Dodelson 3.4

(a) $$\frac{dX_n}{dx} = \frac{x\,\lambda_{np}(x)}{H(x=1)}\left\{ e^{-x} - X_n(1+e^{-x})\right\}, \text{ where } x = Q/T$$

Note that $H(x) \propto a^{-2} \propto T^2 \propto x^{-2}$,

So, $x^2 H(x) = H(x=1)$

$$\frac{dX_n}{dx} = \frac{\lambda_{np}(x)}{x\,H(x)}\left\{ e^{-x} - X_n(1+e^{-x})\right\}$$

$$\frac{dX_n}{dx} + \frac{\lambda_{np}(x)}{x\,H(x)}(1+e^{-x})\,X_n = \frac{\lambda_{np}(x)}{x\,H(x)}e^{-x}$$

Let $$\frac{d\mu(x)}{dx} = \frac{\lambda_{np}(x)}{x\,H(x)}(1+e^{-x})$$

or $$\mu(x) \equiv \int_{x_i}^{x} \frac{dx'\,\lambda_{np}(x')}{x'\,H(x')}\left[1+e^{-x'}\right]$$

where x_i is some initial, very high temperature ($x_i \to 0$), and we let $\mu(x_i) = 0$.

$$\frac{dX_n}{dx} + \frac{d\mu(x)}{dx}X_n = e^{-\mu(x)}\frac{d}{dx}\left[e^{\mu(x)}X_n\right] = \frac{\lambda_{np}(x)}{x\,H(x)}e^{-x}.$$

$$\frac{d}{dx}\left[e^{\mu(x)}X_n\right] = \frac{\lambda_{np}(x)}{x\,H(x)}e^{-x}e^{\mu(x)}$$

$$e^{\mu(x)}X_n(x) = \int_{x_i}^{x} dx'\,\frac{\lambda_{np}(x')\,e^{-x'}}{x'\,H(x')}\,e^{\mu(x')} + X_n(x_i)$$

$X_n(x_i) \to X_n(T\to\infty) = \frac{1}{2}$

$$X_n(x) = \int_{x_i}^{x} dx'\,\frac{\lambda_{np}(x')\,e^{-x'}}{x'\,H(x')}\,e^{\mu(x')-\mu(x)} + \frac{1}{2}e^{-\mu(x)}$$

(b) $$H^2 = \frac{8}{3}\pi G \rho = \frac{8}{3}\pi G g_* \times \frac{\pi^2}{30} T^4 = \frac{4\pi^3}{45} G g_* T^4 = \frac{4}{45}\pi^3 G g_* \frac{Q^4}{x^4}$$

$$x H(x) = \left[\frac{4}{45}\pi^3 G Q^2 g_* \right]^{1/2} \frac{Q}{x}$$

$$\mu(x) = \int_{x_i}^{x} \frac{dx'}{x' H(x')} \lambda_{np}(x') \left[1 + e^{-x'} \right]$$

$$= \int_{x_i}^{x} dx' \times \left[\frac{4}{45}\pi^3 G Q^2 g_* \right]^{-1/2} \frac{x'}{Q} \times \frac{255}{\tau_n x'^5} \left(12 + 6x' + x'^2 \right) \times \left(1 + e^{-x'} \right)$$

$$= \frac{255}{\tau_n Q} \left[\frac{4\pi^3}{45} G Q^2 g_* \right]^{-1/2} \int_{x_i}^{x} \frac{dx'}{x'^4} \left(12 + 6x' + x'^2 \right) \left(1 + e^{-x} \right)$$

$$= -\frac{255}{\tau_n Q} \left[\frac{4\pi^3}{45} G Q^2 g_* \right]^{-1/2} \times \left[\left(\frac{4}{x^3} + \frac{3}{x^2} + \frac{1}{x} \right) + \left(\frac{4}{x^3} + \frac{1}{x^2} \right) e^{-x} \right]_{x_i}^{x}$$

(c) Numerical integral gives $X_n(x = \infty) = 0.148$

Note: when $x \to \infty$, $\mu(x) \to \infty$

the "missing" term $\frac{1}{2} e^{-\mu(x)} \to 0$ has no contribution to $X_n(x \to \infty)$.

Dodelson 3.10

$$n^{(0)}(x) = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \quad , \text{ for } m \gg T$$

$$= g\left(\frac{m^2}{2\pi x}\right)^{3/2} e^{-x} \quad , \text{ where } x = m/T$$

$$H(x) \propto x^{-2} \quad , \text{ so } , \quad x^2 H(x) = H(x=1) = \left[\frac{4}{45}\pi^3 G m^2 g_*\right]^{1/2} m$$

At freeze-out, $\quad n^0(x_f)\langle\sigma v\rangle = H(x_f)$

we can define $\quad \lambda \equiv \dfrac{m^3\langle\sigma v\rangle}{H(x=1)}$

$$LHS = n^0(x_f)\langle\sigma v\rangle = g\frac{m^3}{(2\pi)^{3/2}} x_f^{-3/2} e^{-x_f} \times m^{-3}\lambda H(x=1)$$

$$RHS = H(x_f) = x_f^{-2} H(x=1)$$

so,

$$\boxed{x_f^{-1/2} e^{x_f} = (2\pi)^{-3/2}\lambda g}$$