Dodelson Ch. 2. ex. 2 (20 credits)

(a). $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ in polar coordinates

$$ds^2_{2D} = dr^2 + r^2 d\theta^2$$

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{ii} (\partial_k g_{ji} - \partial_j g_{ki} - \partial_i g_{jk})$$

$$\Gamma^1{}_{jk} = \frac{1}{2} \times 1 \times (\partial_k g_{j1} - \partial_j g_{k1} - \partial_1 g_{jk})$$

$$\Gamma^1{}_{11} = \frac{1}{2} (\partial_1 g_{11}) = 0 .$$

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = \frac{1}{2} (\partial_1 g_{21} - \partial_2 g_{11} - \partial_1 g_{12}) = 0 .$$

$$\Gamma^1{}_{22} = \frac{1}{2} (\partial_2 g_{21} - \partial_2 g_{21} - \partial_1 g_{22}) = -\frac{1}{2} \frac{d}{dr}(r^2) = -r$$

$$\Gamma^2{}_{jk} = \frac{1}{2} r^{-2} (\partial_k g_{j2} - \partial_j g_{k2} - \partial_2 g_{jk})$$

$$\Gamma^2{}_{11} = 0$$

$$\Gamma^2{}_{22} = 0$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{2} r^{-2} (\partial_1 g_{22}) = \frac{1}{2} r^{-2} \frac{d}{dr}(r^2) = \frac{1}{r} .$$

(b) For $\mu = 1$, $\frac{d^2 x^1}{d\tau^2} + \Gamma^1{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$

$$\frac{d^2 r}{d\tau^2} + \Gamma^1{}_{22} \left(\frac{d\theta}{d\tau}\right)^2 = \frac{d^2 r}{d\tau^2} - r \left(\frac{d\theta}{d\tau}\right)^2 = 0 .$$

For $\mu = 2$, $\frac{d^2 x^2}{d\tau^2} + \Gamma^2{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 .$

$$\frac{d^2\theta}{d\tau^2} + 2\Gamma^2{}_{12} \frac{dr}{d\tau} \frac{d\theta}{d\tau} = \frac{d^2\theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} = 0 .$$

$\vec{r} = (r\cos\theta, r\sin\theta)$, $\hat{r} = (\cos\theta, \sin\theta)$, $\hat{\theta} = (-\sin\theta, \cos\theta)$

$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt} \hat{\theta}$, $\frac{d\hat{\theta}}{dt} = -\hat{r} \frac{d\theta}{dt}$, $\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left[\frac{dr}{dt}\hat{r} + r\hat{\theta}\frac{d\theta}{dt}\right]$

$$\vec{a} = \frac{d^2 r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} - r\left(\frac{d\theta}{dt}\right)^2\hat{r} + r\hat{\theta}\frac{d^2\theta}{dt^2}$$

$$= \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\right]r\hat{\theta}$$

$$= 0.$$

i.e. free motion in the plane.

Dodelson Ch. 2. ex. 3 (3 credits)

(a) $g_{\mu\nu}$ is diagonal,

$$g_{00} = -1 - 2\phi, \quad g_{ij} = (1 - 2\phi)\,\delta_{ij}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta} \right]$$

For $\mu = 0$, $\Gamma^{0}{}_{\alpha\beta} = \frac{1}{2} g^{00} \left[\partial_\beta g_{\alpha 0} + \partial_\alpha g_{\beta 0} - \partial_0 g_{\alpha\beta} \right]$

So, $\Gamma^{0}{}_{00} \approx -\frac{1}{2}(1+2\phi)^{-1} \left(\partial_0 g_{00} \right)$

$$= -\frac{1}{2}(1+2\phi)^{-1} \times (-2) \frac{\partial\phi}{\partial t}$$

$$= \frac{\partial\phi}{\partial t}(1+2\phi)^{-1}$$

$$\approx \frac{\partial\phi}{\partial t}$$

because $1 + 2\phi \Rightarrow 1 + 2\phi/c^2 = 1$

$\phi \ll c^2$ because the Newtonian gravitational energy is always much less than the rest energy.

$$\Gamma^{0}{}_{0i} = \frac{1}{2} g^{00} \left[\partial_i g_{00} + \partial_0 g_{i0} - \partial_0 g_{0i} \right]$$

$$= -\frac{1}{2}(1+2\phi)^{-1}(-2)\,\partial\phi/\partial x^i$$

Similarly, $\Gamma^{0}{}_{ij} \approx \partial\phi/\partial x^i$

For $\mu = i$, $\Gamma^{i}{}_{ij} \approx -(\partial\phi/\partial t)\,\delta_{ij}$

$$\Gamma^{i}{}_{\alpha\beta} = \frac{1}{2} g^{ii} \left[\partial_\beta g_{\alpha i} + \partial_\alpha g_{\beta i} - \partial_i g_{\alpha\beta} \right]$$

So,

$$\Gamma^{i}{}_{00} = \frac{1}{2}(1-2\phi)^{-1} \left[\partial_0 g_{0i} + \partial_0 g_{0i} - \partial_i g_{00} \right]$$

$$= \frac{1}{2}(1-2\phi)^{-1} \times 2\,\partial\phi/\partial x^i$$

$$\approx \partial\phi/\partial x^i$$

(b) For $\mu=0$, the time component of the geodesic equation is

$$\frac{d^2x^0}{d\lambda^2} + \Gamma^0{}_{00}\left(\frac{dx^0}{d\lambda}\right)^2 + 2\Gamma^0{}_{0i}\frac{dx^0}{d\lambda}\frac{dx^i}{d\lambda} + \Gamma^0{}_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0.$$

$$\Gamma^0{}_{0i}\frac{dx^0}{d\lambda}\frac{dx^i}{d\lambda} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\lambda}\frac{dx^0}{d\lambda} \sim \mathcal{O}(v^2), \quad \text{because } -\frac{\partial\phi}{\partial x^i} \sim \underbrace{\frac{d^2x^i}{dt^2}}_{\sim\mathcal{O}(v)} \text{ (see c below)}$$

$$\Gamma^0{}_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \sim \mathcal{O}(v^2)$$

These two terms are neglected.

$$\frac{dx^0}{d\lambda} \approx 1 + \mathcal{O}(v^2)$$

$$\Rightarrow \quad \frac{d^2x^0}{d\lambda} + \frac{\partial\phi}{\partial t} = 0$$

$$\frac{d}{d\lambda}\left(\frac{p^0}{m} + \phi\right) = \frac{d}{d\lambda}\left(\frac{dx^0}{d\lambda} + \phi\right) = \frac{d^2x^0}{d\lambda^2} + \frac{\partial\phi}{\partial t}\frac{dt}{d\lambda} + \frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\lambda}$$

$$\approx \frac{d^2x^0}{d\lambda^2} + \frac{\partial\phi}{\partial t} = 0$$

So $p^0 + m\phi$ is conserved.

(c) For $\mu=i$, the space component of the geodesic equation is

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{00}\left(\frac{dx^0}{d\lambda}\right)^2 + 2\Gamma^i{}_{0j}\frac{dx^0}{d\lambda}\frac{dx^j}{d\lambda} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0.$$

It is easy to show that

$$\Gamma^i{}_{0j} = -\frac{\partial\phi}{\partial t}\delta_{ij}$$

$$\Gamma^i{}_{jk} = \mathcal{O}\left(\frac{\partial\phi}{\partial x^i}\right)$$

So, $\Gamma^i{}_{0j}\frac{dx^0}{d\lambda}\frac{dx^j}{d\lambda} \sim \mathcal{O}\left(\frac{dx^i}{dt}\right) \ll \frac{dx^0}{dt}$.

$$\Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} \sim \mathcal{O}(v^2)$$

These two terms can be neglected.

$$\frac{d^2x^i}{d\lambda^2} + \frac{\partial\phi}{\partial x^i}\left(\frac{dt}{d\lambda}\right)^2 = 0. \quad \text{Since } \frac{dt}{d\lambda} = 1 + \mathcal{O}(v^2), \ \frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i}.$$