_{\mu\nu} = -\rho_V g_{\mu\nu}$$
  $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (T^{M}_{\mu\nu} + T^{(V)}_{\mu\nu})$$ (matter and radiation)
  $$\Leftrightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 8\pi G \rho_V g_{\mu\nu} = -8\pi G T^M_{\mu\nu}$$

Einstein equation

varying $g_{\mu\nu}$: $\delta S = (\quad)\delta g_{\mu\nu}$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T^M_{\mu\nu}$$

$(\Lambda = 8\pi G \rho_\Lambda)$

Cosmological Constant Λ

$$S = \int d^4x \sqrt{g}\,(R + \Lambda)$$

是 g 的 2 阶导，是标量，无自由度

真空能 ρ_Λ 和宇宙学常数是等价的
V.E.

引入新物质的结果　修改引力的结果，对应经典的真空能

$\updownarrow$ fine tuning problem

没有物理意义　$\sum_{\omega=0}^{\infty} \hbar\omega \sim \rho_V \to \infty$（需引入截断的量子真空能）

matter $\Leftrightarrow$ geometry
(energy gravitate)

真空能的总量对宇宙没有意义（没有绝对意义）

但存在引力效应 $a(t) = a_0 e^{Ht}$ if Λ dominate
（有绝对意义）$= a_0 e^{\sqrt{\Omega_\Lambda} H_0 t}$ (inflation)

- Einstein's "Greatest" mistake → 事实上并不是错误

  Λ was introduced by Einstein in 1917 to have a static model. $\begin{cases} a = a_E \\ \dot{a} = \ddot{a} = 0 \end{cases}$

  From Friedmann equation

$$\begin{cases} \dfrac{3\ddot{a}}{a} = -4\pi G(3P+\rho) \quad ① \\ \dot{a}^2 + K = \dfrac{8\pi G}{3}\rho a^2 \quad ② \end{cases} \Rightarrow \begin{array}{l} 3P+\rho = 0 \\ K = \dfrac{8\pi G}{3}\rho a^2 \end{array}$$

  If $\rho = \rho_M$, $P_M = 0$, $3P + \rho = \rho_M > 0$, Cannot be Static!

  Introduce ρ_V, such that $\rho = \rho_M + \rho_V$. $P = P_M + P_V = -\rho_V$, $3P + \rho = \rho_M - 2\rho_V$

$$\Rightarrow \begin{array}{l} \rho_M = 2\rho_V > 0 \\ \rho = 3\rho_V \\ K = \dfrac{8\pi G}{3}\rho a^2 > 0 \end{array} \quad \left(K = +1 \Rightarrow a_E = \sqrt{\dfrac{3}{8\pi G\rho}} = \dfrac{1}{\sqrt{8\pi G\rho_V}} = \dfrac{1}{\sqrt{\Lambda}}\right.$$

  Einstein Static Model $\begin{cases} K = +1 \\ \Lambda > 0 \end{cases}$ (un-supported by Hubble's discovery)

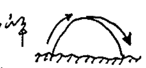
  (also, Einstein's Static Model is not stable solution)

  If $a = a_E + \delta a$, the $\rho_M = 2\rho_V + \delta\rho$, If wanting $\dot{a} = 0$

$$\begin{cases} ② \Rightarrow \begin{array}{l} \delta a > 0 \\ \delta\rho < 0 \end{array} \end{cases}$$

$$① \Rightarrow \frac{\ddot{a}}{a} \propto (-3P+\rho) = -(-3\rho_V + \rho_M + \rho_V) = -\delta\rho > 0$$

  why. 系统是稳定的 now



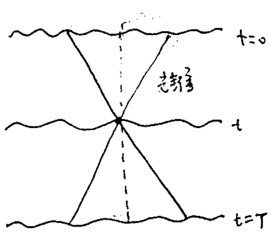
  δa begins to increase if $\delta a > 0$ $(\dot{a} = 0, \ddot{a} > 0)$

  So Einstein's model is unstable.

  If Universe start at $a = a_E$ and $\rho_M = 2\rho_V$, then it expands to infinity (Einstein-Lemaître Model)

  or start at $a = 0$, and approach to a_E (as $t \to \infty$) with enough matter $\rho_M = 2\rho_V$ at $a = a_E$

- Horizons



(1) Particle horizon = distance at which past events can be observed.

$$\underbrace{a(t)}_{\text{proper } d \text{ at } t} \underbrace{\int_0^t \frac{c\,dt'}{a(t')}}_{\text{comoving}} = d_{max}(t)$$

$t \uparrow \rightarrow d_{max}(t) \uparrow$

radiation dominated Universe $a(t) \propto t^{1/2}$

$$d_{max}(t) \simeq 2tc = \frac{c}{H}$$

matter dominated Universe $a(t) \propto t^{2/3}$

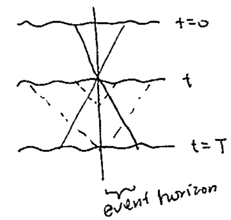
$$d_{max}(t) \simeq 3t = 2/H \quad (c = 1)$$

At today. $d_{max} = \dfrac{c}{H_0}\displaystyle\int_0^1 \dfrac{da'}{a'^2\sqrt{\cdots}}$

use $dt = \dfrac{da'}{H_0 a' \sqrt{\Omega_\Lambda + \Omega_K a'^{-2} + \Omega_m a'^{-3} + \Omega_R a'^{-4}}}$

where $a' = a/a_0$

(2) Event horizon = distances at which it will ever be possible to observe future events.
将来



= the largest distance we will ever be able to travel with speed of light starting from t.

$$\underbrace{d_{Max}(t)}_{\text{proper distance}} = a(t)\int_t^{T\to\infty} \underbrace{\frac{c\,dt'}{a(t')}}_{\text{Comoving distance}}$$

如果不收敛的. 就是 ∞

No Λ, matter dominated, $\int \to \infty$, No event horizon.

With Λ. $a \sim e^{Ht}$, $H = \sqrt{\Omega_\Lambda} H_0$. $d_{Max}(t) \to \dfrac{1}{H}$ $(t \to \infty)$

~ Size of Local Group

10

- Big Bang nucleosynthesis

CMB $z\sim1100$

era / Last year of recombination

$p.e.\gamma.\nu$ DM $\rightsquigarrow$ ($p.e$ H.He light elem) atoms γ,ν.DM

BBN

DM formation

Baryon genesis

CMB形成的过程
(Cosmic Microwave Background)

$p.e.\gamma$ keeps thermal equilibrium

热平衡

快速的能量交换

$e\ \gamma$ Thomonson Scattering (p. γ 的散射弱)

$e.\ p$ Coulomb Scattering 还是大

光子服从玻色统计

$$n_T(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp\frac{h\nu}{k_BT}-1} \quad (c=1)$$

number density of photons

当光子和电子脱耦后, 不再达到热平衡, 为什么还是黑体谱?

Assume there was a time t_L when radiation suddenly went from being thermal equilibrium to free expansion. "Last Scattering" (photon frequency ν_L)

At some later time t, photon frequency ν

$$\nu a(t) = \nu_L a(t_L) \Rightarrow \nu = \nu_L \frac{a(t_L)}{a(t)}$$

(如果无散射, 就是单纯的散射)

Number density at t

$$n(\nu,t)d\nu = \left(\frac{a(t_L)}{a(t)}\right)^3 n_{T(t_L)}(\nu_L)\,d\nu_L$$

Cosmic expansion

自由电子. 核子的散射小得多

$$= \left(\frac{a(t_L)}{a(t)}\right)^3 \frac{8\pi\nu_L^2 d\nu_L}{\exp\frac{h\nu_L}{k_BT(t_L)}-1} = \frac{8\pi\nu^2 d\nu}{\exp\frac{h\nu}{k_BT_{(t)}}-1} = n_{T(t)}(\nu)d\nu$$

仍然是黑体谱 (注意, 空间均匀, 时间上可以有变化, 因而黑体谱不是温度为常数的)

if $\frac{\nu}{T(t)} = \frac{\nu_L}{T(t_L)}$

$$T(t) = T(t_L)\frac{\nu}{\nu_L} = T(t_L)\frac{a(t_L)}{a(t)} \propto \frac{1}{a} = (1+z)$$

这里的T应理解为黑体谱的一个参数, 而非温度.

Note: 但即使自由膨胀的光子仍是黑体谱
但是峰值的波长与a(t)成正比.

1940s. Garmov and Collaborators: Universe is filled with blackbody radiation.

1950s. Alpher + Herman: by BBN, $T_{\gamma,0}\sim5K$

1965年 Princeton group. Peebles $T\sim10K$, by BBN

Penzias and Wilson (已得诺奖) $T\sim3.5K\pm1.0K$ at $\lambda=7.5cm$. isotropic radiation (没有证明是黑体谱)

Companion paper by Princeton group

1966年 Roll and Wilkinson $T\sim3.0\pm0.5K$, @$\lambda=3.2cm$ (仍没有证明是黑体谱)

WMAP的W 由Wilkinson命名

(1989发射) 1990s COBE FIRAS experiment nearly exact blackbody spectrum.

$\lambda\approx0.05\sim0.5cm$

$T=2.725\pm0.002K$

证实了大爆炸理论对CMB的预言

others, different λ

电子散射 CMB光深测量

τ_{es} = 0.17 WMAP1
0.10 WMAP3
0.07 WMAP7~9
⋮

Energy density $\int_0^\infty h\nu n(\nu)d\nu = a_BT^4$

$\frac{1}{c^2}\downarrow$ mass density today

$$a_B = \frac{8\pi^5k_B^4}{15h^3c^3} = 7.566\times10^{-15}\,erg\cdot cm^{-3}\cdot K^{-4}$$

$$\rho_{\gamma,0} = 4.64\times10^{-34}\,g\cdot cm^{-3}$$

$$\Omega_{\gamma} = \frac{\rho_{\gamma,0}}{\rho_{cr}} = 2.47\times10^{-5}h^{-2}$$

$\xrightarrow{\text{number density}}$

$$n_\gamma = \int_0^\infty n(\nu)d\nu = \frac{30\zeta(3)}{\pi^4}\frac{a_B T^3}{k_B} = 20.28\left(\frac{T}{K}\right)^3 cm^{-3}$$

$T=2.725 \Rightarrow n_{\gamma0} = 410\ photons/cm^3$

For radiation

$$\rho_{R,0} = \rho_{\gamma,0}\left(1+3\times\frac{7}{8}\times\left(\frac{4}{11}\right)^{4/3}\right) \approx 7.80\times10^{-34}\,g\cdot cm^{-3}$$

$\gamma+\nu$

3 types of neutrinos

$$\Omega_{R,0} = \frac{\rho_{R,0}}{\rho_{cr}} = 4.15\times10^{-5}h^{-2} \ll \Omega_M, \Omega_\Lambda$$

(与核子的比较, for nucleons

$$n_{B,0} = \frac{\rho_{B0}}{m_N} = \frac{\Omega_B\rho_{cr}}{m_N} = \frac{3\Omega_BH_0^2}{8\pi Gm_N}$$

$$= 1.123\times10^{-5}\Omega_Bh^2\ nucleons/cm^3$$

$$\sim 10^{-5}\ nucleons/cm^3$$

$\frac{n_\gamma}{n_B} = \frac{n_{\gamma,0}}{n_{B,0}}$ 是不变化的

因为数目不变, 体积 $\propto a^3$
$n\propto a^{-3}$

$\rho_R \propto a^{-4}$ 在早期主导
$\rho_m \propto a^{-3}$ 在晚期主导) Epoch of Matter-radiation equality

$$\frac{\rho_R}{\rho_m} \propto \frac{1}{a} \propto T \quad (\propto (1+z))$$

$$\frac{(\rho_R/\rho_M)_{EQ}}{(\rho_R/\rho_M)_0} = \frac{T_{EQ}}{T_{\gamma,0}} \Rightarrow T_{EQ} = T_{\gamma,0}\left(\frac{\Omega_R}{\Omega_m}\right)^{-1}$$

$(\rho_R/\rho_M)_{EQ} \to 1$, $(\rho_R/\rho_M)_0 \to \Omega_R/\Omega_M$

$$= 6.56\times10^4 K \times \underbrace{\Omega_m h^2}_{0.15}$$

$$\sim 10^4 K$$
$$\sim 1 eV$$

T_{CMB} 的温度 $\sim 3000K < T_{EQ}$ $(z\sim1100)$

辐射为主的时期 → 物质为主的时期 → CMB形成
$10^4 K$ → $3000K$
$1+z \sim 3\times10^3$, $1+z\sim 1100$

$$1+z_{EQ} \sim \frac{T_{EQ}}{T_{\gamma,0}} = \frac{\Omega_m}{\Omega_R} \approx 2.4\times10^4\,\Omega_m h^2 \sim 10^4 \times 0.5$$

物理宇宙学 0403/2018 6B301 Lecture 6

复习，考试带A4纸一张，计算器可带

Weinberg chapter 1 $\left\{\begin{array}{l}\text{Friedmann}\\ \text{RW}\\ \text{distance}\end{array}\right.$

Motion of particle in the Universe

- Momentum of particle (matter, photon, ...) $p \propto a^{-1}$

Define the momentum

$$p^\mu := m_0 \frac{dx^\mu}{d\tau} = \left(\frac{E}{c}, \vec{p}\right)$$

While the geodesic motion

$$d\tau^2 = dt^2 - g_{ij}dx^i dx^j$$

In local Cartesian Coordinates $d\tau = dt\sqrt{1-v^2}$. $p^\mu = (\underbrace{\gamma m_0 c, \gamma m_0 v}_{\text{动量 } \vec{p} = \gamma m_0 \vec{v}})$

For massive particle

$$P = |\vec{p}| = m_0\sqrt{g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}$$

Note: in Special Relativity $\vec{p} = \gamma m \vec{v} = m\gamma \frac{d\vec{x}}{dt} = m\frac{d\vec{x}}{d\tau}$

For convinience, choose a spatial coordinate system in which particle position is near origin $\vec{x}=0$. For RW metric

$$g_{00} = -1$$
$$g_{ij} = a^2 \tilde{g}_{ij}$$
$$\tilde{g}_{ij} = \delta_{ij} + K\frac{x^i x^j}{1-K|x|^2} = \delta_{ij} + O(x^2)$$

Geodesic equation

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \longrightarrow \begin{array}{l} RHS = -2\frac{\dot{a}}{a}\frac{dx^i}{d\tau}\frac{dt}{d\tau} \\ LHS = \frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right). \end{array}$$

, 2 comes of exchange of dx^i and dx^0

Connection $\Gamma^0_{ij} = a\dot{a}\delta_{ij} + O(x^2)$, $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$, $\Gamma^i_{jl} = \frac{1}{2}\tilde{g}^{im}\left(\frac{\partial \tilde{g}_{jm}}{\partial x^l} + \frac{\partial \tilde{g}_{lm}}{\partial x^j} - \frac{\partial \tilde{g}_{jl}}{\partial x^m}\right) + \dots = O(x^2)$ → 求导不全消除

The geodesic becomes $\frac{d}{dt}\left(\frac{dx^i}{d\tau}\right) = -\frac{2}{a}\frac{da}{dt}\frac{dx^i}{d\tau}$

or $d\ln\left(\frac{dx^i}{d\tau}\right) = \frac{-2\,da}{a}$

$\Leftrightarrow \frac{dx^i}{d\tau} \propto a^{-2}$

The momentum $P = m_0\sqrt{a^2\left(\frac{dx^i}{d\tau}\right)^2 + O(x^2)} = m_0 a\left(\frac{dx^i}{d\tau}\right) + O(x^2) \propto a^{-1}$

holds for massive particle.

For massless particle

$E = p^0 = m_0\frac{dx^0}{d\tau} \rightsquigarrow \frac{d}{d\tau} = \frac{dx^0}{d\tau}\frac{d}{dx^0} = \frac{E}{m_0}\frac{d}{dt}$

(massless should be/massive should also be)

$$\Rightarrow \frac{d^2x^0}{d\tau^2} = \frac{E}{m_0}\frac{d}{dt}\left(\frac{dx^0}{d\tau}\right) = \frac{E}{m_0}\frac{d}{dt}\left(\frac{E}{m_0}\right) = \frac{E}{m_0^2}\frac{dE}{dt}$$

$$\Rightarrow E\frac{dE}{dt} = -\Gamma^0_{\mu\nu}p^\mu p^\nu$$ (hold for massive and massless particles)

for massless particle

$g_{\mu\nu}p^\mu p^\nu = 0$ (null vector)

$-E^2 + a^2\delta_{ij}p^i p^j = 0$

Note: for photon $E^2 = c^2p^2$, in general relativity. $g_{ij}p^ip^j = p^2$

$-\Gamma^0_{\mu\nu}p^\mu p^\nu = -a\dot{a}\,\delta_{ij}p^ip^j = -\frac{\dot{a}}{a}E^2 \Rightarrow \frac{1}{E}\frac{dE}{dt} = -\frac{da/dt}{a} \Rightarrow E \propto a^{-1}$ for massless particle

$P = \sqrt{g_{ij}p^ip^j} = \sqrt{a^2\delta_{ij}p^ip^j + O(x^2)} = E + O(x^2) \propto a^{-1}$ for massless particle

- CMB谱

退耦合前，CMB光子是黑体谱；而演化到了退耦合后光子是也是黑体谱：
(如果没有光子的弹性散射)

但如果存在非弹性散射，可能存在偏离（比如，重子数密度比光子数少得多，10^{-8}）

For free expansion:

- for photons, $T\propto a^{-1}$ to preserve the Planck black body spectrum $e^{h\nu/k_BT}$
- for non-relativistic particles (e.g. e^-, nuclei), Maxwell-Boltzmann distribution

$$n(p)dp \propto e^{-\frac{p^2}{2mkT}}$$

So that $T\propto a^{-2}$ (because $p\propto a^{-1}$)

(adiabatic cooling after "complete" decoupling $z\sim150$)

Question: At thermal equilibrium $T\propto a^{-1}$ or $T\propto a^{-2}$?

光子数目多，则 $T\propto a^{-1}$
核子/电子数目多，则 $T\propto a^{-2}$
实际光子数目多，所以 $T\propto a^{-1}$

Entropy density is conserved.

(entropy in a comoving volume)

To see this

$$a^{-3}\frac{\partial}{\partial t}(\rho a^3) = -3\frac{\dot a}{a}\underset{\text{pressure}}{P}$$

$$\Leftrightarrow\quad a^{-3}\frac{\partial}{\partial t}\Big((\rho+p)a^3\Big) - \frac{\partial P}{\partial t} = 0$$

Use $\frac{\partial P}{\partial t} = \frac{\partial P}{\partial T}\frac{dT}{dt}$, and relation $\frac{\partial P}{\partial T} = \frac{\rho+P}{T}$ (to be proved later, use partition function)

$$a^{-3}\frac{\partial}{\partial t}\Big((\rho+p)a^3\Big) - \frac{dT}{dt}\frac{\rho+P}{T} = 0$$

$$a^{-3}T\frac{\partial}{\partial t}\Big(\frac{\rho+p}{T}a^3\Big) = 0$$

Define entropy density $\quad \rho\delta V = Ts\delta V - p\delta V$. $s = \frac{\rho+P}{T}$ ✓

So that $s \propto a^{-3}$ (in physical volume)

and $S = \text{const}$ (in comoving volume)

所以,没有热进出的话,熵是守恒的

Baryon number in a comoving volume is conserved.

(只要没有高到粒子反粒子的温度)

So entropy per baryon is also conserved. Define it as $\underset{\uparrow\ \text{dimensionless}}{k_B\sigma}$.

$$T\,d(k_B\sigma) = d\left(\frac{\epsilon}{n_B}\right) + P\,d\left(\frac{1}{n_B}\right)$$

n_B = baryon number density (质子,中子等)

$\frac{1}{n_B}$ = volume per baryon

ϵ = thermal energy density

For an ideal gas of photons and non-relativistic particles (e.g. e^-, protons, He nuclei ...)

Define $\mathcal{N}$ = number of non-relativistic particles per baryon $\sim O(1)$

$$\epsilon = a_B T^4 + \frac{3}{2}n_B\mathcal{N}k_BT$$

$$P = \underset{\text{光子}}{\frac{1}{3}a_BT^4} + \underset{\text{baryon}}{n_B\mathcal{N}k_BT}$$

$$d\sigma = \frac{1}{k_BT}\left(a_B\,d\left(\frac{T^4}{n_B}\right) + \frac{3}{2}\mathcal{N}k_B\,dT + \left(\frac{1}{3}a_BT^4 + n_B\mathcal{N}k_BT\right)d\left(\frac{1}{n_B}\right)\right)$$

将T^4写成$T^3\cdot T$ 　　　将$\frac{1}{n_B}$写为$\frac{T^3}{n_B}T^{-3}$

$$= \frac{1}{k_BT}T\cdot a_B\,d\left(\frac{T^3}{n_B}\right) + \underbrace{\frac{a_B}{k_B}\frac{T^2}{n_B}dT} + \frac{3}{2}\mathcal{N}\frac{dT}{T} + \frac{1}{3}\frac{a_B}{k_B}d\left(\frac{T^3}{n_B}\right) - \underbrace{\frac{a_B}{k_B}\frac{T^2}{n_B}dT}_{\text{cancelled}} + n_B\mathcal{N}d\left(\frac{1}{n_B}\right)$$

$$\simeq \frac{4}{3}\frac{a_B}{k_B}d\left(\frac{T^3}{n_B}\right) + \mathcal{N}\,d\ln\frac{T^{3/2}}{n_B}$$

Now it is easy to show

$$\underset{\text{Constant}}{\sigma} = \underset{\text{photons}}{\frac{4}{3}\frac{a_BT^3}{k_Bn_B}} + \underset{\text{baryons}}{\mathcal{N}\ln\frac{T^{3/2}}{n_BC}} \qquad (*)\ \text{(entropy per baryon)}$$

C: 积分 Constant

而我们已知 $n_\gamma = 0.3702 \times \frac{a_BT^3}{k_B}$, $\frac{n_\gamma}{n_B} = 0.3702\frac{a_BT^3}{k_Bn_B}$. $\frac{n_{\gamma 0}}{n_{B0}} = \frac{410\ \text{photons}/cm^3}{1.123\times10^{-5}\Omega_Bh^2\ \text{nucleus}/cm^3} \sim 10^8$

The first term of $(*) \sim 10^8$, $\mathcal{N} \sim$ constant, if a fractional change in $T^3/n_B \sim 10^{-4}$ (这是对第一项的改变10^4)

So that $\frac{T^{3/2}}{n_B}$ changes by a factor of $e^{10000/\mathcal{N}}$, NOT NATURAL. in logarithm

So, T^3/n_B is constant, or $n_B \propto T^3$ → also because $n_B \propto a^{-3}$ $\Rightarrow T \propto a^{-1}$

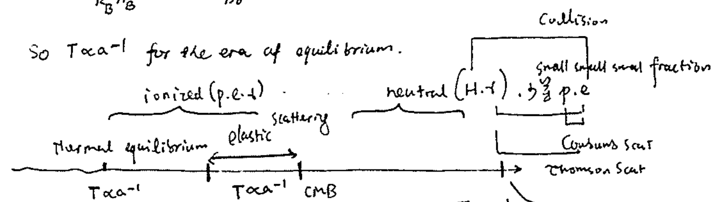
Note: $n_B \propto T^3$ is natural for black body spectrum $T \propto a^{-1}$, $n_B \propto a^{-3} \Rightarrow n_B \propto T^3$

is also natural for elastic scattering (because energy not changed)

C 可以取一个温度平衡时候的值. $C = \frac{T^{3/2}}{n_B}$ at some typical time during the era of equilibrium.

$$\sigma = \frac{4a_BT^3}{k_Bn_B} = 3.60\frac{n_{\gamma 0}}{n_{B0}} = 1.31\times10^8(\Omega_Bh^2)^{-1}$$

★ So $T \propto a^{-1}$ for the era of equilibrium.



Note
前面已证明
$T_\gamma \propto a^{-1}$
$T_b \propto a^{-2}$
decouple
现在进一步证明
couple 的时候
$T_\gamma = T_b \propto a^{-1}$

inelastic scattering $T \propto a^{-1}$

热平衡: 光子与电子交换能量(非弹热交换)
对电子影响大, 对光子影响不大
电子温度跟着 $T_{e,p} \propto a^{-1}$
$z \sim 1100$

(所以, 热平衡时, 但光子可以通过非弹性散射使其他物质 catch up 光子)

$T_{CMB} \propto a^{-1}$
$T_{gas} \propto a^{-2}$ Complete decoupling
中子, 质子 在高温状态
从CMB获得的能量
不足以保 $T \propto a^{-1}$

- Estimation of the "freeze-out" temperature (冷冻)

i.e. The time when photons stopped exchanging energies of order k_BT with electrons.

Scattering rate $\quad \Lambda_\gamma = \sigma_T n_e c = 1.97\times10^{-19}s^{-1}\Omega_Bh^2\,(T/T_{\gamma 0})^3$

$$n_e = n_B\Big(0.76\times\underset{\substack{\uparrow\\ H}}{1} + 0.24\times\frac{2}{4}\Big) = 0.88n_B = 0.88n_{B,0}\left(\frac{T}{T_{\gamma,0}}\right)^3$$

一个baryon对应1个电子
一个He对应2个电子(4个核子)
今天的CMB温度

Note:
单位时间, 单位体积 被散射的光子数 $\frac{dN}{dt\,dV} \simeq n_\gamma\cdot\underbrace{c\sigma_T n_e}_{\Lambda_\gamma}$
单位时间, 每光子的散射次数

At scattering, $\Delta p = \frac{k_BT}{c}$ from/to electrons

$$\Delta E = \frac{(\Delta p)^2}{m} = \frac{(k_BT)^2}{m}$$

光子失去的能量 每次散射 $\frac{\Delta E}{k_BT} \sim \frac{k_BT}{m_e}$ (光子散射机制 Compton 和逆 Compton 散射)

能量传输速率（单位时间，每个光子输能的百分fraction）
The rate of energy transport

$$\Gamma_\gamma = \Lambda_\gamma\left(\frac{\Delta \varepsilon}{k_B T}\right) = \Lambda_\gamma \frac{k_B T}{m_e} \sim 9.0\times10^{-29}s^{-1}\Omega_B h^2\left(T/T_{\gamma 0}\right)^4$$

Compared with the rate of expansion $H = \frac{\dot a}{a} = H_0\sqrt{\Omega_R T^4/T_{\gamma 0}^4} = 2.1\times10^{-20}s^{-1}\left(T/T_{\gamma 0}\right)^2$

tentatively, radiation dominated (*)

if $\underbrace{\Gamma_\gamma H^{-1}}_{\text{cosmic age}} \lesssim 1$, $H^{-1} \propto T^2$, freeze-out.

it corresponds a temperature

$$T_{freeze} = 1.5\times10^4 K\,(\Omega_B h^2)^{-1/2} \sim 10^5$$

注意 $T_{EQ} = 10^4 K < T_{freeze}$

(NOT accurate)

实际的freeze-out比这里的小一个数量级

另一种更不精确的方法 $k_B T_{freeze} \sim 13.6eV$ （不能维持光子至与电子的电离

$T_{freeze} \sim 10^5 K$ 但事实上CMB是连续谱，仍可以电离）

真实的估计是 10^4 K (or, 3000 K)　　Note: 这表明，一般的freeze out energy是估计值的1/10左右

Radiation dominated | Matter dominated

Matter-Radiation equality

$T_{eq} \sim 10^5 K$

$z_{EQ} \sim 2.4\times10^4\Omega_m h^2 \sim 10^4$

CMB形成

$T \sim 3000K$

$z \sim 1100$ 数值计算结果

Note: 同时也是freeze-out的时间 (Recombination)

当光子传出与未传出去直接 kT 的话量，中性的原子得以存在

Change (*) to matter dominated Universe

$$H = \frac{\dot a}{a} = H_0\sqrt{\Omega_M T^3/T_{\gamma 0}^3} = 3.3\times10^{-18}s^{-1}\sqrt{\Omega_m h^2}\left(\frac{T}{T_{\gamma 0}}\right)^{3/2}$$

$$T_{freeze} = 18K\,(\Omega_m h^2)^{1/3}/(\Omega_B h^2)^{2/3} \sim 130 K$$

也不太正确

用一些更精确的方法
↓

- Recombination

（不应称为re，因为这是第一次出现neutral Hydrogen）

$$p + e \leftrightarrows H(1s)$$

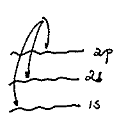
$i = p.e.H$ 满足Maxwell Boltzmann分布

$$n_i = g_i(2\pi\hbar)^{-3}e^{\mu_i/k_BT}\int d^3p\,\exp\left(-\left(m_i + \frac{p^2}{2m_i}\right)/k_BT\right)$$

$g_i \sim$ number of spin states

$\mu_i \sim$ chemical potential

$m_i \sim$ mass



1S

只考虑一级的激发态，忽略其它

<化学统计要求> 的常识　　而光子的化学势为0，$\mu_\gamma=0$，则 $\mu_p+\mu_e=\mu_{1s}$，$g_{e,p}=2$

$m_{1s} \simeq m_p + m_e - B_1$ (13.6eV)　　$g_{1s} = 4$（电子和质子核的自旋，两个态相结合的结果）

积分后部分 $(2\pi\hbar)^{-3}\int d^3p\,\exp\left(\frac{-p^2}{2mk_BT}\right) = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$

所以 $\frac{n_{1s}}{n_p n_e} = \frac{4}{2\times2}e^{-\frac{1}{k_BT}(m_{1s}-m_p-m_e)}\left(\frac{m_e k_BT}{2\pi\hbar^2}\right)^{-3/2} = \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{-3/2}e^{\frac{B_1}{k_BT}}$

化学势约掉了　　$m_p \sim m_{1s}$

$X \equiv \frac{n_p}{n_p+n_{1s}}$ fractional hydrogen ionized

$n_p + n_{1s} \simeq 0.76\,n_B$（氢的占比例）

得到 Saha equation $X(1+SX) = 1$, where $S \equiv \frac{(n_p+n_{1s})n_{1s}}{n_p^2} = 0.76\,n_B\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{-3/2}\times\exp B_1/kT$

$\approx 1.747\times10^{-22}e^{157894/T}\cdot T^{-3/2}\Omega_B h^2$

在早期，$T\to\infty$时，$S\to\frac{1}{\infty}\to0$，$X\to 0+1$（完全ionized的 n_p）

$T\to0$，$S\to\infty$，$X\to0$（完全neutral）

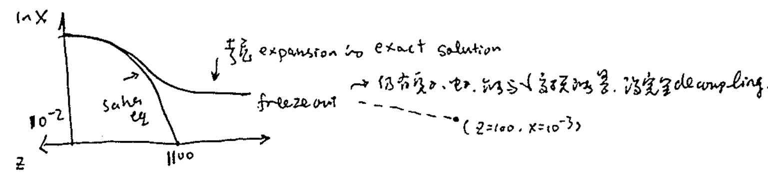
这是不正确的，因为没有考虑宇宙的expansion（考虑宇宙膨胀的话，$T\to0$ 的时候会先冻结）

$X\to97\%$，$T\to4200K$

$X\to1\%$，$T\to3000K$（几乎是neutral）定义为 T_{freeze}

$X\to2\times10^{-12571}$　$T\to2.725K$

真正的，应该用Boltzmann eq（非平衡统计）去解决此问题



注: Boltzmann equation
Dodelson ch 3.3 eq (3.39) second term of RHS
$X e^{2} n_B \alpha^{(2)}$, $n_e + n_H = n_p + n_H = n_B$ (没有乘以 0.76 应该不对)
$\uparrow 0.76 n_B$

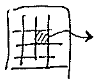
作业
Dodelson 3.7; 3.8
3.8(d) eq (3.68) $\Lambda_\alpha = \frac{H(3\epsilon_0)^3}{(8\pi)^2 n_H}$, $n_H = (1-x)0.76 n_b$

注: $f(\vec{x},\vec{p})$ 是无量纲的
$g_i f(\vec{x},\vec{p}) \frac{1}{(2\pi\hbar)^3}$ 是物理意义上的函数 (单位相空间)

• Boltzmann equation

经典的情况下, 用 $(\vec{x},\vec{p})$ 描述粒子状态, 而分布函数 $f(\vec{x},\vec{p})$ = number of particles in a given region in phase space around $(\vec{x},\vec{p})$
distribution function or occupation number

$$= \frac{1}{e^{(E-\mu)/kT} \pm 1} \begin{cases} -1. & \text{Bose-Einstein} \\ +1. & \text{Fermi-Dirac} \end{cases}$$


$\rightarrow dx\,dp = 2\pi\hbar$

$$\sum_x \sum_p \rightarrow \int d^3x \int \frac{d^3p}{(2\pi\hbar)^3}$$

number density $n_i(\vec{x}) = g_i \int \frac{d^3p}{(2\pi\hbar)^3} f_i(\vec{x},\vec{p})$, g_i is degeneracy

mass density (Energy) $\rho_i(\vec{x}) = g_i \int \frac{d^3p}{(2\pi\hbar)^3} f_i(\vec{x},\vec{p}) E_i(p)$, $E_i(p) = \sqrt{p^2+m_i^2}$

Suppose $1+2 \leftrightarrow 3+4$

e.g. $e^- + p \underset{\text{recombination}}{\overset{\text{ionization}}{\rightleftharpoons}} H + \gamma$

$$\left(\begin{array}{c}\text{The rate of change in abundance of} \\ \text{a given particle}\end{array}\right) = \text{the rate of producing it} - \text{the rate of eliminating it}$$

Boltzmann equation for annihilation

$LHS = \frac{1}{a^3}\frac{d(n_1 a^3)}{dt}$ ← 膨胀的宇宙中, 光看宇宙的膨胀就导致 n_i 改变
n_1 = number density in physical coordinates
$n_1 = \frac{n_1^{com}}{a^3}$ → number density in comoving volume

$\frac{d(n_1 a^3)}{dt}$ = the rate of change in comoving volume
LHS = the rate of change in physical coordinates

Note: 这里讨论的是一般的反应方程, 可应用到其他一些场合, 例如 CMB 的 decoupling

The rate of producing 1 $\propto f_3 f_4 [1 \pm f_1][1 \pm f_2]$
↑ 1 会导致 1 的产生率提升/压低
"+" for boson enhancement
"−" Pauli blocking

The rate of eliminating 1 $\propto f_1 f_2 [1 \pm f_3][1 \pm f_4]$

必须对动量进行积分

$$\int d^4p\, \delta(E^2-p^2-m^2) = \int d^3p \int dE\, \frac{\delta(E-\sqrt{p^2+m^2})}{2E} = \int \frac{d^3p}{2E}$$
with $E = \sqrt{p^2+m^2}$

动量和能量不是独立的
$(-m^2+) E^2 - \vec{p}^2 = p^\mu p_\mu = p^2$
是协变的形式

利用 $\delta(f(x)) = \sum_{x_0} \frac{\delta(x-x_0)}{f'(x_0)}$, $f(x_0)=0$

所以 Boltzmann 方程可以写为 for annihilation

为什么只写一个方程? 因为 $n_B \cdot 0.76 = n_p + n_H$, $n_e = n_p$

$$a^{-3}\frac{d}{dt}(n_1 a^3) = \int \frac{d^3p_1}{2E_1 (2\pi\hbar)^3} \int \frac{d^3p_2}{2E_2 (2\pi\hbar)^3} \int \frac{d^3p_3}{2E_3 (2\pi\hbar)^3} \int \frac{d^3p_4}{(2\pi\hbar)^3 2E_4} \times (2\pi)^4 \delta^3(p_1+p_2-p_3-p_4)\, \delta(E_1+E_2-E_3-E_4) |M|^2$$
$$\times \{\underbrace{f_3 f_4 (1\pm f_1)(1\pm f_2)}_{\text{producing}} - \underbrace{f_1 f_2 (1\pm f_3)(1\pm f_4)}_{\text{annihilation}}\} \quad \cdots (*1)$$

动量、能量守恒
什么时候可以计算 amplitude of the reaction (reversible)
反应速率相同

将上面的方程进一步化简, f 作为 Bose / Fermi 统计函数

(1) 当 $E-\mu > kT$ 时, (interested in) $f = \frac{1}{e^{(E-\mu)/k_BT} \pm 1} \simeq e^{\mu/k_BT} e^{-E/k_BT}$, 此时 $1 \pm f \simeq 1$
产物不决定速率

(2) 一般情况

对 (1) 情况

$$f_3 f_4 (1\pm f_1)(1\pm f_2) - f_1 f_2 (1\pm f_3)(1\pm f_4) = f_3 f_4 - f_1 f_2 = e^{(\mu_3+\mu_4)/k_BT} e^{-(E_3+E_4)/k_BT} - e^{(\mu_1+\mu_2)/k_BT} e^{-(E_1+E_2)/k_BT}$$
$$\underline{\underline{\text{能量守恒}}} \quad e^{-(E_1+E_2)/k_BT} \left\{ e^{(\mu_3+\mu_4)/k_BT} - e^{(\mu_1+\mu_2)/k_BT} \right\} \quad \cdots (*2)$$

若反应速率很快 (time scale < Hubble time)
这时是化学平衡 $\mu_1+\mu_2 = \mu_3+\mu_4$ (对应 Saha e.q.)
若反应速率低, 没有 $\mu_1+\mu_2 = \mu_3+\mu_4$, 这时 $n_1\, n_2\, n_3\, n_4$ 完全由演化决定

又 $n_i = g_i \int \frac{d^3p}{(2\pi\hbar)^3} f_i(p) = g_i e^{\mu_i/k_BT} \int \frac{d^3p}{(2\pi\hbar)^3} e^{-E_i/k_BT}$, $E_i = \sqrt{p^2+m_i^2}$

$$= n_i^{(0)} e^{\mu_i/k_BT}$$
$n_i^{(0)}$ = species-dependent equilibrium number density.
$$\equiv g_i \int \frac{d^3p}{(2\pi\hbar)^3} e^{-E_i/kT}$$
(只有依赖于 T 和 m)

massive particle 非相对论性的
for $m_i \gg T$, $e^{-E_i/k_BT} \simeq e^{(-m_i + \frac{p^2}{2m_i})/k_BT}$
$$n_i^{(0)} = g_i \left(\frac{m_i T}{2\pi\hbar}\right)^{3/2} e^{-m_i/k_BT}$$

for $m_i \ll T$ $e^{-E_i/k_BT} \simeq e^{-p/k_BT}$
相对论性的 光子、中微子 massless
$$n_i^{(0)} = g_i\, T^3/\pi^2$$

可以将 (*2) 写为

$$(*2) = \underbrace{e^{-(E_1+E_2)/k_BT}}_{\text{在 (*1) 中, 与 E.p 有关}} \Big(\underbrace{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}}_{\text{与 E.p 无关, 可以在 (*1) 中提出去}} \Big)$$

Note: 把含 f 的方程转化为含 n 的方程
替换, 的变量原本是 μ_i, T
现在是 n_i, T

Define "thermally averaged" cross-section (含 T)

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3p_1}{(2\pi\hbar)^3 2E_1} \int \frac{d^3p_2}{(2\pi\hbar)^3 2E_2} \int \frac{d^3p_3}{(2\pi\hbar)^3 2E_3} \int \frac{d^3p_4}{(2\pi\hbar)^3 2E_4} \times e^{-(E_1+E_2)/k_BT} \delta^3(p_1+p_2-p_3-p_4)\, \delta(E_1+E_2-E_3-E_4) |M|^2 \cdot (2\pi)^4$$
可以从微观反应, 粒子物理计算

The (*1) Boltzmann eq. Become

$$a^{-3}\frac{d}{dt}(n_1 a^3) = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \quad (*)$$
只含有 μ, 因此与能量无关

量纲分析的大小 $\sim \frac{n_1}{t}$ (timescale of universe expansion), $RHS \sim n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \sim n_1 n_2 \langle \sigma v \rangle$. If the reaction rate $n_2 \langle \sigma v \rangle \gg$ time scale $\frac{1}{t}$ for expansion

the only way to keep equality is $\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}$, $\frac{n_i}{n_i^{(0)}} = e^{\mu_i/k_BT}$ $\Rightarrow \mu_1 + \mu_2 = \mu_3 + \mu_4$

chemical equilibrium eq. (in particle physics)
Saha equation (Recombination/CMB)
nuclear statistical equilibrium (BBN)

Otherwise : out of equilibrium.
($\langle\sigma v\rangle n_2 \gg 1/t$ 不成立)

热平衡时
回到CMB的情况 $1+2 \rightleftarrows 3+4$
$e+p \rightleftarrows H+\gamma$

$$n_i^{(0)} = \int g_i \frac{d^3p}{(2\pi\hbar)^3} e^{-E_i/k_BT} = \begin{cases} g_i\left(\frac{m_iT}{2\pi}\right)^{3/2} e^{-m_i/T} & \text{massive particle} \\ g_i T^3/\pi^2 & \text{massless particle}\end{cases}$$

(只与 m_i, T 有关)

$g_e = 2,\ g_p = 2,\ g_H = 4$

for photon $\mu_\gamma = 0$, $n_\gamma = n_\gamma^{(0)}$

for electron, photon, proton, H $\quad \frac{n_H}{n_H^{(0)}} = \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}}$ ($n_e = n_p$, because of neutrality of the Universe)

define $X \equiv \frac{n_p}{n_p + n_H}$ ← ionized, ← neutral

$n_e = n_p = X(n_p + n_H) = X \cdot 0.76\, n_B$
$n_H = (1-X)\, 0.76\, n_B$
$n_p + n_H = 0.76\, n_B$

所以 $\frac{n_e n_p}{n_H} = \frac{X^2}{1-X} 0.76\, n_B = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$ (忽略氦)

$$= \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_1/T}, \quad \text{where } B_1 = m_e + m_p - m_H = 13.6\,\text{eV}$$

define $S = \frac{1-X}{X^2} = 0.76\, n_B \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{-3/2} e^{B_1/k_B T}$, 这即是 Saha equation $X(1+XS)=1$

(对热平衡成立,
非热平衡不对重写)

非热平衡时

$$n_e^{(0)} n_p^{(0)} \langle\sigma v\rangle \left\{ \frac{n_H}{n_H^{(0)}} - \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} \right\} = a^{-3}\frac{d}{dt}\left(n_1 a^3\right) \qquad (*3)$$

note:
本来有3个未知量 n_H, n_e, n_p.
利用 $n_e = n_p$ (neutral), $n_H + n_p = 0.76\, n_b$ (重子守恒)
消去了两个

$$= 0.76\, n_B \langle\sigma v\rangle \left\{ (1-X)\frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} - 0.76\, n_B X^2 \right\} = 0.76\, n_B \langle\sigma v\rangle \left\{ (1-X)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_1/T} - 0.76\, n_B X^2 \right\}$$

define $\beta \equiv \langle\sigma v\rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_1/T}$ ⟨Ionization Rate⟩

$\alpha^{(2)} \equiv \langle\sigma v\rangle$ ⟨Recombination Rate⟩

$$a^{-3}\frac{d}{dt}\left(n_1 a^3\right) = a^{-3}\frac{d}{dt}\left(X \cdot \underbrace{0.76\, n_B\, a^3}_{n_{B,0} = \text{const}}\right) = 0.76\, n_B \frac{dX}{dt}$$

可将 (*3) 改写为

$$\frac{dX}{dt} = \underbrace{(1-X)}_{\text{neutral fraction}}\beta - X^2 \alpha^{(2)} \times 0.76\, n_B$$

X 为电离度 $\quad$ 来自 $n_e \cdot n_p$, recomb 过程

$\underbrace{\quad}_{\text{ionization}}$ $\underbrace{\quad}_{\text{recombination}}$

原子物理计算给出 $\alpha^{(2)} \simeq 9.78 \frac{\alpha^2}{m_e^2}\left(\frac{B_1}{T}\right)^{1/2} \ln\left(\frac{B_1}{T}\right)$ ← 精细结构常数

当 $\frac{dX}{dt} = 0$ 可得 Saha equation

注 $\alpha^{(2)}$ 中 2 的意思:

$\gamma \leftarrow$ 13.6eV $\quad 2p_{1/2}$, $2s_{1/2}$, 1s

① electric dipole 1-photon (偶极跃迁) $\Delta L = 0, \pm 1$ ($0 \to 0$ 除外) ② $_{2s \to 1s}$ 以2光子放出 才满足 quadrupole (四极跃迁) $E_{\gamma_1} + E_{\gamma_2} = 3Ly\alpha$ 能量可以任意分配, 提供宇宙中大比例的连续谱

如果 直接到 ground state, 产生 γ.
则立即电离另外一个氢原子, X 不会改变;

真正决定 X 的过程是 cascades 多层跃迁 (2 → 1)

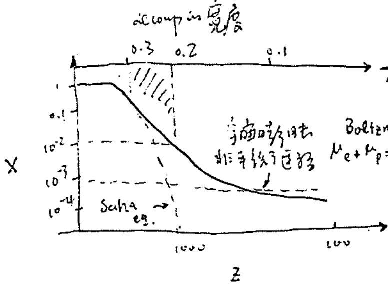
所以 —— 共它态, —— 2, —— 2), —— 1

Lyα由于宇宙膨胀, 能量将下降 不能使氢电离; 或双光子放出 也不能带电离

是较闭关心的

(2 的意义在此. 其他为 n 平衡)

所以 $\alpha^{(2)} = \alpha^{2p \to 1s} + \alpha^{2s \to 1s}$



$z_{decoup} \sim 1100$
$T \sim 3000\,K$
$0.3\,eV$

CMB 的不均匀性

Dipole anisotropy

在运动观测者看来, CMB是不各向同性的
Dipole anisotropy 是可以被消除的.

CMB谱 $n_\gamma(\nu)d^3x$ 不是 Lorentz 不变的, $\nu = \frac{c}{\lambda} = c|p|/h$, $p = \frac{h\nu}{c}$

$d^3p = p^2 dp = \left(\frac{h\nu}{c}\right)^2 \frac{h}{c} d\nu = \left(\frac{h}{c}\right)^3 \nu^2 d\nu$

做 Lorentz 变换时, 相空间数密度是不变的, $\underbrace{N_\gamma(p)}_{\text{number density in phase space}} d^3x\, d^3p$

$$N_\gamma(p) = \frac{1}{2}\frac{n_\gamma\left(c|p|/h\right)}{4\pi h^3 \nu^2/c^3} = \frac{1}{h^3}\frac{1}{e^{|p|c/kT} - 1}$$

↓ 两个极化方向 polarization

从 K 变到 K' 系, $N_\gamma \to N_\gamma'(p') = N_\gamma(p)$ 是不变的

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ |p| \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \gamma & \beta\gamma \\ 0 & & \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ |p'| \end{pmatrix}$$

$\gamma = (1-\beta^2)^{-1/2}$ 是 boost factor
$|p| = \gamma(1+\beta\cos\theta)|p'|$
$\cos\theta = \frac{p_3'}{|p'|}$

所以 $N_\gamma'(p') = N_\gamma(p) = \frac{1}{h^3}\frac{1}{e^{|p'|c/kT'} - 1}$

这样 等效的温度 $T' = \frac{T}{\gamma(1+\beta\cos\theta)}$ 所以, 沿着不同方向的观察者, $N_\gamma'(p')$ 不同

$$\Delta T = T' - T = T\left(-\frac{\beta^2}{6} - \underbrace{\beta P_1(\cos\theta)}_{\text{dipole}} + \underbrace{\frac{2}{3}\beta^2 P_2(\cos\theta)}_{\text{quadrupole}} + \cdots\right)$$

当 $\beta = \frac{v}{c} \simeq 0.0013 = 370\,\text{km/s}/c$

$\beta^2 \sim 10^{-6}$

由于β较小, 所以 dipole 项占主导

而CMB本身的涨落 $\frac{\Delta T}{T} \sim 10^{-5} \gtrsim \beta^2$ from density fluctuation
所以四极项不必考虑

weak e/w

氢原子的电离 $\longrightarrow$ 质子与中子的质量差（核的电离） $\longrightarrow$ dark matter interaction

1eV $\qquad$ 1 MeV

$10^4 K$ $\qquad$ $10^{10} K$

e是非相对论的 $\qquad$ e是相对论的

（10MeV时，可以处理成完全相对论的）

0.1MeV （0.1MeV以下，可以处理成非相对论的）

$10^9 K$

电子和正电子湮灭过程

2018/0424 物理宇宙学 6B301 $\qquad$ 没有作业

lecture 8

Thermal History at $10^4 \sim 10^{11} K$

Assumptions:

① Thermal equalibrium

热平衡，宇宙膨胀的速率远小于散射速率
$H(t) \qquad \sigma n$

② Since $n_b \ll n_\gamma$, $\mu_b = 0$

③ $\rho(T), p(T), S(T)$

所有的物理量都是温度的函数（e.g. 非相对论的电子，相对论，极端相对论电子 $\rho(T)$ 形式可能不同）

之前已证明过

$S(T)a^3 = const$

$S(T) = \frac{\rho(T)+p(T)}{T}$

For species i, distribution function $f(\vec{x},\vec{p}) = \frac{1}{e^{(E-\mu)/k_BT} \pm 1}$ $\begin{cases} - & \text{Bosonic} \\ + & \text{Fermion} \end{cases}$

$$\overset{\mu=0}{=} \frac{1}{e^{\sqrt{p^2+m^2}/k_BT} \pm 1}$$

number density between $(p, p+dp)$

$$n(p,T) = \frac{g_i\, 4\pi p^2}{(2\pi\hbar)^3} f(p,T)$$

mass density

$$\rho(T) = g_i \int \frac{d^3p}{(2\pi\hbar)^3} f(p,T)\, E(p)$$

$$= g_i \int \frac{4\pi p^2 dp}{(2\pi\hbar)^3} f(p,T) E(p)$$

$$= \int_0^\infty n(p,T)\sqrt{p^2+m^2}\, dp$$

degeneracy

$g = \begin{cases} 2, & \text{photon (2 polarization) } \gamma=\bar{\gamma} \\ 6 = 3\times2\times1 & \text{neutrino} \end{cases}$

(3 species ν_e, ν_μ, ν_τ and antineutrino $\bar\nu_e, \bar\nu_\mu, \bar\nu_\tau$; only 1 spin for each. 只有左手中微子，右手反中微子)

$g = \begin{cases} 2\times2=4 & \text{electron (2 spin), positron (2 spin)} \end{cases}$ （强作用不考虑强子）

pressure

$$p(T) = \int_0^\infty n(p,T)\,dp \cdot \frac{p^2}{3\sqrt{p^2+m^2}}$$

entropy density

$$s(T) = \frac{\rho+p}{T} = \frac{1}{T}\int_0^\infty n(p,T)\,dp\left[\sqrt{p^2+m^2} + \frac{p^2}{3\sqrt{p^2+m^2}}\right]$$

Especially for massless particle ($m \ll k_BT$)

e.g. for $E > 10MeV$, $T > 10^{11}K$

The electron can be viewed as massless particle

$$\rho(T) = \begin{cases} \frac{1}{2} g a_B T^4, & \text{Bosons} \\ \frac{7}{8}\times\frac{1}{2} g a_B T^4, & \text{Fermions} \end{cases}$$

（费米子可以有这样的系数）

e.g. for photons $g=2$, $\rho(T) = a_B T^4$, where $a_B = \frac{\pi^2 k_B^4}{15\hbar^3}$

For relativistic particles, define $\mathcal{N}$ such that

$$\rho(T) = \mathcal{N} \times \frac{1}{2} a_B T^4$$

$\mathcal{N}$: number of particle types, spin types, with $\frac{7}{8}$ for fermions

$$p(T) = \frac{1}{3}\rho(T)$$

$$s(T) = \frac{4}{3}\frac{\rho(T)}{T} = \frac{2}{3}\mathcal{N} a_B T^4/T$$

宇宙物理量与时间的关系

Use Friedmann equation $H = \frac{\dot a}{a} = \sqrt{\frac{8\pi G \rho(T)}{3}}$, $S(T) \propto a^{-3}$ we can write

忽略Λ, K

（已知道）在绝热过程中熵守恒，在绝热过程中可以得出 $T \propto a^{-1}$

$$\frac{ds}{s} = -3\frac{da}{a}$$

$$H = -\frac{1}{3}\frac{ds/s}{dt} = \sqrt{\frac{8\pi G}{3}\rho(T)}, \quad dt = \frac{-ds}{s\sqrt{24\pi G\rho}} = -\frac{s'(T)dT}{s(T)\sqrt{24\pi G\rho}}$$

So that $t = -\int \frac{s'(T)dT}{s(T)\sqrt{24\pi G\rho}} + const$ （普适的关系，无论粒子是否为relativistic都成立）

For massless particle $\rho \propto T^4$

$$t = \sqrt{\frac{3}{16\pi G \mathcal{N} a_B}}\frac{1}{T^2} + const$$

<1> $T > 10^{11}K \sim 10MeV$

$k_BT \gg m_ec^2$, $k_BT \ll m_\mu c^2$ （μ子不参与和$\nu+\bar\nu$粒子的相互作用）, $e^+ + e^- \rightleftharpoons 2\gamma$

$$e^+ + e^- \rightleftharpoons \left.\begin{matrix} \nu_e + \bar\nu_e \\ \nu_\mu + \bar\nu_\mu \\ \nu_\tau + \bar\nu_\tau \end{matrix}\right\}$$ 弱相互作用 neutral current reaction $\qquad$ characteristic energy $E \sim 1MeV$

所以在此温度下 plasma: γ, $\nu+\bar\nu$, $e+\bar e$

$$\mathcal{N} = 2 + \frac{7}{8}(2\times3\times1 + 2\times2) = \frac{43}{4}, \quad t = 0.994\,sec\left(\frac{T}{10^{10}K}\right)^{-2} + const$$

e.g. T goes from $10^{11}K$ to $10^{10}K$ (一个数量级) $\quad T: 10^{12}K \to 10^{11}K$

t goes from 0.0094 to 0.994 sec (二个数量级) $\quad t: 10^{-4}\,\text{sec} \to 0.0094$ sec

$\simeq 1s$

⟨2⟩ T is about $10^{10}K \sim 1\,MeV$

这时电子不是极端的相对论，也不是非相对论

另外，弱相互作用 weak cross-section

$$\sigma_{WK} \doteq (\hbar G_{WK} k_B T)^2$$

$$G_{WK} \simeq 1.16 \times 10^{-5}\ GeV^{-2}$$

↑ Coupling Constant of weak interaction

Collision rate of neutrino-electron scattering (中性流过程)

$$\Gamma_\nu \simeq \underset{\propto T^2}{\sigma_{WK}}\ n_e \simeq G_{WK}^2 (k_B T)^5/\hbar$$

↑ 相对论性气体 $\propto T^3$

Expansion rate during radiation-dominated Universe

$$H \simeq \sqrt{G (k_B T)^4/\hbar^3} \propto \sqrt{a^{-4}} \propto \sqrt{T^4}$$

The Ratio

$$\Gamma_\nu / H \simeq G_{WK}^2 \left(\frac{\hbar}{G}\right)^{1/2} (k_B T)^3 \simeq \left(\frac{T}{10^{10}K}\right)^3$$

$$\nu_e \bar{\nu}_e \leftrightarrow e^+ e^- \leftrightarrow \gamma$$

$\begin{cases} T \gtrsim 10^{10}K, & \nu, e, \gamma \text{ are in thermal equilibrium} \\ T < 10^{10}K, & \Gamma_\nu/H \text{ dropped rapidly.} \end{cases}$

neutrino freeze-out (与其他粒子脱耦)

T_ν 仍是黑体谱，但独立地演化 $T_\nu \propto \frac{1}{a}$

entropy density $S \propto a^{-3}$ (after ν freeze-out)

起始时是relativistic，因而

$$S(T) = \underbrace{\frac{4}{3} a_B T^4 / T}_{\text{photon}} + \underbrace{\frac{4}{T} \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3}}_{e^+, e^-,\ g = 2\times2 = 4} \frac{1}{\exp\left(\sqrt{p^2 + m_e^2}/k_B T\right) + 1} \times \left(\sqrt{p^2 + m_e^2} + \frac{p^2}{3\sqrt{p^2 + m_e^2}}\right)$$

$$x = \frac{m_e}{k_B T}, \quad y = \frac{p}{k_B T}$$

$$\overset{\text{积分后}}{=\!=} \frac{4}{3} a_B T^3 S\left(\frac{m_e}{k_B T}\right)$$

$$S(x) = 1 + \frac{45}{2\pi^4} \int_0^\infty y^2 dy \frac{1}{\exp\sqrt{y^2 + x^2} + 1} \times \left(\sqrt{y^2 + x^2} + \frac{y^3}{3\sqrt{y^2 + x^2}}\right) \xrightarrow{x \to \infty} 1$$

只有光子贡献 (totally radiation dominated)

Because $Sa^3 = \text{const}$

So $T^3 S\left(\frac{m_e}{k_B T}\right) a^3 = \text{const}$

$$T_\nu \propto \frac{1}{a} \propto T S^{1/3}\left(\frac{m_e}{k_B T}\right)$$

or

$$\frac{T_\nu}{T S^{1/3}\left(\frac{m_e}{k_B T}\right)} = \frac{1}{S^{1/3}(0)} = \left(\frac{4}{11}\right)^{1/3}$$

Note: Before freeze-out, $k_B T \gg m_e c^2$

$T_\nu = T$

which Corresponds to $S(0) = \frac{1}{2}\left(2 + \frac{7}{8} 2 \times 2\right) = \frac{11}{4}$

$e^\pm$

↑ 光子的 spin-2 简并

or

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T S^{1/3}\left(\frac{m_e}{k_B T}\right) \xrightarrow[S(\infty)=1]{\text{At present } k_B T \ll m_e c^2 \text{ (or at recombination)}} \left(\frac{4}{11}\right)^{1/3} T$$

$$\left(\frac{T}{T_\nu}\right)_{\text{present}} \simeq 1.401$$

$T_{CMB} = 2.725K, \quad T_\nu = 1.945K$

T_{CMB}: 在 $10^3K \sim 0.1eV$ 脱耦

T_ν ↑ cosmic neutrino background (很难直接探测到) 在 $1MeV \sim 10^{10}K$ 脱耦

Note: 也即，$e^\pm$ 与电子一起加热，使得 $T_{(a)}$ 的演化跟 $T_{\nu(a)}$ 不同

Total energy density $\to H, a, T(t)$

$$\rho(T) = \underbrace{a_B T^4}_{\gamma} + \underbrace{6 \times \frac{a_B T_\nu^4}{2}}_{\nu} \times \frac{7}{8} + 4 \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{\sqrt{p^2 + m_e^2}}{\exp\sqrt{p^2 + m_e^2}/k_B T + 1}$$

↑ g and fermion

$$= a_B T^4 \mathcal{E}\left(\frac{m_e}{k_B T}\right)$$

$$\mathcal{E}(x) = 1 + \frac{21}{8}\left[\left(\frac{4}{11}\right)^{1/3} S^{1/3}\Big(\underbrace{\frac{m_e}{k_B T}}_{x}\Big)\right]^4 + \frac{30}{\pi^4} \int_0^\infty \frac{y^2\sqrt{y^2 + x^2}\, dy}{\exp\sqrt{y^2 + x^2} + 1}$$

$$t = -\int \frac{S'(T) dT}{S(T)\sqrt{24\pi G \rho(T)}} + \text{const} = t_e \int \left(3 - \frac{x S'(x)}{S(x)}\right) \mathcal{E}^{-1/2}(x) dx$$

$$t_e \simeq \left(24\pi G c^6 a_B m_e^4 / k_B^4\right)^{-1/2} = 4.37\ \text{sec}$$

e.g.

$T(K)$	T/T_ν		t (sec)
10^{11}	1	ν与e 不完全 decouple	0
10^{10}	1.008		0.998
10^9	1.345	ν与e 完全 decouple	168 ~ 3 minutes
3×10^8	1.401		1980 ~ 30 minutes

⟨3⟩ $T < 10^9 K \sim 0.1\,MeV \ll 2 m_e c^2$ 电子是非相对论

$$\frac{\Gamma_\nu}{H} \sim \left(\frac{T}{10^{10}K}\right)^3 \ll 1$$, neutrino Completely freeze-out with T_ν

energy density $\rho(T) = \underbrace{a_B T^4}_{\gamma} + \underbrace{\frac{1}{2} a_B T_\nu^4 (3 \times 2 \times 1 \times \frac{7}{8})}_{\nu, \bar{\nu}}$

相对论性气体 number/energy 的 3 倍

$$= a_B T^4 \left(1 + \frac{7}{8} \times 3 \times \left(\frac{4}{11}\right)^{4/3}\right)$$

$$\simeq 3.363 \times \frac{1}{2} a_B T^4$$

for relativistic situation $t \propto T^{-2}$, $\quad t = 178\,\text{sec}\left(\frac{T}{10^{10}K}\right)^{-2} + \text{const}$

T (K)	T/T_ν	t (sec)
3×10^8	1.401	180
10^8		$1.78\times10^4 \simeq 4.9$ hours
10^6		$1.78\times10^8 \simeq 5.6$ yrs
10^4 K ~ 1eV		360000 yrs $z\sim10^3$ (K ~ decoupling)

考虑 baryon的贡献

$\rho = \rho_m + \rho_b$

取 $\Omega_m h^2 = 0.15$

· 周四 5/10 7:20-9:45 补课

· HW ch3 Ex 4, 10

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Big Bang Nucleosynthesis (BBN)

$\gtrsim 1$ MeV

- Relativistic Particle : $\underbrace{\gamma, e^\pm}_{\text{thermal equilibrium}}$, $\{\nu_e, \nu_\mu, \nu_\tau\}$
  - 并不完全相对论，但很近似所以影响不大
  - $\nu_e \rightleftharpoons \nu_e$ (> MeV, in equilibrium with other)
  - (< MeV ν 与其余脱离，但与其他的联系 out-of-contact)
- Non-relativistic particles : baryons (p, n)
  - $\bar{p}p^+$, $\bar{n}n^+$ 在 GeV 以上是平等的（湮）；GeV 以下会湮灭，剩余一些 p^+ 和 n
  - 用 $\eta_b = \frac{n_b}{n_\gamma}$ 来描述，$\eta_b = \frac{n_{b0}}{n_{\gamma 0}} = 5.5\times10^{-10}\left(\frac{\Omega_b h^2}{0.020}\right)$
  - 是非常小的值，湮灭后剩的一点

0.1 MeV,

Energy Conservation

$m_p + E_{\bar\nu} = m_n + E_{e^+}$ (可以把 m_p, m_n 用能量算)

$Q = m_n - m_p = 1.293$ MeV

反应可以中子转化

$p + \bar{\nu}_e \rightleftharpoons n + e^+$ → lepton number conservation（不是守恒的）: $L_e, L_{\nu_e} = 1$; $L_{e^+}, L_{\bar\nu_e} = -1$

交叉对称 $p + e^- \rightleftharpoons n + \nu_e$ } (p, n) thermal equilibrium ⟨*1⟩

$n \rightleftharpoons p + e^- + \bar{\nu}_e$

由于 $Q = 1.293$ MeV，所以当温度降至 0.1 MeV 时，会向 p 转化导致中子数目减少

称为 neutron → proton Conversion：最后，中子数目足够少时，$n \to p$ 反应速率与 hubble 膨胀可比拟时不再是简单的转化过程

0.07 MeV

light element synthesis

可与 $n \to p$ 同时发生（在不同温度）

$$\begin{cases} p + n \rightleftharpoons D + \gamma \\ D + D \rightleftharpoons {}^3He + n \\ {}^3He + D \rightleftharpoons {}^4He + p \\ \vdots \end{cases}$$

温度足够高时，每产生一个D，就会被γ打回去，所以必须温度足够低才可产生D

对于一个核子，可以描述以下结果 nucleus

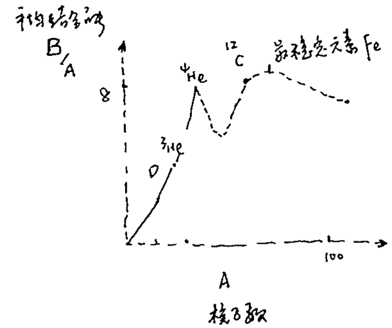
Z protons 元素种类

A protons + neutrons 同位素种类

$A - Z$ neutron

$B = Z m_p + (A-Z) m_n - m$ binding energy

e.g. $B_D = m_p + m_n - m_D = 2.22$ MeV



$\vdots$

$D + D \rightleftharpoons {}^3He + n$

${}^3He + D \rightleftharpoons {}^4He + p$ ${}^4He$ 已是 local maximum，下一个能形成的是 ${}^{12}C$

${}^4He + {}^4He + {}^4He \rightleftharpoons {}^{12}C + \gamma$

$\vdots$

3个 He 碰撞几率小，所以宇宙中难形成。但恒星中可以形成（恒星形成的 C 与 O 结合形成 CO，可用来探测恒星形成区）

Note: 为什么 $B_D > Q$，反而晚一些才开始？ (Note: m_p 非常大)

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T}$$

$$\Rightarrow \frac{n_D}{n_b} \sim \frac{n_b}{\underset{\to\, \propto T^3}{n_\gamma}} \times \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T} \quad \text{(类似于 Saha eq)}$$

$$\sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}$$

由于 η_b 很小，$e^{B_D/T}$ 只有 T 很小时才使 n_D 数目很大

Neutron Abundance

反应方程 ⟨*1⟩ 前两个反应，后面第3个衰变反应在之后考虑中子衰变就可以

$n_n e^{-t/\tau_n}$, $\tau_n \to 900$ sec, t: 100 sec for 0.1 MeV，所以可以暂不考虑

$$\underset{1}{n} + \underset{2}{l} \rightleftharpoons \underset{3}{p} + \underset{4}{l}$$

在 0.1 MeV 时 e, ν_e 等化学势都忽略不计

The Boltzmann equation

$$a^{-3}\frac{d}{dt}\left(a^3 n_1\right) = n_1^{(0)} n_2^{(0)}\langle\sigma v\rangle\left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right]$$

- in this case $1 \Rightarrow n$, $3 \Rightarrow p$
- for lepton (e, ν), $n_i^{(0)} = \dfrac{n_i}{e^{\mu/kT}} = \begin{cases} g_i\left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} & m_i \gg T \ (\text{non-rel}) \\ g_i\, T^3/\pi^2 & m_i \ll T \ (\text{rel}) \end{cases}$

$$\mu = 0 \Rightarrow n_i/n_i^{(0)} = 1$$

Now the Boltzmann equation becomes

$$a^{-3}\frac{d}{dt}\left(a^3 n_n\right) = n_n^{(0)} n_\ell^{(0)}\langle\sigma v\rangle\left[\frac{n_p}{n_p^{(0)}} - \frac{n_n}{n_n^{(0)}}\right] = n_\ell^{(0)}\langle\sigma v\rangle\left[n_p\frac{n_n^{(0)}}{n_p^{(0)}} - n_n\right]$$

而 $\dfrac{n_n^{(0)}}{n_p^{(0)}} = \left(\dfrac{m_n}{m_p}\right)^{3/2} e^{-Q/T} \simeq e^{-Q/T}$

$g_n = g_p = 2$

$Q = m_n - m_p = 1.293\ \text{MeV}$

质量差的两部分是: $\left(\dfrac{m_p+\delta_m}{m_p}\right)^{3/2} = \left(1+\dfrac{\delta_m}{m_p}\right)^{3/2} = 1+\dfrac{3}{2}\dfrac{\delta_m}{m_p} \doteq 1+10^{-3}$

但指数上要考虑

Define $X_n = \dfrac{n_n}{n_n+n_p} = \dfrac{1}{1+n_p/n_n}$, $X_n(EQ) = \dfrac{1}{1+n_p^{(0)}/n_n^{(0)}} \simeq \dfrac{1}{1+1} = \dfrac{1}{2}$ $\quad T \gg Q$ $(T \gg 1\,\text{MeV})$

$n_n = X_n n_b$

$n_p = (1-X_n) n_b$

高温时, $n_n^{(0)} n_\ell^{(0)}\langle\sigma v\rangle$ 很大, 如此有 $\dfrac{n_p}{n_p^{(0)}} = \dfrac{n_n}{n_n^{(0)}}$

低温时 $n_n^{(0)} n_\ell^{(0)}\langle\sigma v\rangle$ 很小, 可以有 $\dfrac{n_p}{n_p^{(0)}} \neq \dfrac{n_n}{n_n^{(0)}}$

代入原来的方程, 注意

$\dfrac{d}{dt}\left(a^3 n_b\right) = 0$, 则有

$\text{LHS} = n_b\dfrac{dX_n}{dt}$

记 $\lambda_{np} = n_\ell^{(0)}\langle\sigma v\rangle$, 则

$$\text{RHS} = \lambda_{np} n_b\left((1-X_n)e^{-Q/T} - X_n\right)$$

Now the Boltzmann equation

$$\frac{dX_n}{dt} = \lambda_{np}\left[(1-X_n)e^{-Q/T} - X_n\right]$$

由于时间 t 与温度 T, scale factor a 有关系, 换成 T 更方便 : define $x = Q/T$ evolution variable

$$\frac{d}{dt} = \frac{d}{dx}\frac{dx}{dT}\frac{dT}{dt} = xH\frac{d}{dx}$$

$\dfrac{dx}{x} = -\dfrac{dT}{T}$

$T \propto a^{-1}$

neutrino 还没有脱耦, 先考虑温度

$dx/da = x/a$

原来的 Boltzmann 方程又可化为

$$xH\frac{dX_n}{dx} = \lambda_{np}\left[(1-X_n)e^{-x} - X_n\right]$$

$x^2 H = \text{constant} = H(x=1)$ for radiation dominated Universe

$$\Rightarrow \frac{dX_n}{dx} = \frac{x\lambda_{np}}{H(x=1)}\left[(1-X_n)e^{-x} - X_n\right] = \frac{x\lambda_{np}}{H(x=1)}\left[e^{-x} - X_n\left(1+e^{-x}\right)\right]$$

注意 $H(x=1) = H(T=Q) = \sqrt{\frac{8}{3}\pi G\rho} = \sqrt{\frac{8}{3}\pi G\frac{\pi^2}{30}g_* T^4} = \sqrt{\frac{4\pi^3 G Q^4}{45}}\sqrt{g_*} = 1.13\ \text{sec}^{-1}$

这里 $\rho = \frac{\pi^2}{30}T^4$

加上中微子 $(\times 7/8)$, $e^\pm$

$\rho = g_*\frac{\pi^2}{30}T^4$

$g_*(T=1\,\text{MeV}) = 2 + \left[\underbrace{2\times 2}_{e^\pm} + \underbrace{3\times 2\times 1}_{\nu}\right]\frac{7}{8} \simeq 10.75$

- If chemical equilibrium, $X_n = \dfrac{e^{-x}}{1+e^{-x}} = \dfrac{1}{1+e^{x}} \overset{T\gg Q}{\simeq} e^{-x}$ 表低温极限
- $\lambda_{np}(x) = n_\ell^{(0)}\langle\sigma v\rangle = \dfrac{255}{\tau_n x^5}\left(12+6x+x^2\right)$

τ_n 是 neutron 的平均寿命

$\tau_n \simeq 886.7\ \text{sec}$ (neutron lifetime)

上面的方程可化为 $\dfrac{dX_n}{dx} + \dfrac{x\lambda_{np}(x)}{H(x=1)}\left(1-e^{-x}\right)X_n = \dfrac{x\lambda_{np}(x)}{H(x=1)}e^{-x}$

利用积分因子法 $f^{-1}\dfrac{d}{dx}fX_n = \dfrac{dX_n}{dx} + \dfrac{d\ln f}{dx}X_n$

希望 $\dfrac{d\ln f}{dx} = \dfrac{x\lambda_{np}(x)}{H(x=1)}\left(1-e^{-x}\right)$

则可取 $f(x) = f(x_i)e^{\mu(x)}$, $\mu(x) = \displaystyle\int_{x_i}^{x}dx'\frac{x'\lambda_{np}(x')}{H(x=1)}\left(1-e^{-x'}\right)$

可取 $f(x_i) = 1$, 而 $x_i = \dfrac{Q}{T_i}$ at some early time

T_i 不能太高, 否则有其他过程

最后方程为 $\dfrac{d}{dx}\left(e^{\mu(x)}X_n(x)\right) = \dfrac{x\lambda_{np}(x)}{H(x=1)}e^{\mu(x)-x}$

得到的解是 $X_n(x) = e^{-\mu(x)}\left[\displaystyle\int_{x_i}^{x}dx'\frac{x'\lambda_{np}(x')}{H(x=1)}e^{-x'}e^{\mu(x')} + X_n(x_i)\right]$

初始条件 $X_n(x_i) = \frac{1}{2}$

所以 $X_n(x) = \frac{1}{2}e^{-\mu(x)} + \displaystyle\int_{x_i}^{x}dx'\,\frac{x'\lambda_{np}(x')e^{-x'}}{H(x=1)}e^{\mu(x')-\mu(x)}$

答案里少了这一项, 是否很小?

$\dfrac{x'}{H(x=1)} = \dfrac{1}{x' H(x')}$

对比书中其他的方程 $\mu(x)$ 相当于光深 (因为方程形式相同)

可得到答案给出的结果

如果要考虑 n 的衰变，则 $X_n(x)=X_n(\frac{Q}{T})e^{-t/\tau_n}$

先求解 $t=\int dt=\int\frac{da}{a\,\dot a/a}\propto a^2$ (radiation dominated)

figure out coeffs

$\rho=g_*\frac{\pi^2}{30}T^4$, $X_n\simeq e^{-Q/T}$ ($Q\simeq 1\,\mathrm{MeV}$)

脱耦发生于 $T\simeq 0.1\,\mathrm{MeV}$，$g_*(T=0.1\,\mathrm{MeV})=2+\frac{7}{8}\times2\times1=3.75$

Dodelson 书里面是 3.36

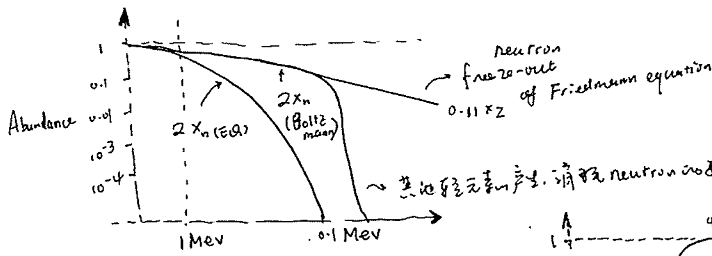
可能源于这里的是：光子的温度和中微子温度不同

在 $T\simeq 0.1\,\mathrm{MeV}$，时间 $t\simeq 132\,\mathrm{sec}\left(\frac{0.1\,\mathrm{MeV}}{T}\right)^2$

ν_e，没有其他中微子

$\tau_n\simeq 886.7\,\mathrm{sec}$，$t/\tau_n$ 很小

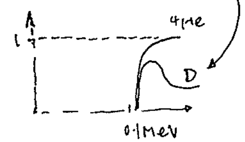
最后可以给出 $X_n(x)$ 的曲线



$X_n(T\simeq0.1\,\mathrm{MeV})=0.11$ (0.15×0.74)

$e^{-t/\tau_n}=e^{-132/887}=0.74$

～ 其他轻元素的产生，消耗 neutron 的时间



化学平衡开始打破

但宇宙的膨胀还继续下去

$n_n=X_b n_b\simeq 0.11 n_b$, $n_p\simeq 0.89 n_b$

Define

$$\frac{n_D}{0.89 n_b}=\eta_b\frac{3}{4}\left(\frac{4\pi}{m_pT}\right)^{3/2}e^{B_D/T}$$

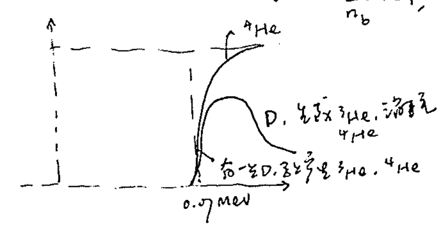
$n_\gamma\eta_b\propto\eta_bT^3$

$$\frac{n_D}{n_b}\sim\eta_b\left(\frac{T}{m_p}\right)^{3/2}e^{B_D/T}$$

want $\frac{n_D}{n_b}\sim1$, $\Rightarrow$ $\ln\eta_b+\frac{3}{2}\ln\frac{T_{nuc}}{m_p}\simeq-B_D/T$

$T_{nuc}\simeq0.07\,\mathrm{MeV}$

重要核合成温度



最后，可认为 n 已经几乎全部转化为了 4He。

$X_{^4He}\simeq 2X_n(T_{nuc})$ 含两个中子

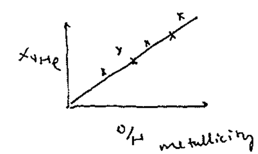
$=0.22$

更仔细的计算，多级的 couple

$$X_{^4He}\simeq 0.2262+0.0135\ln\left(\frac{\eta_b}{10^{-10}}\right)$$

(star 内产生 He 比较少，忽略)

实验观测的



← 外推到 O/H = 0，得到 $X_{^4He}\simeq0.22-0.25$

观测与理论值高度吻合，是大爆炸的证据。

BBN

Light element abundance

$$\begin{cases}p+n\to D+\gamma\\ D+D\to{}^3He+n\\ {}^3He+D\to{}^4He+p\\ \vdots\end{cases}$$

D 作为上一级的产物，又是下一级的反应物

为简单起见，可以一步一步计算，分开看 先产生 D

产生 D 足够以后，再消耗

$p+n\to D+\gamma$

1 2 3 4

$B_D=m_p+m_n-m_D=2.22\,\mathrm{MeV}$

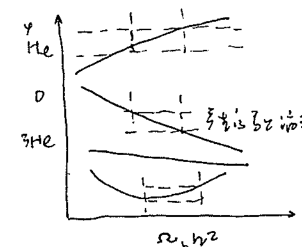
化学反应还未脱耦，下一级反应就发生，所以不用 Boltzmann eq.

$m_n\simeq m_p\simeq\frac{1}{2}m_D$

$$(\mu_\gamma=0)\Rightarrow\frac{n_D}{n_pn_n}=\frac{n_D^{(0)}}{n_p^{(0)}n_n^{(0)}}=\frac{3}{4}\left(\frac{m_DT/2\pi}{\frac{m_pT}{2\pi}\frac{m_nT}{2\pi}}\right)^{3/2}e^{-B_D/T}=\frac{3}{4}\left(\frac{m_D\cdot2\pi}{m_nm_pT}\right)^{3/2}e^{-B_D/T}=\frac{3}{4}\left(\frac{4\pi}{m_pT}\right)^{3/2}e^{B_D/T}$$

$g_p=g_n=2$, $g_D=3$ ↓ $\frac12,\frac12$ couple，自旋为 1 (有 3 态)

理论上，其他的元素也可计算



测量以上的数据，得到 Ω_b 的数密

$\simeq0.022\pm\ldots$

05.11 / 2018 周四晚 物理学院 6B301

DARK MATTER

理论模型 { Modified Gravity（$f(R)$ 连续可调参数的一类理论）; MOND; WIMP（可以是著名 LSP，即 light supersymmetry particle；但不仅是超对称模型的某些）}

实验 { 直接探测 pandax（利用与重核碰撞得到其反冲）; 间接探测 } ⇒ 没有迹象

下面我们假设暗物质是存在的；在比较高温时，各种 DM particle 都还存在，在低温时，由于宇宙的膨胀，所以会 freeze-out（脱离平衡）；其剩余丰度是可预言的。

假定 Dark matter 存在湮灭 $x+x \rightleftharpoons l+\bar{l}$（某种轻粒子），写出 Boltzmann 方程

$$a^{-3}\frac{d}{dt}(n_x a^3) = \langle\sigma v\rangle n_x^{(0)2}\left(\left(\frac{n_l}{n_l^{(0)}}\right)^2 - \frac{n_x^2}{n_x^{(0)2}}\right)$$

$\frac{n_l}{n_l^{(0)}} \simeq 1$ for relativistic

$$= \langle\sigma v\rangle\left[n_x^{(0)2} - n_x^2\right]$$

- $n_x \propto a^{-3} \propto T^3$

  $n_x/T^3 = \text{const}$ if no conversion/annihilation

- Define $Y \equiv \frac{n_x}{T^3}$, time variable $x = \frac{m}{T} \propto a$.

  So $Y \propto a^{-3}$, $Y_{EQ} = \frac{n_x^{(0)}}{T^3}$, $\frac{dx}{x} = \frac{da}{a}$, $\frac{dY}{dt} = xH\frac{dY}{dx}$

  $LHS = a^{-3}\frac{d}{dt}\left(T^3a^3 Y\right) = T^3\frac{dY}{dt}$

  equation: $\frac{dY}{dt} = T^3\langle\sigma v\rangle\left[Y_{EQ}^2 - Y^2\right]$

  $$\Rightarrow \frac{dY}{dx} = \frac{T^3\langle\sigma v\rangle}{xH}\left[Y_{EQ}^2 - Y^2\right]$$

Define $\lambda = \frac{m^3\langle\sigma v\rangle}{H(T=m)}$ ← annihilation rate / expansion rate

$H = \sqrt{\frac{8}{3}\pi G\rho} \propto a^{-2} \Rightarrow x^2H = \text{const} = H(m)$

So $\frac{T^3\langle\sigma v\rangle}{xH} = \frac{m^3\langle\sigma v\rangle}{H(T=m)}\frac{T^3}{m^3}\frac{H(T=m)}{xH}$

$$= \frac{\lambda}{x^4}\frac{H(m)}{H} = \frac{\lambda}{x^2}$$

equation:

$$\frac{dY}{dx} = \frac{-\lambda}{x^2}\left(Y^2 - Y_{EQ}^2\right)$$

记最终 freeze-out 时，$Y = Y_\infty = Y(x=\infty)$，此时 $x \gg 1$

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}Y^2$$

$$\frac{1}{Y_\infty} - \frac{1}{Y_f} = \frac{\lambda}{x_f}$$

今天的 abundance $Y_\infty \ll Y_f$（T 很小），所以

$$Y_\infty \simeq \frac{x_f}{\lambda}$$

一般来说 $x_f = m/T_{\text{free-out}} \simeq 10$，所以 $Y_\infty \simeq \frac{10}{\lambda}$

今天的暗物质的 density（已经 freeze out）

$$\rho_x = m n_{x,0} = m Y_\infty T_0^3 \simeq m Y_\infty T_0^3 \left(\frac{a_1^3T_1^3}{a_0^3T_0^3}\right)$$

$Y_\infty = \frac{10}{\lambda}$，$T_0^3$ 上有 $\frac{T_0^3}{30}$。注意：这里的 T_0 不是指现有光子的温度今天的温度（decouple 后，电子与光子等湮灭加热光子，但暗物质有自己的温度）

注：$\lambda = \frac{m^3\langle\sigma v\rangle}{H(T=m)}$，$H(T=m) = \sqrt{\frac{8\pi G}{3} g_* \frac{\pi^2}{30} m^4}$

↑ 相对论性粒子的自由度

这个温度下，主要是相对论性粒子，$g_* \simeq 100$（$m \simeq 100\,\text{GeV}$）

$$\Omega_x \simeq 0.3h^{-2}\left(\frac{x_f}{10}\right)\left(\frac{g_*(m)}{100}\right)^{1/2}\frac{10^{-39}\text{cm}^2}{\langle\sigma v\rangle}$$

这就给出了湮灭截面的大小（带这么大的 $\langle\sigma v\rangle$，速度，温度无关）

对暗物质直接探测，需要的是 σ（弹性碰撞截面），与 $\langle\sigma v\rangle$ 不一样

宇宙学的一般历史和相关过程

$T \sim 1\,\text{eV} \times 0.1$ 光子和电子脱耦 $\xrightarrow{\text{更早}}$ $T \sim 1\,\text{MeV} \times 0.1$ 原子核的合成 $\xrightarrow{\text{更早}}$ $T \sim 1\,\text{GeV} \times 0.1$ 产生重子物质一个时刻，p^+ 的数目比 $\bar{p}$ 多一些 → 更早的时刻 inflation model

neutron 衰变 $\frac{1}{2}$ → 轻元

↓

注 D，^{3}He，^{4}He 等

宇宙结构的形成

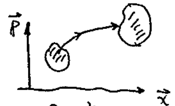
均匀、各向同性的背景 → 小尺度的扰动
非线性结构形成 → 大尺度的扰动
线性扰动理论

反应率方程 → 扰动的 metric → 影响的
with metric → 扰动的 metric 下的 Boltzmann 方程，对应各种物质的相空间分布演化

分布函数 $f(t,\vec{x},\vec{p})$

$\frac{df}{dt}=0$ for non-collision

$\frac{df}{dt}=C[f]$ for collision



注意这里采用的是 lagrangian perspective，是全导数。

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x^i}\dot{x}^i+\underbrace{\frac{\partial f}{\partial p}\frac{dp}{dt}}_{\text{动量大小变化}}+\underbrace{\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}}_{\text{动量方向变化}}$$

$\dot{x}^i$：经典力学 $\frac{p^i}{m}$，运动学项

$\frac{\partial f}{\partial p^i}\frac{dp^i}{dt}$ → 经典力学 $F^i=-\partial_i V$，动力学项

在广义相对论下，动力学项用 geodesic equation 给出

$ds^2=-dt^2+a^2(t)\,d\vec{x}^2$ （后面都假定宇宙是 flat FLRW 的）

$$g=\begin{pmatrix}-1 & \\ & a^2(t)\delta_{ij}\end{pmatrix}$$

想要改变 $g_{\mu\nu}$（perturb），可行的方法是

$$g_{\mu\nu}=\begin{pmatrix}-1-2\Psi(t,\vec{x}) & A^i(t,\vec{x})\\ \vec{A}=A^i(t,\vec{x}) & a^2(1+2\Phi)\delta_{ij}+\tilde{h}_{ij}\end{pmatrix}=\begin{pmatrix}-1 & \\ & a^2\delta_{ij}\end{pmatrix}+\begin{pmatrix}-2\Psi & \\ & a^2 2\Phi\delta_{ij}\end{pmatrix}+\begin{pmatrix}0 & A^i\\ A^i & \tilde{h}_{ij}\end{pmatrix}$$

Scalar perturbation (Ψ,Φ)

vector perturbation

tensor perturbation

$h_{ij}=a^2 2\Phi(t,\vec{x})\delta_{ij}+\tilde{h}_{ij}$

$a^2 2\Phi(t,\vec{x})\delta_{ij}$：trace identity；$\tilde{h}_{ij}$：trace less，$\mathrm{Tr}\,\tilde{h}_{ij}=0$

对物质演化，Scalar perturbation 最重要（物质的唯一来源，是引力的重要来源）

对某些奇异物质，vector perturbation 也重要

对原初引力波，tensor perturbation 重要

我们考虑简单的重要的 $g_{ij}=a^2(t)\delta_{ij}(1+2\Phi(t,\vec{x}))$

$g_{00}=-1-2\Psi(t,\vec{x})$

$\Psi,\Phi\ll 1$

Ψ 对应牛顿的引力势，是负的，给出牛顿 $\vec{a}=-\nabla\Psi$

在相对论下，分布函数 $f(x^\mu,p^\mu)$

momentum $p^\mu\equiv\frac{dx^\mu}{d\lambda}$

- square of momentum $\underline{P}^2=g_{\mu\nu}\underline{P}^\mu\underline{P}^\nu=0$ for photon

$$g_{\mu\nu}\underline{P}^\mu\underline{P}^\nu=-(1+2\Psi)p^{0\,2}+p^2$$

$$p^2=g_{ij}p^ip^j \quad (\text{空间部分})$$

所以 $p^0=\frac{p}{\sqrt{1+2\Psi}}\overset{\Psi\ll 1}{=}p(1-\Psi)$ （重要的结果） (★)

- 所以 $f(t,x^i,p,\hat{p}^i)$，$\hat{p}^i$ 动量方向，$\delta_{ij}\hat{p}^i\hat{p}^j=1$ (definition)

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x^i}\frac{dx^i}{dt}+\frac{\partial f}{\partial p}\frac{dp}{dt}+\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}$$

动量大小引入有关 零阶 一阶 2阶小量

一阶小量 小量 零阶

对于零阶的分布函数 $f^{(0)}=f^{(0)}(p)$

$$\frac{\partial f}{\partial \hat{p}^i}=\frac{\partial f^{(0)}}{\partial \hat{p}^i}=0$$

所以一阶的 $\frac{\partial f}{\partial \hat{p}^i}$ 是小量，又 $\frac{d\hat{p}^i}{dt}$ 是小量，最后一项是2阶小量

先求 $\frac{dx^i}{dt}=\frac{dx^i}{d\lambda}\Big/\frac{dt}{d\lambda}=p^i/p^0$

而另一方面 $p^i=C\hat{p}^i$

$p^2=g_{ij}p^ip^j=C^2\delta_{ij}a^2(1+2\Phi)\hat{p}^i\hat{p}^j=a^2(1+2\Phi)C^2$

所以 $C=a^{-1}p(1-\Phi)$

所以 $p^i=\frac{1-\Phi}{a}p\hat{p}^i$ (★)

最后 $\frac{p^i}{p^0}=\frac{(1-\Phi)\hat{p}^i}{a(1-\Psi)}=\frac{\hat{p}^i}{a}(1-\Phi+\Psi)=\frac{dx^i}{dt}$ (★)

意义，在 overdense region $\Psi<0$，所以 $\frac{dx^i}{dt}$ 变大

再求 $\frac{dp}{dt}$，因为 $\frac{\partial f}{\partial p}$ 是零阶的，所以 $\frac{dp}{dt}$ 需展开到一阶。

（这是动力学项，所以要用 geodesic）

$$\frac{dp^0}{d\lambda}=-\Gamma^0_{\alpha\beta}p^\alpha p^\beta \Rightarrow \quad lhs=\frac{dt}{d\lambda}\frac{dp^0}{dt}=p^0\frac{dp^0}{dt}=p(1-\Psi)\frac{d}{dt}\big(p(1-\Psi)\big)$$

$$=p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi)-p\frac{d\Psi}{dt}\right]$$

$$=p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi)-p\left(\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x^i}\frac{dx^i}{dt}\right)\right]$$

（$\frac{\hat{p}^i}{a}$ 保留零阶项）

$$-rhs=\frac{1}{2}g^{0\mu}(\partial_\alpha g_{\mu\beta}+\partial_\beta g_{\mu\alpha}-\partial_\mu g_{\alpha\beta})p^\alpha p^\beta$$

μ,α,β 都是哑标的，加起来是 $2\partial_\alpha g_{0\beta}$

$$=\frac{1}{2}g^{00}\left(2\partial_\alpha g_{0\beta}p^\alpha p^\beta-\frac{\partial}{\partial t}g_{\alpha\beta}p^\alpha p^\beta\right)$$

$g^{00}=-(1-2\Psi)$

$$=-\frac{1}{2}(1-2\Psi)\left(-4\partial_\alpha\Psi p^\alpha p^0-\frac{\partial}{\partial t}g_{\alpha\beta}p^\alpha p^\beta\right)$$

For photon

$$p^\mu=\left(p(1-\Psi),\ \frac{p\hat{p}^i}{a}(1-\Phi)\right)$$

$$\frac{dx^i}{dt}=p^i/p^0=\frac{\hat{p}^i}{a}(1-\Phi+\Psi)$$

$$\frac{dp}{dt}=-p\left[H+\frac{\partial\Phi}{\partial t}+\frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

$$f(p)=f^{(0)}(p)-p\frac{\partial f^{(0)}}{\partial p}\Theta(t,\vec{x},\hat{p}^i)$$

$$= -\frac{1}{2}(1-2\psi)\left(-4\frac{\partial\psi}{\partial t}p^{0\,2} - 4\frac{\partial\psi}{\partial x^i}p^ip^0 - \frac{\partial g_{00}}{\partial t}(p^0)^2 - \frac{\partial g_{ij}}{\partial t}p^ip^j\right)$$

$$-2\frac{\partial\psi}{\partial t} \qquad \delta_{ij}\frac{\partial}{\partial t}a^2(1+2\Phi)$$

$$= \delta_{ij}\left(2a^2H(1+2\Phi) + 2a^2\frac{\partial\Phi}{\partial t}\right)$$

$$= 2a^2\delta_{ij}\left(H+2H\Phi+\frac{\partial\Phi}{\partial t}\right)$$

修正0阶

$$= -\frac{1}{2}(1-2\psi)\left(-4\frac{\partial\psi}{\partial t}p^2 - 4\frac{\partial\psi}{\partial x^i}\frac{p^2}{a}\hat{p}^i + 2\frac{\partial\psi}{\partial t}p^2 - 2a^2\left(H+2H\Phi+\frac{\partial\Phi}{\partial t}\right)\frac{1-2\Phi}{a^2}p^2\,\underbrace{\delta_{ij}\hat{p}^i\hat{p}^j}_{1}\right)$$

保留一阶

所以 $\text{rhs} = \frac{p^2}{2}(1-2\psi)\left(-2\frac{\partial\psi}{\partial t} - \frac{4p^2}{ap^2}\frac{\partial\psi}{\partial x^i}\hat{p}^i - 2\left(H+2H\Phi-2H\Phi+\frac{\partial\Phi}{\partial t}\right)\right)$

$$= p^2(1-2\psi)\left(-\frac{\partial\psi}{\partial t} - \frac{2p^2}{ap^2}\frac{\partial\psi}{\partial x^i}\hat{p}^i - H - \frac{\partial\Phi}{\partial t}\right)$$

$$= p^2\left(-\frac{\partial\psi}{\partial t} - \frac{\partial\Phi}{\partial t} - \frac{2}{a}\frac{\partial\psi}{\partial x^i}\hat{p}^i - H + 2H\psi\right)$$

令 lhs = rhs 则有

$$\frac{dp}{dt}(1-\psi) = p\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{p}{a}\hat{p}^i + \underbrace{p(1+\psi)\,\text{rhs}/p^2}$$

$$= p(1+\psi)\left(-H+2H\psi - \frac{2}{a}\frac{\partial\psi}{\partial x^i}\hat{p}^i - \frac{\partial\psi}{\partial t} - \frac{\partial\Phi}{\partial t}\right)$$

$$= p\left(-H+2H\psi - \frac{2}{a}\frac{\partial\psi}{\partial x^i}\hat{p}^i - \frac{\partial\psi}{\partial t} - \frac{\partial\Phi}{\partial t} - H\psi\right)$$

$$= p\left[\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a} - H + 2H\psi - \frac{2}{a}\frac{\partial\psi}{\partial x^i}\hat{p}^i - \frac{\partial\psi}{\partial t} - \frac{\partial\Phi}{\partial t} - H\psi\right]$$

$$= p\left[-H + H\psi - \frac{\partial\Phi}{\partial t} - \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

$$\frac{dp}{dt} = p(1+\psi)\left[-H+H\psi - \frac{\partial\Phi}{\partial t} - \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

$$= p\left[-H+H\psi - \frac{\partial\Phi}{\partial t} - \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a} - H\psi\right]$$

$$= -p\left[H + \frac{\partial\Phi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a}\right] \qquad (\star)$$

Hubble flow 导致的红移减小 $p\propto a^{-1}$

$\frac{\partial\psi}{\partial x^i}<0$ 如果减掉入势阱

将以上结果代入 $\frac{d}{dt}$ 表达式

$\frac{dx^i}{dt}$ ~ 与 metric 有关的项，展开

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a} + \frac{\partial f}{\partial p}\frac{dp}{dt} + O(\text{二阶})$$

$$= \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x^i}}_{O(\Theta)}\frac{\hat{p}^i}{a} - p\frac{\partial f}{\partial p}\left(H + \frac{\partial\Phi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a}\right)$$

[将这些项保留和加和的项展开后的 $\frac{df}{dt}$]

For photon $f^{(0)}(p) = \left(\exp\frac{p}{T(t)} - 1\right)^{-1}$

更一般地，T 与 t, x 有关，0阶时忽略之

$$T(t,\vec{x},p,\hat{p}^i) = T(t)\left(1+\Theta(t,\vec{x},\hat{p})\right)$$

非均匀性 非各向同性

对应. 不考虑运动学大小的流法. 忽略掉（因为可与电子散射）

In general case $f(p) = \left(\exp\left(\frac{p}{T(t)(1+\Theta)}\right)-1\right)^{-1} = f^{(0)}(p) + \frac{\partial f^{(0)}}{\partial T}T(t)\Theta = f^{(0)}(p) - p\frac{\partial f^{(0)}}{\partial p}\Theta(t,\vec{x},\hat{p})$ $(\star)$

实可验证 $\frac{\partial f^{(0)}}{\partial T}T = -p\frac{\partial f^{(0)}}{\partial p}$ (easy)

下面只需计算 $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x^i}$, $\frac{\partial f}{\partial p}$ 等.

$$\left.\frac{df}{dt}\right|_{0^{th}\text{ order}} = \frac{\partial f^{(0)}}{\partial t} - p\frac{\partial f^{(0)}}{\partial p}H = 0 \quad (\text{for collisionless})$$

应该是有其他的物理定律告知没有 collision$^{(0)}$

$$= T\frac{\partial f^{(0)}}{\partial T}\frac{1}{T}\frac{dT}{dt} - pH\frac{\partial f^{(0)}}{\partial p} = -p\frac{\partial f^{(0)}}{\partial p}\cdot\frac{1}{T}\frac{dT}{dt} - pH\frac{\partial f^{(0)}}{\partial p} = -p\frac{\partial f^{(0)}}{\partial p}\left(\frac{1}{T}\frac{dT}{dt}+H\right) = 0$$

$$\Rightarrow -\frac{1}{T}\frac{dT}{dt} = \frac{1}{a}\frac{da}{dt} \Rightarrow T\propto a^{-1}$$

这是符合的. 因为相对光子能量有 $T\propto a^{-1}$.

也就是说, 即使不平衡, 在0阶(无碰撞)下也有 $T\propto a^{-1}$!

$$\left.\frac{df}{dt}\right|_{1^{st}\text{ order}} = \overbrace{-p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t}}^{-p\Theta\frac{\partial^2 f^{(0)}}{\partial p\,\partial t}} - p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} - p\frac{\partial f^{(0)}}{\partial p}\left(\frac{\partial\psi}{\partial x^i}\frac{\hat{p}^i}{a}+\frac{\partial\Phi}{\partial t}\right) + \underbrace{(-pH)(-\Theta)\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right)}_{A}$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} + \underbrace{\Theta p\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right)\frac{1}{T}\frac{dT}{dt}}_{B} + \dots$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}\right]$$

$(\star)$

注意

$$p\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) = p\frac{\partial f^{(0)}}{\partial p} + p^2\frac{\partial^2}{\partial p^2}f^{(0)}$$

第2项利用

$$-p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{\partial^2 f^{(0)}}{\partial T\,\partial p}\frac{dT}{dt}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial f^{(0)}}{\partial T}\right) = -\frac{\partial}{\partial p}\left(\frac{1}{T}p\frac{\partial f^{(0)}}{\partial p}\right) = -\frac{1}{T}p\frac{\partial^2 f^{(0)}}{\partial p^2} - \frac{1}{T}\frac{\partial f^{(0)}}{\partial p}$$

注意

A+B

$$= p\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right)\Theta\left(\frac{1}{T}\frac{dT}{dt}+H\right)$$

$$= 0$$

(利用 0^{th} order Boltzmann eq.)

总之 $\frac{df}{dt}$ 的形式已经展开到一阶

$$\frac{df}{dt} = C[f] \quad \text{collision terms from Compton Scattering}$$

光子的 Boltzmann 方程的线性形式

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b\right]$$

(引入Compton散射后的光子的一阶)

Other types of matter

Boltzmann equation
$$\begin{cases} \dfrac{df_{dm}}{dt} = 0 & \text{cold dark matter} \\ \dfrac{df_e}{dt} = C[f] & \text{Baryon} \end{cases}$$

和前面处理较的不同在于 $E^2 = p^2 + m^2$

($g_{\mu\nu}P^\mu P^\nu = -m^2$ for baryons/cdms)

1. CDM

✓将f改写成

$$f_{dm}(t,\vec{x},p,\hat{p}^i) = f_{dm}(t,\vec{x},E,\hat{p}^i)$$

E与p有确定关系

$$0 = \frac{df_{dm}}{dt} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{dm}}{\partial E}\frac{dE}{dt} + \underbrace{\frac{\partial f_{dm}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}}_{\text{二阶}}$$

f是0阶只与p有关(与方向无关)

FLRW metric 下是直线运动

$$= \frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E}\left[H\frac{p^2}{E} + \frac{\partial \Phi}{\partial t}\frac{p^2}{E} + \frac{\hat{p}^i}{a}p\frac{\partial \Psi}{\partial x^i}\right] \quad (:1)$$

多一个　改为E　改　加　加

$$P^\mu = \frac{dx^\mu}{d\lambda}$$
$$p^2 = g_{ij}P^iP^j$$

对光子
$$P^\mu = \left(p(1-\Psi), \frac{p\hat{p}^i}{a}(1-\Phi)\right)$$

对 baryon/cdm
$$P^\mu = \left(E(1-\Psi), \frac{p\hat{p}^i}{a}(1-\Phi)\right)$$

Here $E = \sqrt{p^2+m^2}$

除非对 f_{dm} 作黑体的假设，否则不能求解上面的方程。

我们先定义一些可观测量，然后希望求解这些可观测量的演化：

$$n_{dm}(t,\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} f_{dm} \qquad \text{cdm 数密度 (分布函数的零阶矩)}$$

$$v^i(t,\vec{x}) \equiv \frac{1}{n_{dm}}\int \frac{d^3p}{(2\pi)^3} f_{dm}\frac{p\hat{p}^i}{E} \qquad \text{速度 (分布函数的一阶矩)}$$

在 $\hat{p}^i$ 空间

(注意，这里的 v^i 的定义是 $\frac{dx^i}{d\eta}$，并且略去 $(1-\Phi+\Psi)$ 项)

要知道0阶矩，要知道1阶(多阶)混合　所以对不同阶的矩分别计算演化

要知道1阶矩，要知道2阶(到达)混合(被忽略掉了，所以close)

$$\int \frac{d^3p}{(2\pi)^3} \times (:1) \Rightarrow 0^{th} \text{ order momentum equation, including } n_e v^i$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{p\hat{p}^i}{E} \times (:1) \Rightarrow 1^{th} \text{ order momentum equation, including } v^i, \hat{p}^{i2} \text{ 忽略}$$

✓0阶矩方程：

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}\underbrace{\int \frac{d^3p}{(2\pi)^3} f_{dm}\frac{p\hat{p}^i}{E}}_{n_{dm}v^i} - \left(H + \frac{\partial \Phi}{\partial t}\right)\overbrace{\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{dm}}{\partial E}\cdot\frac{p^2}{E}}^{-3n_{dm}}$$
$$- \frac{1}{a}\frac{\partial \Psi}{\partial x^i}\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{dm}}{\partial E}p\hat{p}^i}_{\text{二阶小量，略去.}} = 0$$

第二部分 $\propto v^i \cdot \frac{\partial \Psi}{\partial x^i} \propto$ 二阶量

$E\,dE = p\,dp$

$$\frac{\partial f_{dm}}{\partial E}\frac{p^2}{E} = \frac{\partial f_{dm}}{\partial p}\cdot p$$

$$\int \frac{d^3p}{(2\pi)^3}\cdot\frac{\partial f_{dm}}{\partial p}\cdot p \quad \text{各向同性}$$
$$= \frac{4\pi}{(2\pi)^3}\int p^3dp\frac{\partial f_{dm}}{\partial p}$$
$$= -\frac{1}{2\pi^2}\int dp\, f_{dm}\frac{dp^3}{dp}$$
$$= -3\int \frac{4\pi}{(2\pi)^3}dp\cdot p^2 f_{dm}$$
$$= -3n_{dm}$$

⇓

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(n_{dm}v^i) + 3n_{dm}\left(H + \frac{\partial \Phi}{\partial t}\right) = 0$$

对 n_{dm} 展开，有 $n_{dm} = n_{dm}^{(0)}[1+\delta(\vec{x},t)]$

0^{th}-order:
$$\frac{\partial n_{dm}^{(0)}}{\partial t} + 3n_{dm}^{(0)}\frac{\dot{a}}{a} = 0$$
$$\Rightarrow \frac{d}{dt}\left(n_{dm}^{(0)}a^3\right)a^{-3} = 0$$
$$\Rightarrow n_{dm}^{(0)} \propto a^{-3} \quad \text{和我们的认识一样} \quad (:)$$

(·是对t)

1^{th}-order:
$$n_{dm}^{(0)}\frac{\partial \delta}{\partial t} + \underbrace{\frac{\partial n_{dm}^{(0)}}{\partial t}\delta}_{①} + \frac{1}{a}\frac{\partial}{\partial x^i}\left(n_{dm}^{(0)}v^i\right) + 3n_{dm}^{(0)}\frac{\partial \Phi}{\partial t} + \underbrace{3n_{dm}^{(0)}\delta\frac{\dot{a}}{a}}_{②} = 0$$

与 x^i 无关

① + ② = 0 (利用 0^{th}-order 方程)

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}v^i + 3\frac{\partial \Phi}{\partial t} = 0$$

换为 conformal time

$$\frac{\partial \delta}{\partial \eta} + \frac{\partial}{\partial x^i}v^i + 3\frac{\partial \Phi}{\partial \eta} = 0$$
$$\Leftrightarrow \dot{\delta} + \frac{\partial v^i}{\partial x^i} + 3\dot{\Phi} = 0$$

变换到 Fourier 空间

$$\dot{\tilde{\delta}} + i\vec{k}\cdot\vec{v} + 3\dot{\tilde{\Phi}} = 0 \quad (:2)$$

留到作业

1阶矩方程：

$$\frac{\partial(n_{dm}v^i)}{\partial t} + 4Hn_{dm}v^i + \frac{n_{dm}}{a}\frac{\partial \Psi}{\partial x^i} = 0$$

对 n_{dm} 展开，0^{th}-order 不存在

$$\frac{\partial n_{dm}^{(0)}}{\partial t} + 3Hn_{dm}^{(0)} = 0$$

1^{th}-order ($n_{dm} \to n_{dm}^{(0)}$)
$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0$$
$$\Leftrightarrow \frac{\partial v^i}{\partial \eta} + \frac{\dot{a}}{a}v^i + \frac{\partial \Psi}{\partial x^i} = 0 \quad (\text{· 是对 conformal time})$$
$$\Leftrightarrow \dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0 \quad (:3)$$

$\tilde{v}$ is irrotational
$$\tilde{\vec{v}} = \tilde{v}\hat{k}$$

(:3) 也可以写为没有方向的形式
$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0$$

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Einstein 下节课

metric perturbation → matter perturbation

Boltzmann 本节课

06/06/2018 Quiz
范围 第一章到第三章 Dodelson
从现在之前的内容 (?)

HW
Derive the Boltzmann eq for CDM
from $\frac{df_{dm}}{dt}=0$
to (:2) and (:3)
Fill all skipped steps

Boltzmann Equation for photons

$$\begin{cases} g_{00}(t,\vec{x}) = -1-2\Psi(t,\vec{x}) \\ g_{0i}=0 \\ g_{ij}(t,\vec{x}) = a^2\delta_{ij}(1+2\Phi(t,\vec{x})) \end{cases}$$

Scalar perturbation metric

four momentum

$$p^\mu = \left(p(1-\Psi),\ p\hat{p}^i\frac{1-\Phi}{a}\right)$$

spatial magnitude

$f(t,\vec{x},p,\hat{p}^i)$ distribution function

$$= f^{(0)}(p) - p\frac{\partial f^{(0)}}{\partial p}\Theta(t,\vec{x},\hat{p})$$

$$T(t,\vec{x},\hat{p}^i) = T(t)\left(1+\Theta(t,\vec{x},\hat{p}^i)\right)$$

$\frac{df}{dt}=0$ (collisionless) ~ 每一阶都是0

全微分，是 Lagrangian 表述

$$\begin{cases} \left.\frac{df}{dt}\right|_{0^{th}\text{-order}} = 0 \iff T(t)\propto a^{-1} \\ \left.\frac{df}{dt}\right|_{1^{th}\text{-order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \end{cases}$$

在宇宙早期 (e.g. CMB)，或足够大尺度 $\Theta,\Psi,\Phi \ll 1$

可以将 Boltzmann 方程线性化

对光子，由于会与电子发生 Compton 散射，所以粒子数不守恒

更一般地

$$\frac{df}{dt} = C[f]$$

先将 $C[f]$ 的形式写出，然后就可以展开，并写出 metric perturbation 的形式

Compton Scattering

$$\begin{matrix} e+\gamma & \rightleftharpoons & e+\gamma \\ \vec{q}\ \ \vec{p} & & \vec{q}'\ \ \vec{p}' \end{matrix}$$

光子的分布函数 $f(\vec{p})$

电子的分布函数 $f_e(\vec{q})$

← 产生过程

$$C[f(\vec{p})] = \sum_{\vec{q},\vec{q}',\vec{p}'}|\text{Amplitude}|^2\left\{f_e(\vec{q}')f(\vec{p}')(1-f_e(\vec{q}))(1+f(\vec{p}))\right.$$

Pauli Blocking　Bose Enhancement

，统计项全部忽略，因 $f, f_e \ll 1$

$$= \frac{1}{p}\int\frac{d^3q}{(2\pi)^3 2E_e(\vec{q})}\int\frac{d^3q'}{(2\pi)^3 2E_e(\vec{q}')}\int\frac{d^3p'}{(2\pi)^3 2E(\vec{p}')}\ \left. - f_e(\vec{q})f(\vec{p})(1-f_e(\vec{q}'))(1+f(\vec{p}'))\right\}$$

光子的 $\frac{1}{2E(\vec{p})}$

来自于δ函数 $\int d^4p\,\delta(E^2-p^2-m^2) = \int\frac{d^3p}{2E(p)}$

$$|\mathcal{M}|^2(2\pi)^4\cdot\delta^3(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\cdot\delta(E(\vec{p})+E_e(\vec{q})-E(\vec{p}')-E_e(\vec{q}'))$$

$$\left\{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right\}$$

可以将三重的 $\vec{q}'$ 积分消掉 (利用 δ^3 函数)

可以将剩下的 $\vec{q}$ 或 $\vec{p}'$ 的一个自由度消去 (利用 δ 函数)

$E_e^{(q)} = \frac{q^2}{2m_e}+m_e$

$E(p) = p$

对 Compton Scattering, leading term

$$|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2,\quad \sigma_T:\ \text{Thomson Scattering cross-section}$$

最后

由光子分布函数 f 带来

$$C[f]_{1^{st}\text{-order}} = -p\frac{\partial f^{(0)}}{\partial p}(n_e\sigma_T)\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]$$

$$\Theta_0 = \frac{1}{4\pi}\int\Theta(t,\vec{x},\hat{p}')d\Omega'$$

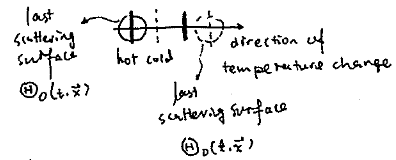
monopole of Θ

对方向的平均值

bulk baryon velocity

大尺度的整体运动

由 f_e 带来



结合 (?) $\left.\frac{df}{dt}\right|_{1^{st}\text{ order}}$ 与 $C[f]|_{1^{st}\text{-order}}$ 中一阶项:

$$\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left(\Theta_0-\Theta(\hat{p})+\hat{p}\cdot\vec{v}_b\right)$$

也是 $t,\vec{x}$ 的函数

引入 Conformal time　$dt = a\,d\eta$，则方程左边改写为

$$\left(\frac{\partial\Theta}{\partial\eta}+\hat{p}^i\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial\eta}+\hat{p}^i\frac{\partial\Psi}{\partial x^i}\right)\frac{1}{a}$$

这是 Θ, Φ 中的线性方程，在 Fourier 空间 $\frac{\partial}{\partial x^i}\to ik^i$

特点是可叠加性 …

无非线性的 Coupling

不同的 k-mode 独立演化 (尺度)

大尺度的线性性来自于线性近似，小尺度，线性演化不再成立；

出现非线性项 $\Theta(\vec{x})\Phi(\vec{x}) \xrightarrow{\text{Fourier}} \int\frac{d^3k'}{(2\pi)^3}\tilde{\Theta}(\vec{k}')\tilde{\Phi}(\vec{k}-\vec{k}')$

有卷积，表明不同的 mode 是相互耦合的。

对方程作 Fourier 变换，$\Theta\to\tilde{\Theta}$．

$\frac{\partial}{\partial\eta}\to\ \dot{}$

$$\dot{\tilde{\Theta}} + i\underbrace{\hat{p}\cdot\vec{k}}_{=k\mu}\tilde{\Theta} + \dot{\tilde{\Phi}} + i\hat{p}\cdot\vec{k}\tilde{\Psi} = \underbrace{n_e\sigma_T a}_{=-\dot{\tau}}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \hat{p}\cdot\tilde{\vec{v}}_b\right]$$

$\hat{p}\cdot\frac{\vec{k}}{k} = \mu = \cos\theta(\hat{p},\hat{k})$

$\vec{k}$ 即是平面波方向

或强度改变方向

Compton 散射的光深

$\nabla\times\vec{v}_b = 0 \Rightarrow i\vec{k}\times\tilde{\vec{v}}_b(\vec{k}) = 0$

bulk velocity irrotational

$\vec{v}_b(\vec{k}) \parallel \vec{k}$，$\simeq \tilde{\vec{v}}_b = \tilde{v}_b\hat{k}$

$\hat{p}\cdot\tilde{\vec{v}}_b = \tilde{v}_b\mu$

因 $\tau = \int_t^{t_0} n_e\sigma_T\cdot c\,dt' = \int_\eta^{\eta_0} c\,d\eta' n_e\sigma_T a'$

$n_e\sigma_T a = -\frac{\partial\tau}{\partial\eta} = -\dot{\tau}$

2. Baryon (e^-, p^+)

仍有第一阶矩的方程 $\frac{\partial n_b}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(n_b v_b^i) + 3n_b\left(H + \frac{\partial \Phi}{\partial t}\right) = \int C[f]$

差别在于右边有散射项

由于电子、质子之间有很强的库仑耦合，可以写到一起 $1+\delta = \frac{n_e}{\bar{n}_e} = \frac{n_p}{\bar{n}_p} = \frac{\rho_e}{\bar{\rho}_e} = \frac{\rho_b}{\bar{\rho}_b} = \frac{\rho_p}{\bar{\rho}_p}$

库仑散射 $e+p \rightleftharpoons e+p$

$\vec{q}\ \vec{Q} \quad \vec{q}'\ \vec{Q}'$

Compton Scattering $e+\gamma \rightleftharpoons e+\gamma$

$\vec{q}\ \vec{p} \quad \vec{q}'\ \vec{p}'$

对电子而言.

$$\frac{df_e}{dt}(\vec{q}) = \langle C_{ep}\rangle_{QQ'q'} + \langle C_{e\gamma}\rangle_{pp'q'}$$

$$\frac{df_p}{dt}(\vec{Q}) = \langle C_{ep}\rangle_{qQ'q'}$$

记号 $\langle \text{some}\rangle_{pp'q'} = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3q'}{(2\pi)^3}$

$$C_{e\gamma} = (2\pi)^4\delta^4(p+q-p'-q')\frac{|\mathcal{M}|^2}{8E(p)E(p')E_e(q)E_e(q')}\left\{f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\right\}$$

对电子 $\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(n_e v_b^i) + 3n_e\left(H + \frac{\partial\Phi}{\partial t}\right) = \int C[f]$

bulk flow
耦合在一起

令 $n_e \equiv \int\frac{d^3q}{(2\pi)^3}f_e(\vec{q})$，则右边变为 $\underbrace{\langle C_{ep}\rangle_{QQ'qq'}}_{=0} + \langle C_{e\gamma}\rangle_{pp'qq'}$

由于 $C_{e\gamma}$ 对 $(ff-ff)$ 反对称 $p\leftrightarrow p'$, $q\leftrightarrow q'$
对其他部分对称.

因此积分一定为0 for lowest order
and all order

所以 0阶矩的方程与CDM一样

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0$$

同样，质子的方程也是如同一样，所以合写为b.

对一阶矩方程 × $\int\frac{d^3q}{(2\pi)^3}\frac{p\hat{p}^i}{E}$，$C[f]$的积分并不是完全为0的

而是 $\langle C_{e\gamma}\vec{q}\rangle_{pp'q'q} = \langle C_{e\gamma}\vec{p}\rangle_{pp'q'q}$

注意 $\langle C_{e\gamma}\vec{p}\rangle$ 事实上是一阶矩，定义 $\Theta_1 = i\int\frac{d\mu}{2}\mu\,\Theta(\mu)$

方程可写为

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi}/\hat{k} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}[3i\tilde{\Theta}_1 + \tilde{v}_b] \quad \text{(见Dodelson)}$$

一般地. $\Theta_l = \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\,\Theta(\mu)$

用了球谐函数或Legendre展开

HW
Dodelson Ch5
ex 5 (+8 选择做)
试证明一定要[illegible]. 可和ID ex3结果和书上比较

05/23/2018 物理学院南楼 6B301

How matter perturbation affect metric perturbation? (Einstein field equation)

For scalar perturbation

$$\begin{cases} g_{00} = -1-2\Psi \\ g_{0i} = 0 \\ g_{ij} = a^2\delta_{ij}(1+2\Phi) \end{cases}$$

- Compute Christoffel Symbol $\Gamma^\alpha_{\mu\nu}$
- Compute Ricci tensor $R_{\mu\nu}$
- Compute Ricci Scalar R
- Compute Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$
- Compute energy-momentum tensor perturbation $G_\mu{}^\nu = 8\pi T_\mu{}^\nu$
- Seperate out the linear perturbation $\delta G_\mu{}^\nu = 8\pi\,\delta T_\mu{}^\nu$ (see Dodelson)

Skip the calculations, For Scalar perturbation

$$\delta G^0{}_0 = -6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}$$

$$T^0{}_0 = -\rho = -\sum_{\text{all species } i} g_i\int\frac{d^3p}{(2\pi)^3}E_i(p)\,f_i(\vec{x},\vec{p},t)$$

$E_i(p) = \sqrt{p^2+m^2}$

For photons $f(\vec{x},\vec{p},t) = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$

$$f^{(0)}_{(p)} = [e^{p/T(t)} - 1]^{-1}$$

$$T(t,\vec{x},\hat{p}) = T(t)[1+\Theta(t,\vec{x},\hat{p})]$$

zeroth order:

$$2\int\frac{d^3p}{(2\pi)^3}p f^{(0)}(p) \equiv \rho_\gamma(t)$$ average photon energy density (不含扰动项) 仅与 a(t) 有关的项

first order:

$$-2\int\frac{d^3p}{(2\pi)^3}\, p\times \underbrace{p\frac{\partial f^{(0)}}{\partial p}}_{\text{和大小有关 } p}\underbrace{\Theta(t,\vec{x},\hat{p})}_{\text{和方向有关 } \hat{p}} = -2\underbrace{\int\frac{dp\times 4\pi}{(2\pi)^3}p^4\frac{\partial f^{(0)}}{\partial p}}_{} \times \underbrace{\int d\Omega\,\Theta(t,\vec{x},\hat{p})\frac{1}{4\pi}}_{\text{monopole } \Theta_0(t,\vec{x})} = 4\rho_\gamma\Theta_0$$

$d^3p = p^2 dp\, d\Omega$

$$\int\frac{dp\cdot p^4}{(2\pi)^3}\frac{\partial f^{(0)}}{\partial p}$$

分部积分 $\frac{p^4 f^{(0)}}{(2\pi)^3}\Big|_0^\infty - 4\int\frac{p^3 dp\, f^{(0)}(p)}{(2\pi)^3}$

边界项为0 $-\frac{4}{4\pi}\int\frac{d^3p}{(2\pi)^3}p f^{(0)}(p)$

$$= -\frac{4}{4\pi}\frac{1}{2}\rho_\gamma(t)$$

对 energy-momentum tensor 的贡献

$$T^0{}_0 = -\rho_\gamma(1+4\Theta_0) \quad \text{(photons)}$$

适用 massless particle (radiation) 都可以用

评论: 光子 $\rho \propto T^4$

$$\frac{\delta\rho}{\rho} = 4\frac{\delta T}{T} = 4\Theta$$

所以 4倍的关系很明显, 另外应该对角度平均

For neutrino.

$$T^0{}_0 = -\rho_\nu(1+4\mathcal{N}_0) \quad \text{(neutrinos)}$$

中微子温度 no fluctuation

$$T_\nu = T_\nu(t)(1+\mathcal{N}(t,\vec{x},\hat{p}))$$

$$\mathcal{N}_0 = \int \mathcal{N}(t,\vec{x},\hat{p})\frac{d\Omega}{4\pi} \quad \text{monopole}$$

For dark matter

$$T^0{}_0 = -\rho = -\rho_{dm}(1+\delta)$$

dm density fluctuation

Note: a function of only t and $\vec{x}$, but not $\hat{p}$

For baryon

$$T^0{}_0 = -\rho_b(1+\delta_b)$$

baryon 因为库仑耦合, 所以假定 $\delta_b = \delta_{proton} = \delta_{electron}$

For dark energy

$$T^0{}_0 = -\rho_\Lambda \quad \text{(No perturbation)}$$

for Cosmological Constant

Time-Time Component perturbation of $T^0{}_0$

$$\delta T^0{}_0 = -\left[4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0 + \rho_{dm}\delta + \rho_b\delta_b\right]$$

Einstein field equation $\delta G^0{}_0 = 8\pi G\delta T^0{}_0$ then gives:

$$-6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2} = -8\pi G[\cdots]$$

$\div(-2)$ $\Leftrightarrow$

$$3H\Phi_{,0} - 3\Psi H^2 + \frac{k^2\Phi}{a^2} = 4\pi G[\quad]$$

Conformal time $d\eta = \frac{dt}{a}$, $\frac{\partial\Phi}{\partial t} = \frac{1}{a}\frac{\partial\Phi}{\partial\eta} = \frac{1}{a}\dot{\Phi}$ ("$\cdot$" denotes derivative w.r.t. η)

$$H = \frac{da}{a\,dt} = \frac{1}{a^2}\dot{a}$$

$\times a^2$ $\Leftrightarrow$

$$3\frac{\dot{a}}{a}\dot{\Phi} - 3\Psi\left(\frac{\dot{a}}{a}\right)^2 + k^2\Phi = \underbrace{4\pi G a^2[\cdots]}_{}$$

$$= 3\frac{\dot{a}}{a}\underbrace{[\dot{\Phi} - \Psi\frac{\dot{a}}{a}]}_{\text{G.R effect of Hubble expansion}} + k^2\Phi$$

$4\pi G a^2 \times$ density perturbation $= 4\pi G a^2\delta\rho$

评论: for static universe $(a \equiv 1)$

$$k^2\Phi = 4\pi G\delta\rho \quad \text{(波数空间)}$$

$$-\nabla^2\Phi = 4\pi G\delta\rho \quad \text{(实空间的 Poisson 方程)}$$

评论: G.R.效应在什么时候比较显著?

$$3\Psi\left(\frac{\dot{a}}{a}\right)^2 \gtrsim k^2\Phi \quad \text{Hubble terms wins}$$

$\Psi \simeq \Phi$, $3\sim 1$, $\frac{\dot{a}}{a} \gtrsim k$ 即可, 亦即

$$\underbrace{\frac{a}{k}}_{} \gtrsim H^{-1}$$

Hubble scale $c\cdot H^{-1}$

$\frac{1}{k}$ comoving scale

$\frac{a}{k}$ physical size

亦即, 尺度的大小 ~ event horizon scale 时, G.R.效应比 Newtonian 重要

尺度的大小 ≲ Hubble Scale 时, G.R.效应不重要

(N-body simulation 一般不考虑 G.R.效应)

而只把G.R.的效应暗含在时空背景即a(t)内

The Spatial Component

$$G^i{}_j = \overset{\text{diagonal}}{A\delta^i{}_j} + \overset{\text{off-diagonal}}{k_i k_j \frac{(\Phi+\Psi)}{a^2}}$$

已经做了 Fourier 变换
$(t,\vec{x}) \leftrightarrow (t, i\vec{k})$

Consider longitudinal traceless part of $G^i{}_j$

$$(\underbrace{\hat{k}_i \hat{k}^j - \tfrac{1}{3}\delta_i{}^j}_{\text{projection operator}})\underbrace{\delta^i{}_j}_{\text{identity matrix}} = \hat{k}_i\hat{k}_i - \tfrac{1}{3}\delta_i{}^i = 1-1=0$$

$(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j) G^i{}_j$: longitudinal (non-traceless) part $\quad \nabla_j[\cdots]^{ij}=0$ 实空间
$k_j[\cdots]^{ij}=0$ fourier空间

$\hat{k}^i\hat{k}_i - \frac{1}{3}\delta_i{}^i = 0$: Traceless $\quad k_j(\cdots) = (k_j \hat{k}_i \hat{k}^j - \frac{1}{3}k_j\delta_i{}^j) G^i{}_j$

Trace $\quad (\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i{}^j) G^i_j = \frac{\Phi+\Psi}{a^2}(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i{}^j)k^i k_j = K^2 - \frac{1}{3}K^2 = \frac{2}{3}K^2$

$$= \frac{2K^2(\Phi+\Psi)}{3a^2}$$

Energy-momentum Tensor

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & & \\ & p & \\ & & p \\ & & & p \end{pmatrix} \quad \text{for non-perturbative situation}$$

$$P_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x},\vec{p},t) \frac{p^2}{3E_i(p)}$$

$$T^i{}_j = g\int \frac{d^3p}{(2\pi)^3} f(\vec{x},\vec{p},t)\frac{p^i p_j}{E(p)} \qquad p^i p_i = \tfrac{1}{3}p^2 \text{ 就回到对角情况 (no summation)}$$

fourier ↓

$$T^i_j(t,\vec{k}) = \cdots f(\vec{k},\vec{p},t)\cdots$$

Consider longitudinal, traceless part of $G^i{}_j$:

$$(\hat{k}_i\hat{k}^j - \tfrac{1}{3}\delta_i{}^j) T^i{}_j = \sum_{\text{all species}} g\int\frac{d^3p}{(2\pi)^3} f(\vec{k},\vec{p},t)\frac{1}{E(p)}(\hat{k}_i\hat{k}^j - \tfrac{1}{3}\delta_i{}^j)p^i p_j$$

$\mu^2p^2 - \frac{1}{3}p^2 = (\mu^2 - \frac{1}{3})p^2 = P_2(\mu)p^2 \cdot \frac{2}{3}$ ← Legendre polynomial $l=2$ (quadrupole)

Note $\hat{k}_i p^i = \hat{k}_i \hat{p}^i p = \mu p$, $\mu = \frac{\hat{k}\cdot\vec{p}}{p}$

$$= \sum_{\text{all species}} g\int\frac{d^3p}{(2\pi)^3} f(\vec{k},\vec{p},t)\frac{2P_2(\mu)p^2}{3E(p)}$$

For photons

$$f = \underbrace{f^{(0)}(p,t)}_{\text{引入上标,为 background term}} - p\frac{\partial f^{(0)}}{\partial p}(p,t)\underbrace{\Theta(t,\vec{k},\hat{p})}_{\text{扰动项}}$$

Note:
$\Theta_l = \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\Theta(\mu)$
$\Theta_1 = i\int_{-1}^1 \frac{d\mu}{2}\mu\Theta(\mu)$

$g_i = 2$

$$(\hat{k}_i\hat{k}^j - \tfrac{1}{3}\delta_i^j)\delta T^i{}_j = -2\int\frac{d^3p}{(2\pi)^3}p\frac{\partial f^{(0)}}{\partial p}\Theta\,\frac{2P_2(\mu)p^2}{3p} = -2\int\underbrace{\frac{dp\cdot p^2}{(2\pi)^3}p^2\frac{\partial f^{(0)}}{\partial p}}_{-\frac{1}{(2\pi)^3}\int 4p^3 f^{(0)}dp} \times \underbrace{\int\frac{2}{3}d\Omega\,\Theta(\hat{p})P_2(\mu)}_{\text{只与}\Theta\text{有关}} = -\frac{8}{3}\rho_\gamma\Theta_2$$

$$\overbrace{-2\pi\int 2\cdot\tfrac{2}{3}\tfrac{d\mu}{2}\Theta(\hat{p})P_2(\mu)}^{-2\pi\frac{4}{3}\Theta_2}$$

$$-\frac{1}{(2\pi)^3}\int 4p^3 f^{(0)}dp = -\frac{1}{\pi}\int\frac{d^3p}{(2\pi)^3}pf^{(0)} = -\frac{1}{2\pi}\rho_\gamma$$

Einstein Equation

$$K^2(\Phi+\Psi) = -32\pi G a^2 \rho_\gamma \Theta_2$$

加上中微子的贡献

$$K^2(\Phi+\Psi) = -32\pi G a^2[\rho_\gamma\Theta_2 + \rho_\nu \mathcal{N}_2]$$

dm, baryon 的 ρ 只与大小有关, 与方向无关, 没有四极矩.

由于四极矩很小, 所以经常有 $\Phi \simeq -\Psi$

Small talk about tensor perturbation

$$g_{ij} = a^2(\delta_{ij} + h_{ij})$$

h_{ij} = trace × $\mathbb{1}$ (标量) + traceless $\mathcal{H}_{ij}$ (张量)

For tensor perturbation

$$\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

只在 x y 平面扰动
也可以在其他 2 个方向扰动

2 freedom parameter $(h_+, h_\times)$ for a given direction z (propagation direction)
所以经常说引力波是横波

给出 metric perturbation 之后,

$$\delta G^1{}_1 - \delta G^2{}_2 = 3Hh_{+,0} + \underset{\uparrow\,\text{对时导数}}{h_{+,00}} + \frac{K^2 h_+}{a^2} = a^{-2}[\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + K^2h_+] \quad (\text{Conformal time})$$

通常 matter 没有 tensor perturbation, 所以 (引力波无源)

$$\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + K^2 h_+ = 0$$

同样 $\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + K^2 h_\alpha$ for $\alpha = \times$

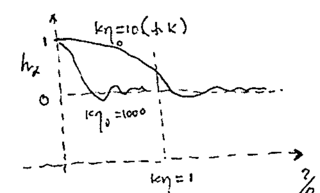
对于无膨胀的空间 $\dot{a}=0$ (Static Universe), 即 经典波动方程 $\ddot{h}_\alpha + K^2 h_\alpha = 0$

解为 $h_\alpha(\vec{x},\eta) = \int d^3k\, e^{i\vec{k}\cdot\vec{x}}[\underset{+z\text{方向}}{Ae^{ik\eta}} + \underset{-z\text{方向}}{Be^{-ik\eta}}]$

$ck\eta = \omega\eta$, $\omega = ck$. (与光相同的色散)

所以 Speed of gravitational wave = c.

对于有 Hubble expansion, 多一个粘滞项



tensor perturbation 在宇宙早期不变
进入 hubble scale 时 不再守恒

$k\eta c = \frac{kc\cdot t}{a} \sim \frac{kcH^{-1}}{a}$, $k\eta \simeq 1$ means $\underset{\text{physical size}}{\frac{a}{k}} \simeq \underset{\text{hubble size}}{cH^{-1}}$ 早期的 → 后面的
(大尺度的引力波被压制)

At epoch of decoupling $\frac{\eta}{\eta_0} \simeq 0.02$

if $k\eta_0 \lesssim 100$，才有显著的影响（大尺度,原初的引力波才显著）

由标量场引起

A small talk about Scalar perturbation

$$\begin{cases} g_{00} = -1-2\psi \\ g_{0i} = 0 \\ g_{ij} = (1+2\Phi)\, a^2 \delta_{ij} \end{cases} \xrightarrow{\text{more general case}} \quad \begin{aligned} g_{00} &= -(1+2A) \\ g_{0i} &= -a B_{,i} \rightarrow \frac{\partial B}{\partial x^i} \\ g_{ij} &= a^2\left(\delta_{ij}[1+2\psi] - 2E_{,ij}\right) \end{aligned}$$

$$E_{,ij} \rightarrow \frac{\partial E}{\partial x^i \partial x^j}$$

更一般的情况 (A, B, ψ, E)

四个scalars，四个自由度

- Gauge choice

general covariance

物理定律在任何坐标变换下形式一样，e.g. $x^\mu \rightarrow \tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} x^\nu$

从理论上来讲，这就产生了一定的自由度，所以广义相对论也是 gauge theory.

$A(x^\mu) \rightarrow \tilde{A}(\tilde{x}^\mu) = A(x^\mu)$

scalar

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}$$

contains some gauge freedom

如果某个坐标变换

$$t \rightarrow \tilde{t} = t + \xi^0(t, \vec{x})$$

$$x^i \rightarrow \tilde{x}^i = x^i + \delta^{ij}\xi_{,j}(t, \vec{x})$$

ξ^0, $\xi_{,j}$ 有两个自由度，可以适当选择使 $\tilde{B}=0$, $\tilde{E}=0$

最后只剩 A, ψ 两个自由度 (conformal Newtonian Gauge)

$$\left.\begin{aligned} A &\rightarrow \tilde{A} = A - \frac{1}{a}\dot{\xi}^0 \\ \psi &\rightarrow \tilde{\psi} = \psi - H\xi^0 \\ B &\rightarrow \tilde{B} = B - \frac{\xi^0}{a} + \dot{\xi} \end{aligned}\right\} \text{最后得到的结果是不变的}$$

$$E \rightarrow \tilde{E} = E + \xi$$

也有不同的 gauge，但在具体问题分析上可以选择方便（当然，最后的可观察量是一样）

e.g. $\Phi_A = A + a^{-1}\frac{\partial}{\partial\eta}[\dot{a}(\dot{E}-B)]$

$\Phi_B = -\psi + aH(B-\dot{E})$ 是 gauge-invariant 的.

6月27日，周三

期末考试 take home

HW Dodelson Ch6 ex4

Initial Conditions

(1) At early times, $k \sim \frac{1}{\lambda}$ (对于给定的 k), $k\eta \ll 1$

$\eta = \int_0^t \frac{dt}{a}$ 代表 comoving horizon size = comoving size of causal patch (所有因果联系的区域)

$k\eta = \frac{\eta}{\lambda} = \text{comoving horizon} / (\text{comoving scale of interest}) \ll 1$

(2) Higher moments than monopole can be neglected at early times.

Boltzmann equation

$\dot{\Theta}_0 + \dot{\Phi} = 0$ (photons)

$\dot{N}_0 + \dot{\Phi} = 0$ (neutrinos)

$\dot{\delta} + 3\dot{\Phi} = 0$ (DM)

$\dot{\delta}_b + 3\dot{\Phi} = 0$ (Baryon)

e.g. 对光子

$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left(\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right)$

$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$

$\dot{\Theta} = \frac{d\Theta}{d\eta} \simeq \frac{\Theta}{\eta}$

$\frac{\dot{\Theta}}{\Theta} \simeq \frac{\Theta/\eta}{ik\mu\Theta} \sim \frac{1}{i\mu}\frac{1}{k\eta} \gg 1$

Take the monopole

$\dot{\Theta}_0 + \dot{\Phi} = 0$

Einstein Equation

e.g. 对 DM，忽略 ikv (很小)

$\underbrace{k^2\Phi}_{\text{忽略}=0} + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left[\underbrace{\rho_{dm}\delta + \rho_b\delta_b}_{\text{早期时\ 这两项不为主}=0} + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]$

($\dot{\Phi}$: 对 η 求导；$\frac{\dot{a}}{a} \sim \frac{1}{\eta}$)

Radiation dominated $\eta \propto a$

$\frac{\dot{a}}{a} = \frac{1}{\eta}$

定义 ρ 是总的 density，利用 0^{th}-order Einstein Equation $\left(\frac{1}{a}\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$ ($\frac{1}{a^2\eta^2}$)

把 Einstein Equation 改写为 $\frac{1}{\eta}\dot{\Phi} - \frac{1}{\eta^2}\Psi = \frac{2}{\eta^2}\left(\frac{\rho_\gamma\Theta_0 + \rho_\nu N_0}{\rho}\right)$

定义 $f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}$，则改写为 $\eta\dot{\Phi} - \Psi = 2\left[(1-f_\nu)\Theta_0 + f_\nu N_0\right]$

再对 η 求一阶导数 $\eta\ddot{\Phi} + \dot{\Phi} - \dot{\Psi} = 2\left[(1-f_\nu)\dot{\Theta}_0 + f_\nu\dot{N}_0\right] = 2\dot{\Theta}_0 = -2\dot{\Phi}$

$\dot{\Theta}_0 + \dot{\Phi} = 0$

$\dot{N}_0 + \dot{\Phi} = 0$

Use $\Phi = -\Psi$ for higher momentum negligible

$\eta\ddot{\Phi} + 4\dot{\Phi} = 0 \Rightarrow \frac{d\dot{\Phi}}{\dot{\Phi}} = -\frac{4}{\eta}d\eta$

$\Phi = \underbrace{C_1\eta^{-3}}_{\text{decay}} + \underbrace{C_2}_{\text{constant}}$

第2项占主导，因为第一项 ($C_1\eta^{-3}$) 衰减很快，所以早期宇宙 $\Phi(\eta) = \text{constant}$.

代回原来的方程有

$\Phi(\eta) = \Phi = 2\left[(1-f_\nu)\Theta_0 + f_\nu N_0\right] = 2\Theta_0$ 在宇宙早期也是 constant

For dark matter $\frac{n_{dm}}{n_\gamma} = \frac{n_{dm}^{(0)}}{n_\gamma^{(0)}}\frac{1+\delta}{1+3\Theta_0} \simeq \frac{n_{dm}^{(0)}}{n_\gamma^{(0)}}(1+\delta-3\Theta_0)$

(number density)

可假定 $\frac{n_{dm}}{n_\gamma} = \frac{n_{dm}^{(0)}}{n_\gamma^{(0)}}$ (称为 adiabatic perturbation) 早期的 DM 和光子紧密 couple 在一起

或者 $\frac{n_{dm}}{n_\gamma} \neq \frac{n_{dm}^{(0)}}{n_\gamma^{(0)}}$ (称为 iso curvature perturbation)

在 adiabatic perturbation 下

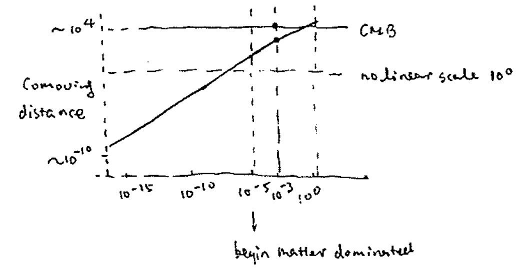
$\underbrace{\delta_b = \delta}_{\text{couple}} = 3\Theta_0$ (δ: DM)

再将 Boltzmann 方程和 Einstein 方程保留到 1^{st} order (因为 v_b 和 v 是一阶的，所以也需要)，得出它们初始条件

1^{st}-order $\Theta_1 = N_1 = \frac{iv_b}{3} = \frac{iv}{3} = -\frac{k\Phi}{6aH}$

Horizon Problem

(CMB 的温度场是均匀的，各向同性的 (今天观察到，同时 z=1100 时也是均匀各向同性的))



• $\underbrace{\eta}_{\text{Comoving Horizon}} = \int_{t=0}^{t(a)} \frac{dt'}{a'} = \int_0^a \frac{da'}{a'^2 H(a')} = \int_0^a \frac{da'}{a'} \frac{1}{a' H(a')}$ → Comoving Hubble radius

$\frac{1}{H} \doteq$ physical Hubble radius = largest physical size of local causal patch (最大的可能存在因果联系区域)

$\simeq$ time during $\frac{a}{2} \to a$ (这也并不是从 t=0 开始传播的)

因为 $\Delta t = \int_{a/2}^a \frac{da'}{a' H(a')} = \frac{1}{H(a)}\ln 2 \simeq \frac{1}{H(a)}$

近似地，可认为在 Radiation Dominated Universe，$a \propto \eta$，$H = \frac{\dot{a}}{a^2} = \frac{1}{a\eta}$，$\frac{1}{Ha} \propto \eta$，也就是 comoving hubble radius 正比于 conformal time

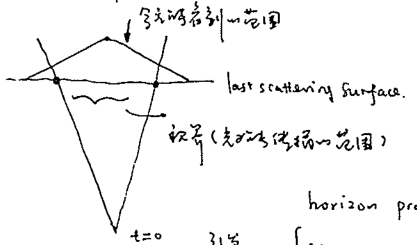
(注意不是 the situation of CMB)

$$\frac{(a_e H_e)^{-1}}{(a_0 H_0)^{-1}} = a_e = \frac{T_0}{T_e} = \frac{3K}{10^{15}\,GeV} \simeq \frac{10^{-4}\,eV}{10^{15}\times 10^{9}\,eV} \simeq 10^{-28}$$ ，所以早期的 comoving size 很小.

10^{15} GeV: some early time, most inflation model work

$$\frac{(a_{CMB} H_{CMB})^{-1}}{(a_0 H_0)^{-1}} = a_{CMB} = 10^{-3}$$ 在 CMB 时期，将有 10^3 个 patch 是处于 un-connected 的，没有因果联系

- 要从 physical size 理解



引发 inflation 的三个问题
- horizon problem
- flatness problem
- monopole problem → Guth and Tye

在 GUT (grand unification theory) 里面，纳入了强、弱、电相互作用

SU(5) 内含：SU(3), SU(2) 和 U(1)

预言了质子可以衰变（目前的实验否定了这个结论）

预言了很多的 monopole，为什么观测不到？

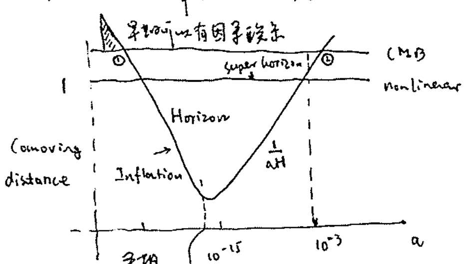
如果短时间内宇宙膨胀很快，使 monopole 被稀释 (1 per horizon size) 就可以解决.

initial condition problem

最初的密度涨落哪里来？

将 quantum fluctuation 拉伸到大尺度，成为大尺度结构的种子

- Inflation 如何解决 Horizon Problem



end of inflation

10^{15} GeV（已经在标准模型之外）$\frac{(a_e H_e)^{-1}}{(a_0 H_0)^{-1}} \simeq 10^{-28}$ (radiation dominated)

对于任意 scale，总可以找到一个时间 (cross time)，使 horizon > scale ($k\eta > 1$)，然后再 leave horizon ① ($k\eta < 1$)，最后 re-enter the horizon ②

$\frac{1}{aH}$ = Comoving size of local causal patches (Hubble radius)

$$\text{need } \frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \Rightarrow \frac{d}{dt}(aH) = \frac{d}{dt}\left(a\frac{da/dt}{a}\right) = \frac{d^2a}{dt^2} > 0 \quad \text{(inflation)}$$

For radiation, matter (DM, baryon), all cause $\frac{d^2a}{dt^2} < 0$.

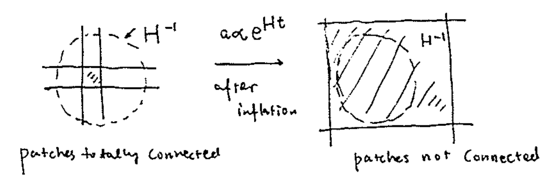
某一些特别的场有可能可以. For cosmological constant $H = \text{const}$, $a \propto e^{Ht}$, $a(t) = a_e e^{H(t-t_e)}$ $(t < t_e)$

$$\text{则 } \frac{1}{aH} = \frac{1}{Ha_e} e^{-H(t-t_e)}$$

e-folding

$$\begin{cases} (aH)^{-1}_b = \frac{1}{Ha_e} e^{H(t_e - t_b)} \simeq 10^{-28} (H_0 a_0)^{-1} e^{H(t_e - t_b)} \\ (aH)^{-1}_b \simeq (a_0 H_0)^{-1} \end{cases} \Rightarrow H(t_e - t_b) = \ln(10^{28}) \simeq 64 \gtrsim 60$$ （估计的误差）（事实上可以更早）

从物理尺度上来看，physical Hubble radius H^{-1}



如何驱动 inflation：want $\frac{d^2a}{dt^2} > 0$

$$\text{Friedmann equation} \begin{cases} \frac{d^2a/dt^2}{a} = -\frac{4}{3}\pi G(\rho + 3p) \\ \frac{(da/dt)^2}{a^2} = \frac{8}{3}\pi G\rho \end{cases}$$ conservation law

$$\frac{d^2a}{dt^2} > 0 \Rightarrow \rho + 3p < 0 \Rightarrow p < -\frac{1}{3}\rho$$ ($\rho > 0$)

→ 负的 pressure：体积增加，外部对其做功

→ matter $p \simeq 0$, radiation $p = \frac{1}{3}\rho$: can not act.

e.g. for vacuum energy, $\rho = \text{const}$

$$d(\rho V) = \underbrace{\rho dV}_{\text{const }\rho} = -pdV + \underbrace{0}_{\text{绝热}}$$

标准模型里的 Fermi, vector field 都不太可能产生 $P < 0$

目前唯一已知的 scalar field → Higgs，并不知道其是否可以作为 candidate for inflation

在 GUT 里（Standard），有多种 scalar field 可以使用 $\mathcal{L} = \frac{1}{2}\left(\frac{\partial\Phi}{\partial t}\right)^2 - V(\Phi)$

scalar field 给了对称性约束小，想怎么搞都可以

"Scalar field is a bitch"

— 2018, MaoYi

- Scalar field as the "inflaton"

$$\mathcal{L}(\phi, \dot\phi, t) = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi)$$

$$T^{(0)\alpha}{}_{\beta} = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}$$

$\phi \longleftrightarrow x$

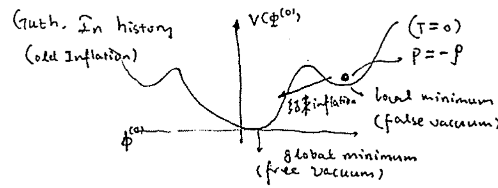
field strength ⟷ particle position

$$\rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi^{(0)})$$

$$P = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi^{(0)})$$ need $P < -\frac{1}{3}\rho$，即 V 比较显著地大于 T

φ 的演化可以看做粒子在势能中的演化轨迹

Guth, In history (old Inflation)

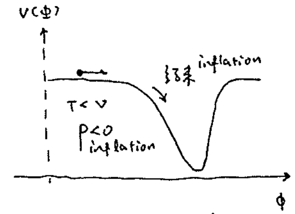


"Guth"

"Old Inflation"：依靠量子隧穿越过势垒，结束 inflation

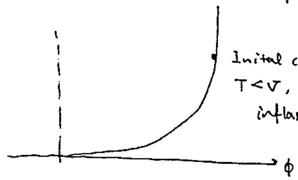
Not work：结束 inflation 是随机的，不同的空间点的 φ(x) 不能同时结束，宇宙不会均匀

Linde 1982
(new inflation/
slow roll inflation/
small field inflation)



slow roll + adiabatic perturbation
是标准的暴涨理论

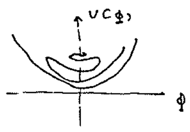
1980s
Linde 提出，任何势都可以产生一段时间的 inflation
(chaotic inflation/
large field inflation)



原则上，也可以有多个 inflation scalar field. Inflation model 是一大类理论
而场的演化则是由 Euler-Lagrangian Equation $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$ (推到弯曲空间) 决定

$$\frac{d^2\phi^{(0)}}{dt^2} + \underbrace{3H\frac{d\phi^{(0)}}{dt}}_{\text{阻尼的来源}} + V'(\phi) = 0$$

Hubble friction $\propto \dot\phi^{(0)}$



change t to η

$$\Rightarrow \quad \ddot\phi^{(0)} + 2aH\dot\phi^{(0)} + a^2 V' = 0$$

Conformal time

$$\eta = \int_{a_{\text{before inflation}}}^{a} \frac{da}{a^2 H} = \int_{a_b}^{a_e \,(\text{end of inflation})} + \int_{a_e}^{a} \frac{da}{a^2 H}$$

$$= \eta_e + \eta_{(a_e \to a)}$$

η_e (<0) ; $\eta_{(a_e\to a)}$ → 平移可调

$$\eta_{(a_e\to a)} = \frac{1}{H}\left(\frac{1}{a_e} - \frac{1}{a}\right) \simeq \frac{-1}{Ha}$$

下标是开始时刻

$$\eta_e = \frac{-1}{H}\left(\frac{1}{a_b} - \frac{1}{a_e}\right) = \frac{-1}{Ha_b}$$

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{ Matter couples scalar perturbation → 数学上更复杂，有与物质的 Coupling
Gravitational wave couples tensor perturbation → 数学上更简单，没有与物质 (scalar perturbation) 的耦合
(原初宇宙阶段)

Gravitational Wave (Tensor Perturbation)

tensor perturbation $g_{ij} = a^2 \begin{pmatrix} 1+h_+ & h_\times \\ h_\times & 1-h_+ \end{pmatrix}$, $(h_+, h_\times)$ are two indep. modes.

Evolution of tensor perturbation

$$\ddot h + 2\frac{\dot a}{a}\dot h + k^2 h = 0$$

→ no coupling with matter (scalar perturbation) from gravitational wave/tensor mode ($h = h_+$ or $h_\times$)

define $\tilde h = \frac{ha}{\sqrt{16\pi G}} \Rightarrow \frac{\dot h}{\sqrt{16\pi G}} = \frac{\dot{\tilde h}}{a} - \frac{\tilde h \dot a}{a^2}$ and $\frac{\ddot h}{\sqrt{16\pi G}} = \frac{\ddot{\tilde h}}{a} - \frac{2\dot{\tilde h}\dot a}{a^2} - \frac{\tilde h \ddot a}{a^2} + 2\tilde h \frac{\dot a^2}{a^3}$

Then the equation can be simplified as

$$\ddot{\tilde h} + \Big(k^2 - \underbrace{\frac{\ddot a}{a}}_{\text{not constant}}\Big)\tilde h = 0$$

仍然不是简谐振子，但已经很接近

Lets write $\hat{\tilde h}(\vec k, \eta) = v(k,\eta)\hat a + v^*(k\eta)\hat a^\dagger$, then $\ddot v + \left(k^2 - \frac{\ddot a}{a}\right)v = 0$

During inflation $\eta \cong -\frac{1}{aH}$ ($a_e \gg a$) (a_e: end of inflation)

$$\Rightarrow \dot a = \frac{da}{d\eta} = a^2 H, \quad \ddot a \simeq 2a\dot a H = 2a^3H^2, \quad \frac{\ddot a}{a} = 2a^2H^2 \simeq \frac{2}{\eta^2}$$

H = constant during inflation

$$\Rightarrow \ddot v + \left(k^2 - \frac{2}{\eta^2}\right)v = 0$$

简谐振子的解是 $v = \frac{e^{-ik\eta}}{\sqrt{2k}}$，但常数变化后变为 $v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)$ ★(引力波的通解)

$\frac{i}{k\eta}$: important only for $k|\eta| < 1$, super horizon

→ 这里重新定义了零点为 $\eta_e = 0$
所以 $\eta < 0$

仿照其它的解法，可以量子化，简单的方法是利用 $\hat a, \hat a^\dagger$

$$[\hat a, \hat a] = [\hat a^\dagger, \hat a^\dagger] = 0$$
$$[\hat a, \hat a^\dagger] = 1$$
$$\hat a|0\rangle = 0$$

$|0\rangle$: vacuum (no modes)

Compared with Simple Harmonic Osillator $\ddot x + \omega^2 x = 0$

$$\hat x(t) = v(\omega,t)\hat a + v^*(\omega,t)\hat a^\dagger, \quad v \propto e^{-i\omega t}$$

$$\hat p(t) = \frac{d\hat x}{dt} = \dot v\hat a + \dot v^*\hat a^\dagger \propto -i\omega(v\hat a - v^*\hat a^\dagger)$$

if we require $[\hat x, \hat p] = i\hbar = i$, we may require the normalization

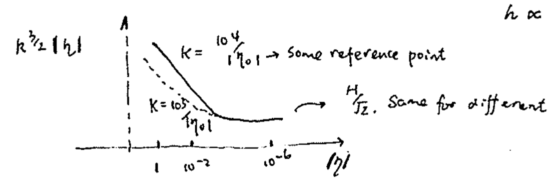
$$v(\omega,t) = \underbrace{\frac{1}{\sqrt{2\omega}}}_{\text{normalization}} e^{-i\omega t}$$

如果不考虑量子化，则有

$$\tilde h(k,t) = \frac{ha}{\sqrt{16\pi G}}, \quad \text{所以} \quad h(k,t) = \frac{\sqrt{16\pi G}}{a}\tilde h(k,t)$$

For subhorizon (Early during inflation, $k|\eta| \gg 1$), $v = \frac{e^{-ik\eta}}{\sqrt{2k}}$, $|v|$ = constant

$$h \propto 1/a \propto H\eta$$



For super horizon mode (After leaving the horizon, $k|\eta| \ll 1$), $v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) = \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{i}{k|\eta|}$

$$h \propto \frac{v}{a} \simeq \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{i}{k|\eta|} \times H|\eta| = \frac{e^{ik\eta}Hi}{\sqrt{2k^{3/2\times 2}}}$$

$$\left|\left(\frac{v}{a}\right)\right| k^{3/2} = \left|\frac{iH}{\sqrt 2}e^{-ik\eta}\right| = \frac{H}{\sqrt 2}$$

所以. leave horizon 后 的 不同 mode 的 amplitude 是 不同 的, 可以以 leave horizon 到 re-enter 之间 的 径直 时间 作为 initial condition.

For simple harmonic oscillator, $\langle 0|\hat{x}|0\rangle = \langle 0| va + v^* a^\dagger |0\rangle = 0$

所以. fluctuation 的 平均值 是 0

$$\langle 0|\hat{x}^2|0\rangle = \langle 0| (va + v^* a^\dagger)(va + v^* a^\dagger) |0\rangle$$

$$= vv^* \langle 0|aa^\dagger|0\rangle = vv^* \langle 0| \underbrace{[a,a^\dagger]}_{1} + a^\dagger a |0\rangle = |v(\omega,t)|^2$$

所以. fluctuation 的 方差 不是 0

variance

In the real tensor perturbation $v(\omega,t)$ 强立的

$\Updownarrow$

$\hat{h}(\vec{k},\eta)$

连续的. 所以 不能 自交

$$\langle \hat{h}^\dagger(\vec{k},\eta)\, \hat{h}(\vec{k}',\eta) \rangle = |v(k,\eta)|^2 (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$$

各 mode 是 独立 演化 的. 因此 有 $\delta^{(3)}(\vec{k}-\vec{k}')$ 存在

而 $h = \frac{\sqrt{16\pi G}}{a}\hat{h}$, 因而 h 的 variance

$$\langle \hat{h}^\dagger(\vec{k},\eta)\, \hat{h}(\vec{k},\eta) \rangle = \frac{16\pi G}{a^2} |v(\vec{k},\eta)|^2 \delta^{(3)}(\vec{k}-\vec{k}')(2\pi)^3$$

$$= (2\pi)^3 \delta^3(\vec{k}-\vec{k}') \underbrace{P_h(k)}_{\text{definition of power spectrum}}$$

For gravitational wave

$$P_h = 16\pi G \frac{|v(k,\eta)|^2}{a^2} \xrightarrow[\text{horizon}]{\text{after leaving the}} 16\pi G \frac{H^2}{2k^3} = \frac{8\pi G H^2}{k^3}$$

不同的 mode, 离开 horizon 的 时间 是 不同 的, 所以 H 为 是 那时 的 Hubble Parameter.

如果 要 考虑 不同 的 时间 的 不同 的 H. 由于 $|\eta| = \frac{1}{aH}$ (leaving time)

$$k^{-1} = (aH)^{-1} \quad \text{comoving hubble radius} \Big\} \; k|\eta| = 1$$

Comoving Scale of interest

所以 Hubble parameter 可以 记为 $H(-\frac{1}{k})$. Or define

$$P_h = A_T \, k^{n_T - 3} \qquad n_T \simeq 0 \text{ in inflation}$$

Tensor Perturbation; Generic Parametrization

Inflation "固化" 了 tensor perturbation, 可以 在 观测 量 (CMB-B mode) 中 体现 出来

若 能 够 探测 出 这一 信号, 则 我们 可以 把 对 宇宙 的 观测 推 到 极 早期 宇宙 (inflation) 阶段. 及 检验 高 能 阶段.

Scalar Perturbation

metric perturbation $\rightleftarrows$ Scalar field perturbation

比较 复杂. 更 方便 的 做法 是 利用 gauge transformation 简化 计算, 可以 在 conformal Newtonian gauge 更 简单.

思路 的 方法 是 在 某一 个 时刻. 取 gauge 使 spatial fluctuation 为 0. (Spatially flat Slicing gauge)

- Scalar field perturbation around a smooth background.

Scalar field $\Phi(\vec{x},t) = \Phi^{(0)}(t) + \delta\Phi(\vec{x},t)$

在 inflation 阶段. 可以 达到 均匀 各向同性

on a FLRW metric, $g_{00} = -1$, $g_{0i} = 0$, $g_{ij} = a^2\delta_{ij}$

Energy momentum tensor $T^{\mu\nu}{}_{;\mu} = T^{\mu}{}_{\nu;\mu} = 0 \Rightarrow \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0$

$$\Rightarrow T^\mu{}_{0;\mu} = \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_0 - \Gamma^\alpha{}_{0\mu} T^\mu{}_\alpha = 0$$

$$\Rightarrow \frac{\partial \delta T^0{}_0}{\partial t} + i k_i \delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i = 0$$

By definition

$$T^0{}_0 = g^{00}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2} g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - V \quad \text{potential } V(\phi)$$

$$\Rightarrow \delta T^0{}_0 = -\frac{\dot\phi^{(0)}}{a^2}\delta\dot\phi - V'\delta\phi$$

$$\delta T^i{}_0 = \frac{i k_i}{a^3}\dot\phi^{(0)}\delta\phi$$

$$\delta T^i{}_j = \delta_{ij}\left(\frac{\dot\phi^{(0)}}{a^2}\delta\dot\phi - V'\delta\phi\right)$$

The Conservation equation becomes

$$\delta\ddot\phi + 2aH\,\delta\dot\phi + k^2\delta\phi = 0 \qquad \text{和引力波的方程形式完全一样}$$

$$(aH = \dot a/a)$$

$$P_h = \frac{8\pi G H^2}{k^3} = 16\pi G \frac{H^2}{2k^3}$$

$$P_{\delta\phi} = \frac{H^2}{2k^3} \qquad (\text{normalization 不一样})$$

我们 希望 得到 Conformal Newtonian gauge 下 的 metric perturbation Φ

方法 是 寻找 gauge invariant 的 物理 量. 在 Spatially flat Slicing 下, $g_{ij} = \delta_{ij} a^2$. 加入 一些 perturbation

使 metric 改为 $ds^2 = -(1+2A)\,dt^2 - 2aB_{,i}\,dx^i dt + a^2\delta_{ij}\,dx^i dx^j$, 在 此 种 gauge 下. 可以 定义 不 变 量

保持 spatial part unperturbed

因 有 2个 gauge freedom

Bardeen's Velocity $\quad v = ikB - \frac{ik\,\dot\phi^{(0)}\delta\phi}{(\rho+P)a^2} \quad$ (defined at Spatially flat Slicing)

0th order pressure and density

$$\zeta = -\Phi_H - \frac{iaH}{k} v$$

is a gauge invariant

In spatially flat slicing $\zeta = -\frac{aH}{\dot{\phi}^{(0)}}\delta\phi$

Because $P_{\delta\phi} = \frac{2H}{k^3}$. we can show $P_\zeta = \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi} = \left.\frac{2a^2H^3}{k^3\dot{\phi}^{2(0)}}\right|_{k=aH}$

回到 Conformal Newtonian gauge 下面

dipole of temperature fluctuation

$$\zeta = -\frac{aH\delta\phi}{\dot{\phi}^{(0)}} \rightarrow \zeta = -\frac{3aH}{k}\Theta_1 - \Psi = -\frac{3}{2}\Psi = \frac{3}{2}\Phi$$

$$ds^2 = -(1+2\Psi)\,dt^2 + a^2(1+2\Phi)\,\delta_{ij}\,dx^i dx^j$$

可以利用近似 $\Phi \simeq -\Psi$ (higher moment ignored)

In initial condition we know $\Theta_1 = \frac{-k\Phi}{6aH} = \frac{k\Psi}{6aH}$

Now we know

$$P_\Phi = \left(\frac{2}{3}\right)^2 P_\zeta = \left(\frac{2}{3}\right)^2 \left.\frac{2a^2H^3}{k^3\dot{\phi}^{2(0)}}\right|_{k=aH} = \left.\frac{8}{9}\frac{a^2H^3}{k^3\dot{\phi}^{(0)}}\right|_{k=aH} = \left.\frac{8\pi G H^2}{9\epsilon k^3}\right|_{k=aH}$$

由于 $\dot{\phi}^{(0)}$ 并不是可观测量，我们用之前定义的与 slow roll 有关的 parameter 来改写 (ϵ)

最重要的特点是 $P_\Phi \propto \frac{1}{k^3}$, or if we consider H at leaving horizon, and write P_Φ

In a more general form

n_s close to 1.0 与离开 horizon 的时间有关

$$P_\Phi = A_S K^{-3} \times K^{n_s-1}$$

Inflation 的重要预言是 $k^3 P_\Phi \cong$ const (Scale invariant)

Harrison - Zeldovich scale invariant power spec.

<1> WMAP 最早发现 $n_s \simeq 0.96$，符合接近 1（同时，也是第一个做出 CMB lensing 的，但是 Tes 测得极不准）

而 $n_T \simeq 0$ 仍没有被证实

<2> 注意，这里的 ϕ field 是 harmonic oscillator，量子化之后是高斯的场。

高斯场的三点以上的关联函数一定是 0。因此三点关联可以验证高斯场理论，限制 inflation

<3> 另外，P_Φ 的绝对量值的测量也可以约束 Inflation model.

- Consistency Relation 极重要的关系.

$$n_T = -2\epsilon$$

$$n - 1 = -4\epsilon - 2\delta$$

Scalar

这里的 ϵ 定义是 $\epsilon = \left.\frac{d\ln H}{d\ln k}\right|_{k=aH} = \frac{k}{H} \times \cdots$

是 Hubble Constant 随着时间演化的度量。

不同的尺度，Leaving horizon 的时间不一样。

需 initial condition, 可以由 inflation 预言

计算 density, temperature fluctuation (linear)

non-linear scale 用 re-normalization 的技巧.

N-body simulation

对于 CMB, linear 就够了.

宇宙学探针 (需考虑各种观测误差.)

eg. LSS, weak-lensing, Lyα forest, 21 cm

LSS ↓ 间接   weak-lensing ↓ 直接看质量

Astrostatistics. MCMC. Machine learning

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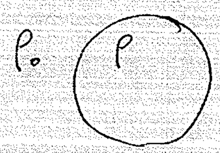
inflation 之后 离开 horizon



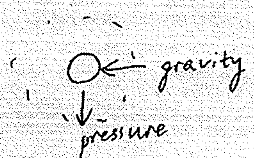
sperical collapse model

$\delta > \delta_{cr} = 14\pi^2 \approx 200$ (推导略)

只有这样高密度才能保证形成坍缩, 形成 halo

如何在早期形成这么高密度?

高红移 95% 物质在星系之外 (?)



Toy model:

$$\ddot{\delta} - (\text{gravity} - \text{pressure})\,\delta = 0$$

若只有引力: $\delta \propto e^{\alpha t}$ enhancement

若只有 pressure: $\delta \propto e^{i\beta t}$ oscillatory

Gravity instability.

坍缩会越来越快

更精确地:

$\delta(\vec{k})$ $\quad k^{-1} \quad (aH)^{-1}$

k^{-1}: comoving scale; $(aH)^{-1}$: comoving horizon size

若 $k^{-1} > (aH)^{-1}$, 则δ不会增加 (super-horizon)

primodial

$$P_\Phi \propto k^{-3}\left(\frac{k}{H_0}\right)^{n-1} \qquad n=1$$

$$\Rightarrow \left[k^3 P_\Phi\right]^{1/2} \propto \left(\frac{k}{H_0}\right)^{\frac{n-1}{2}}$$

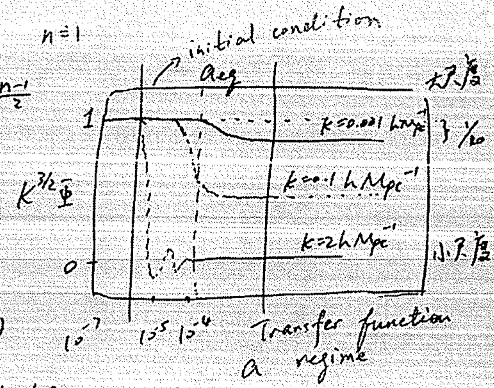
k↑ → early entering → more suppression

enter the horizon:

$$k = aH = cH_0\sqrt{\Omega_m a^{-3} + \Omega_R a^{-4} + \Omega_\Lambda}$$

$\propto a^{-1/2}$ (matter dominated)

$\propto a^{-1}$ (radiation dominated)



$$\underbrace{\Phi(\vec{k}, a) = \Phi_p(\vec{k})}_{\text{primodial}} \times \underbrace{\{\text{Tensor function}(k)\}}_{\text{scale dependence}} \times \underbrace{\{\text{Growth function}(a)\}}_{\text{dependence on a}}$$

大尺度一开始的 scale 一样.

原初 fluctuation 是 scale-free.

进入 horizon 时刻不同.

∴ 今天测到的是 scale-dependent

Final Wed 27/6 10 am 办公室拿卷子

No Internet!

Thurs 28/6 10 am 交卷

HWs last-minute turn-in

Tuesday Jun 19 10 am

$$T(k) = \frac{\Phi(k, a_{late})}{\Phi_{large\text{-}scale}(k, a_{late})}$$ an epoch well after the transfer function regime

$$= \frac{\Phi(k, a_{late})}{\frac{9}{10}\Phi_p(\vec{k})}$$

For matter dominated universe

$$\frac{\Phi(a)}{\Phi(a_{late})} \equiv \frac{D_1(a)}{a} \quad a > a_{late}$$

$D_1(a) \propto a$

growth function

$$\Phi(k, a) = \Phi(k, a_{late}) \cdot \frac{D_1(a)}{a} = \frac{9}{10}\Phi_p(k)\, T(k)\, \frac{D_1(a)}{a}$$

$a > a_{late}$

并不直接可测 potential

$$k^2\Phi = 4\pi G a^2 \left[\delta_m \rho_m + 4\rho_r \cancel{\Theta_{r,0}} + \frac{3aH}{k}\left(i\rho_m v_m \cancel{+ 4\rho_r \Theta_{r,1}}\right)\right]$$

no radiation 忽略不计 忽略高阶项

$$\Phi = \frac{4\pi G a^2 \rho_m}{k^2} \quad a > a_{late}$$

$$\rho_m = \Omega_m \rho_{cr} a^{-3}, \quad \rho_{cr} = \frac{3}{8\pi G} H_0^2$$

$$4\pi G \rho_m = 4\pi G \rho_{cr} \Omega_m a^{-3} = \frac{3}{2} H_0^2 \Omega_m a^{-3}$$

$$\Phi = \frac{1}{k^2} \cdot \frac{3}{2} H_0^2 \Omega_m a^{-1} \delta$$

$$\delta(\vec{k}, a) = \frac{k^2 \Phi(k, a)\, a}{\frac{3}{2} H_0^2 \Omega_m} \quad a > a_{late}$$

$$\delta(\vec{x}) \longleftrightarrow \delta(k)$$

$$\langle \delta(\vec{k})\, \delta^*(\vec{k}') \rangle \equiv \underbrace{(2\pi)^3 \delta^3(\vec{k} - \vec{k}')}_{\to V\delta_{\vec{k}\vec{k}'}} P(k)$$ 宇宙体积 V

power spectrum $P(k)$ $= \langle |\delta|^2 \rangle$

不同 k mode 之间无关

Roughly speaking

实数空间

$\rho(\vec{x}_1) \xrightarrow{\vec{x}} \rho(\vec{x}_2)$

$$\text{Probability} = \langle \rho(\vec{x}_1) \cdot \rho(\vec{x}_1 + \vec{x}) \rangle$$
$$= \bar{\rho}^2 \langle (1 + \delta(\vec{x}_1))(1 + \delta(\vec{x}_1 + \vec{x})) \rangle$$
$$= \bar{\rho}^2 (1 + \langle \delta(\vec{x}_1)\, \delta(\vec{x}_1 + \vec{x}) \rangle)$$
$$= \bar{\rho}^2 (1 + \xi(r))$$

Correlation function $\xi(r) \equiv \langle \delta(\vec{x}_1)\, \delta(\vec{x}_1 + r) \rangle$

可证 $$\xi(r) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} P(k)$$

$$\delta(\vec{k}, a) = \frac{k^2 a}{\frac{3}{2} H_0^2 \Omega_m} \cdot \frac{9}{10} \Phi_p(\vec{k}) \cdot T(k) \frac{D_1(a)}{a}$$

$$= \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \Phi_p(\vec{k})\, T(k)\, D_1(a) \quad a > a_{late}$$

$$P_\delta(k, a) = \frac{9}{25} \frac{k^4}{\Omega_m^2 H_0^2} \cdot T^2(k)\, D_1^2(a) \cdot P_\Phi \longrightarrow P_\Phi = \langle |\Phi_p(k)|^2 \rangle$$

$$= \frac{9}{25} \frac{k^4}{\Omega_m^2 H_0^2} \cdot T^2(k)\, D_1^2(a)\, A_s k^{n-4}$$

$$P_\Phi \propto k^{-3} \left(\frac{k}{H_0}\right)^{n-1}$$

$$P_\Phi \equiv \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0}\right)^{n-1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2$$

$$= 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2$$

s.t. 代入后计算简单

δ_H: density fluctuation at Horizon crossing at $k = H_0$

$D_1(a=1)$: Growth function at today.

$$(P_\Phi = A_s k^{n-4})$$

$$\delta(\vec{k}, a) = \frac{k^2 \Phi(\vec{k}, a)\, a}{\frac{3}{2} H_0^2 \Omega_m}$$

Dimensionless power spectrum

$$\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}$$

$$\because \int_{k_1}^{k_2} \frac{d^3k}{(2\pi)^3} P(k) = \int \frac{4\pi k^2 dk}{(2\pi)^3} P(k) = \int \underbrace{\frac{k^3 P(k)}{2\pi^2}}_{\Delta^2(k)} \cdot \underbrace{\frac{dk}{k}}_{d(\ln k)}$$

$$\Delta_\Phi = \frac{25}{9}\left(\frac{k}{H_0}\right)^{n-1} \delta_H^2 \frac{\Omega_m^2}{D_1^2(a=1)}$$

$$\propto k^{n-1}$$

For $n=1$ Δ_Φ has no k dependence. i.e. scale-independent
scale free

在 inflation 提出以前，有人猜到了这个特征.

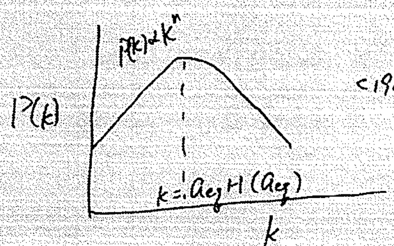
Harrison-Zal'dovich-Peebles spectrum $(n=1)$

$$\Delta_\delta = \frac{k^3 P_\delta}{2\pi^2} = \left(\frac{k}{H_0}\right)^{n+3} \delta_H^2\, T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2$$

For $k=H_0$, at today, at large scale $T(k)=1$

$$\Delta_\delta = \delta_H$$

At large scale $T(k)=1$ $\quad P_\delta \propto k^n \doteq k$



turn-over 可用于判断宇宙学模型

<1980s standard CDM $\Omega_m=1$, flat

ΛCDM : $\Omega_m=0.3$, $\Omega_\Lambda=0.7$ flat

看似 Λ 只对晚期有影响.

但也同时 Ω_m $\quad a_{eq} = \frac{\Omega_R}{\Omega_m}$

$\Lambda\text{CDM} \to \Omega_m \downarrow \to a_{eq} \uparrow \to$ later time eq.

$\to$ larger horizon size $\to$ smaller turnover k

$(aH)_{eq}$

non-linear scale

$\Delta(k_{nl})=1$ 即 linear scale 中假设

fluctuation $\ll 1$ 才可作展开

$k=0.1 \Longleftrightarrow P(k) = 3\text{-}4\times10^3$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$

at today $(a=1)$ $\quad k=0.1\,h\,\text{Mpc}^{-1}$ $\quad P(k)\sim10^4$

$$\Delta^2 \sim 1$$

$k_{nl} = 0.2\ h\,\text{Mpc}^{-1}$ at today

$\therefore k>0.2$ 不可用 linear scale.

必须在很大尺度 LSS 才可用 linear scale

而高红移星系 $\quad P_\delta \propto D_1^2(a)$

$D_1(a) = a$ for matter dominated.

高红移，对应 k 的 non-linear limit 可以大一些

$$P(k,a) = P(k,a=1)\left(\frac{D_1(a)}{D_1(a=1)}\right)^2 \doteq P(k,a=1)\,a^2$$

for matter dominated. $D_1(a)=a$

$$\Delta^2(k,a) \doteq \Delta^2(k,a=1)\,a^2$$

$$1 = \Delta^2(k_{nl},a) = \Delta^2(k_{nl},a=1)\,a^2$$

eg. $a=0.1$ $\quad z=9$ $\quad a^2=0.01$ $\quad 1 = \Delta^2(k_{nl},a=0.1) \Rightarrow$

$\Delta^2(k_{nl},a=1) = 100$

$\Rightarrow k_{nl}=1$. $P(k)\doteq10^3$

$\Rightarrow \frac{k^3P(k)}{2\pi^2} \doteq 10^2$

$\therefore$ 早期宇宙 21cm mapping

可以 probe 很大范围 linear scale.