Dodelson Chapter 2. Ex. 6 (1 credit)

$$R_{ij} = \delta_{ij}(2\dot{a}^2 + a\ddot{a})$$

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$$

$$R_{ij} - \frac{1}{2} g_{ij} R = \delta_{ij}[2\dot{a}^2 + a\ddot{a}] - \frac{1}{2} a^2 \delta_{ij} 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]$$

$$= -2\delta_{ij} a^2 \left[\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]$$

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}, \quad \text{so} \quad T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^2 P & & \\ & & a^2 P & \\ & & & a^2 P \end{pmatrix}$$

$$8\pi G T_{ij} = 8\pi G a^2 P \delta_{ij}$$

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij} \Rightarrow \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G P$$

Dodelson Ch.2. Ex.8 (4 credits)

(a)
$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \boxed{g_{ij}} \end{pmatrix}$$

where $g_{ij} = a^2 \tilde{g}_{ij}$, $a = a(t)$ and $\tilde{g}_{ij}$ is the pure spatial part.

Note that $\tilde{g}_{ij}$ is NOT diagonal in quasi-Cartesian coordinates.

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2}(\tilde{g}^{-1})^{ij} \end{pmatrix}$$

For $\mu=0$.

$$\Gamma^0{}_{00} = 0$$

$$\Gamma^0{}_{0i} = \frac{1}{2} g^{00} [\, \partial_i g_{00} + \partial_0 g_{i0} - \partial_0 g_{0i} \,] = 0.$$

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00} [\, \partial_j g_{i0} + \partial_i g_{j0} - \partial_0 g_{ij} \,]$$
$$= \frac{1}{2} \times (-1)(-1)\, 2a\dot{a}\, \tilde{g}_{ij} = H g_{ij}$$

For $\mu = i$

$$\Gamma^i{}_{00} = \frac{1}{2} g^{ik} (\partial_0 g_{0k} + \partial_0 g_{0k} - \partial_k g_{00}) = 0$$

$$\Gamma^i{}_{0j} = \frac{1}{2} g^{ik} (\partial_0 g_{jk} + \partial_j g_{0k} - \partial_k g_{0j})$$
$$= \frac{1}{2} a^{-2} (\tilde{g}^{-1})^{ik}\, 2a\dot{a}\, \tilde{g}_{jk} = H \delta^i_j$$

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{ij})$$
$$= \tilde{\Gamma}^i{}_{jk} \equiv \frac{1}{2} \tilde{g}^{il} (\partial_j \tilde{g}_{kl} + \partial_k \tilde{g}_{jl} - \partial_l \tilde{g}_{ij})$$

(b) $R_{00} = \Gamma^{\alpha}{}_{00,\alpha} - \Gamma^{\alpha}{}_{0\alpha,0} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{00} - \Gamma^{\alpha}{}_{\beta 0}\Gamma^{\beta}{}_{0\alpha}$

$$= -\Gamma^{i}{}_{0i,0} - \Gamma^{i}{}_{j0}\Gamma^{j}{}_{0i}$$

$$= -3\frac{dH}{dt} - H^2\delta^{i}{}_{j}\delta^{j}{}_{i}$$

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2$$

$$R_{00} = -3\left(\frac{\ddot{a}}{a} - H^2\right) - 3H^2 = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = \Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{ij} - \Gamma^{\alpha}{}_{\beta j}\Gamma^{\beta}{}_{i\alpha}$$

$$= \Gamma^{0}{}_{ij,0} + \Gamma^{k}{}_{ij,k} - \Gamma^{k}{}_{ik,j} + \Gamma^{k}{}_{0k}\Gamma^{0}{}_{ij} + \Gamma^{l}{}_{kl}\Gamma^{k}{}_{ij}$$
$$- \Gamma^{k}{}_{0j}\Gamma^{0}{}_{ik} - \Gamma^{0}{}_{kj}\Gamma^{k}{}_{i0} - \Gamma^{l}{}_{kj}\Gamma^{k}{}_{il}$$

$$= \Gamma^{0}{}_{ij,0} + \Gamma^{k}{}_{0k}\Gamma^{0}{}_{ij} - \Gamma^{k}{}_{0j}\Gamma^{0}{}_{ik} - \Gamma^{0}{}_{kj}\Gamma^{k}{}_{i0} + \tilde{R}_{ij}$$

$$= \left(H^2 + \frac{\ddot{a}}{a}\right)g_{ij} + 3H^2 g_{ij} - H^2 g_{ij} - H^2 g_{ij} + \tilde{R}_{ij}$$

$$= \left(\frac{\ddot{a}}{a} + 2H^2\right)g_{ij} + \tilde{R}_{ij}$$

where $\tilde{R}_{ij} \equiv \Gamma^{k}{}_{ij,k} - \Gamma^{k}{}_{ik,j} + \Gamma^{l}{}_{kl}\Gamma^{k}{}_{ij} - \Gamma^{l}{}_{kj}\Gamma^{k}{}_{il}$

$$= \tilde{\Gamma}^{k}{}_{ij,k} - \tilde{\Gamma}^{k}{}_{ik,j} + \tilde{\Gamma}^{l}{}_{kl}\tilde{\Gamma}^{k}{}_{ij} - \tilde{\Gamma}^{l}{}_{kj}\tilde{\Gamma}^{k}{}_{il}$$

is the purely spatial Ricci tensor.

To find $\tilde{R}_{ij}$, we need to compute $\tilde{\Gamma}^{i}{}_{jk}$ first.

$$\tilde{g}_{ij} = \delta_{ij} + \frac{K x^i x^j}{1 - K\vec{x}^2}$$ in quasi-Cartesian coordinates.

$$\partial_k \tilde{g}_{jl} = \frac{K}{(1-K\vec{x}^2)^2}\left\{(1-K\vec{x}^2)\left[\delta_k{}^j x^l + \delta_k{}^l x^j\right] + 2K x^j x^l x^k\right\}$$

$$\tilde{\Gamma}^i{}_{jk} = \frac{1}{2}\tilde{g}^{il}\left[\partial_k \tilde{g}_{jl} + (k \leftrightarrow j) - (k \leftrightarrow l)\right]$$

$$= \frac{K}{1-K\vec{x}^2}\,\tilde{g}^{il}\,\tilde{g}_{jk}\,x^l$$

Since $\tilde{g}^{il}\,\tilde{g}_{lj} = \delta^i{}_j$

$$(x^i\,\tilde{g}^{il})\,\tilde{g}_{lj} = x^j$$

Then it's easy to show that $x^i\,\tilde{g}^{il} = x^l(1-K\vec{x}^2)$

$\ulcorner$ <u>Proof</u>: $(x^i\,\tilde{g}^{il})\,\tilde{g}_{lj} = x^l(1-K\vec{x}^2)\left(\delta_{lj} + \frac{Kx^l x^j}{1-K\vec{x}^2}\right)$

$$= x^j(1-K\vec{x}^2) + K\vec{x}^2 x^j$$

$$= x^j \quad \lrcorner$$

$$\tilde{\Gamma}^i{}_{jk} = K\,\tilde{g}_{jk}\,x^i$$

For simplicity, at $\vec{x}=0$, it's easy to show that

$$\tilde{R}_{ij} = 2K\,\delta_{ij} = 2K\,\tilde{g}_{ij}$$

This expression is 3D-invariant and can be generalized to the general case.

Therefore,

$$R_{ij} = g_{ij}\left(\frac{\ddot{a}}{a} + 2H^2\right) + \frac{2K}{a^2}\,g_{ij}$$

$$= g_{ij}\left(\frac{\ddot{a}}{a} + 2H^2 + \frac{2K}{a^2}\right)$$

where $K = -\Omega_k H_0^2$.

(c) $$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$$

$$= -R_{00} + g^{ij} g_{ij} \left( \frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2} \right)$$

$$= \frac{3\ddot{a}}{a} + 3\left( \frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2} \right)$$

$$= 6\left( \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right)$$

$$R_{00} - \frac{1}{2} g_{00} \mathcal{R} = -3\frac{\ddot{a}}{a} + 3\left( \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right) = 3\left( H^2 + \frac{k}{a^2} \right)$$

$$8\pi G T_{00} = 8\pi G \rho$$

$$\Rightarrow \quad H^2 + \frac{k}{a^2} = \frac{8}{3}\pi G \rho$$

$$H^2 = \frac{8}{3}\pi G \rho - \frac{k}{a^2}$$