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物理72

Physical Cosmology
Final exam, Spring 2018
(Take-home 25 hours, 50 points)

47/50

16

Problem 1. "Age of the Universe" (16 points = 2 x 8)

(a) Write down the formula for the age of the Universe t_0 (with the scale factor from $a = 0$ to 1) in which the ratio of the energy densities to the critical density today are Ω_γ (radiation), Ω_m (matter), Ω_k (curvature) and Ω_Λ (cosmological constant), and the Hubble rate today is H_0.

(b) Compute the dimensionless age of the Universe $H_0 t_0$ for the matter-dominated case ($\Omega_m = 1,\ \Omega_\gamma = \Omega_k = \Omega_\Lambda = 0$).

(c) Compute the dimensionless age of the Universe $H_0 t_0$ for the radiation-dominated case ($\Omega_\gamma = 1,\ \Omega_m = \Omega_k = \Omega_\Lambda = 0$).

(d) Compute the dimensionless age of the Universe $H_0 t_0$ for the empty Universe case ($\Omega_k = 1,\ \Omega_\gamma = \Omega_m = \Omega_\Lambda = 0$).

(e) Compute the dimensionless age of the Universe $H_0 t_0$ for the currently favored case ($\Omega_k = 0,\ \Omega_\gamma \approx 0,\ \Omega_m = 0.3,\ \Omega_\Lambda = 0.7$). Hint: This integral can be done analytically, so feel free to use Mathematica or an integral book or simply do it yourself.

(f) For the above cases (b) – (e), give the age of the Universe t_0 in Gigayears. Assume a dimensionless Hubble parameter $h = 0.7$.

(g) Let $t(z)$ denote the age of the Universe at redshift z. Compute $H_0 t(z)$ for the matter-dominated case ($\Omega_m = 1,\ \Omega_\gamma = \Omega_k = \Omega_\Lambda = 0$).

(h) Compute the age of the Universe $t(z)$ numerically for the currently favored cosmology ($\Omega_k = 0,\ \Omega_\gamma \approx 0,\ \Omega_m = 0.3,\ \Omega_\Lambda = 0.7$) when the CMB radiation was released ($z = 1100$), and corresponding to the most distant quasars ever observed ULAS J1342+0928 ($z = 7.54$).

Solution:

(a) The formula for the age of the Universe is $t_0 = \int_0^1 \frac{da}{aH} = \frac{1}{H_0}\int_0^1 \frac{da}{a\sqrt{\Omega_\gamma a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}}$ ✓

(b) For matter dominated case ($\Omega_m = 1$, $\Omega_{others} = 0$) $t_0 H_0 = \int_0^1 \frac{da}{a\sqrt{\Omega_m a^{-3}}} = \frac{2}{3}$ ✓

(c) For radiation dominated case ($\Omega_\gamma = 1$. $\Omega_{others} = 0$) $t_0 H_0 = \int_0^1 \frac{da}{a\sqrt{\Omega_\gamma a^{-4}}} = \frac{1}{2}$ ✓

(d) For empty Universe ($\Omega_k = 1$, $\Omega_{others} = 0$) $t_0 H_0 = \int_0^1 \frac{da}{a\sqrt{\Omega_k a^{-2}}} = 1$ ✓

(e) For current favored case ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$. $\Omega_{others} \simeq 0$) $t_0 H_0 = \int_0^1 \frac{da}{a\sqrt{0.3a^{-3} + 0.7}} \overset{\text{by Mathematica}}{=} 0.964$ ✓

(f) For the case $h = 0.7$, $H_0 = h \cdot 100\ km/s/Mpc$. $\frac{1}{H_0} \simeq 14\ Gyr$

The time for (b) ~ (e) is

	(b)	(c)	(d)	(e)
t_0/Gyr	9.3	7.0	14.0	13.5

✓

(g) For matter dominated Universe, let's compute $t(a) H_0$ first:

$$t(a) H_0 = \int_0^a \frac{da'}{a' \sqrt{\Omega_m a'^{-3}}} = \frac{2}{3} a^{3/2}$$

Using the relation $a = \frac{1}{1+z}$ we obtain

$$t(z) H_0 = \frac{2}{3} \frac{1}{(1+z)^{3/2}}$$

(h) Again we set $h = 0.7$, or $\frac{1}{H_0} \simeq 14\ Gyr$. The age of the Universe

$$t(z) = \frac{1}{H_0} \int_0^{\frac{1}{1+z}} \frac{da}{a \sqrt{0.3 a^{-3} + 0.7}}$$

By numerical calculation, we obtain

$$t(z=1100) = \frac{1}{H_0} \times 3.3 \times 10^{-5} = 4.7 \times 10^{-4}\ Gyr$$

✓

$$t(z=7.54) = \frac{1}{H_0} \times 0.049 = 0.68\ Gyr$$

6 Problem 2. "Evolution of the scale factor for a Universe with $\Omega_\Lambda = 0$, $\Omega_m > 1$, and $\Omega_k < 0$": (6 points = 3+3)

(a) Show that the evolution of the scale factor of a **closed** Universe (only matter and curvature) can be worked out analytically to yield the following parametric expressions:

$$a = \frac{1}{2}\left(\frac{\Omega_m}{\Omega_m - 1}\right)(1-\cos\alpha),$$
$$t = \frac{1}{2H_0}\frac{\Omega_m}{(\Omega_m-1)^{3/2}}(\alpha - \sin\alpha).$$

The curve $a(t)$ is a *cycloid*, i.e., the same curve that solved the brachistochrone problem.

(b) For the above closed Universe, show that a photon leaving the origin at the Big Bang arrives back at the same place at the Big Crunch, i.e., just barely has the time to go circumnavigate the Universe once. Hint: The photon trajectory is defined by $d\tau = 0$ which gives an expression for dr/dt that you can integrate.

Solution: (a) For the closed Universe $\Omega_m > 1$, $\Omega_k = 1 - \Omega_m < 0$.

$$\frac{1}{a}\frac{da}{dt} = H = H_0\sqrt{\Omega_m a^{-3} + \Omega_k a^{-2}}$$

It can be simplified to

$$\int \frac{\sqrt{a}\,da}{\sqrt{\Omega_m + \Omega_k a}} = \int H_0\,dt$$

lhs can be done by transformation $a = \frac{\Omega_m}{-\Omega_k}\sin^2\eta$, then we obtain

$$\begin{cases} \text{lhs} = \int \frac{\sqrt{a}\,da}{\sqrt{\Omega_m + \Omega_k a}} = \frac{\left[\frac{\Omega_m}{-\Omega_k}\right]^{3/2}}{\sqrt{\Omega_m}} \int 2\sin^2\eta\,d\eta = \frac{\Omega_m}{(\Omega_m - 1)^{3/2}}\cdot\frac{1}{2}\cdot(2\eta - \sin 2\eta) \\ \text{rhs} = H_0 t \end{cases}$$

(here we assume $a(t=0) = 0$, to cancel the integral constant)

This imediately gives

$$\begin{cases} a = \frac{\Omega_m}{\Omega_m - 1}\frac{1-\cos 2\eta}{2} \overset{\alpha = 2\eta}{=\!=} \frac{\Omega_m}{\Omega_m - 1}\frac{1-\cos\alpha}{2} \\ t = \frac{1}{2H_0}\frac{\Omega_m}{(\Omega_m-1)^{3/2}}(2\eta - \sin 2\eta) \underset{\alpha = 2\eta}{=\!=} \frac{1}{2H_0}\frac{\Omega_m}{(\Omega_m-1)^{3/2}}(\alpha - \sin\alpha) \end{cases}$$ ✓

(b) From $d\tau = 0$ we obtain $\int \frac{c\,dt}{a} = \int \frac{dr}{\sqrt{1-kr^2}}$, where $k = \frac{(\Omega_m - 1)H_0^2}{c^2}$

$$\begin{cases} \text{lhs} = \int \frac{c\,dt}{a} = \int \frac{c\cdot\frac{1}{2H_0}\frac{\Omega_m}{(\Omega_m-1)^{3/2}}\,d(\alpha - \sin\alpha)}{\frac{\Omega_m}{\Omega_m-1}\frac{1-\cos\alpha}{2}} = \frac{c}{H_0}\frac{1}{(\Omega_m-1)^{1/2}}\alpha \\ \text{rhs} = \frac{1}{\sqrt{k}}\arcsin\sqrt{k}\,r \end{cases}$$

$$\Rightarrow r = \frac{c}{H_0\sqrt{\Omega_m - 1}}\sin\alpha$$ ✓

For the Big Crunch $\alpha = 2\pi$, $a = 0$, we have $r = 0$

So the photon arrives back.

6 Problem 3. "Angular scale of the Horizon" (6 points = 2 x 3)

This problem considers a flat $\Omega_m = 1$ Universe, with no radiation.

(a) Calculate the particle horizon scale at $z = 1100$ in *proper* distance, and in *comoving* distance, respectively.

(b) Calculate the comoving distance from $z = 1100$ to us at present.

(c) What is the angular scale θ subtended by this scale today? Express your result in degrees and in terms of angular frequency $\ell = 2\pi/\theta$. That the CMB is smooth above this scale is known as the horizon problem; causal physics generates anisotropies below this scale – in particular the CMB acoustic peaks.

Solution :

(a) The comoving particle horizon at redshift z is given by

$$\Delta\chi(z) = \int_0^{t(z)} \frac{c\,dt}{a} = \int_0^{\frac{1}{1+z}} \frac{c\,da}{H_0 E a^2} = \int_0^{\frac{1}{1+z}} \frac{c\,da}{H_0\, a^{1/2}} = \frac{2c}{H_0} \frac{1}{(1+z)^{1/2}}$$

The proper distance is then

$$\Delta d(z) = a(z)\chi(z) = \frac{2c}{H_0} \frac{1}{(1+z)^{3/2}}$$

at $z = 1100$, they are

$$\Delta d(1100) = \frac{c}{H_0} \times 5.5 \times 10^{-5}$$

$$\Delta\chi(1100) = \frac{c}{H_0} \times 6.0 \times 10^{-2}$$

✓

(b) The comoving distance from z to present is

$$\chi(z) = \int_{\frac{1}{1+z}}^{1} \frac{c\,da}{a^2 H} = \frac{c}{H_0} \int_{\frac{1}{1+z}}^{1} \frac{da}{a^{1/2}} = \frac{2c}{H_0}\left(1 - \frac{1}{(1+z)^{1/2}}\right)$$

At $z = 1100$

$$\chi(z) = \frac{c}{H_0} \times 1.94$$

✓

(c) The angular scale

$$\theta \simeq \frac{\Delta\chi(1100)}{\chi(1100)} \simeq 0.031 = 1.78 \text{ degrees}$$

✓

angular frequency

$$\ell = \frac{2\pi}{\theta} = 203.2$$

✓

Problem 4. "Nonstandard nucleosynthesis." (5 points)

Under which of the following suppositions would primordial nucleosynthesis have produced less ^{4}He than the standard model? Less ^{2}H (deuterium)?

a) Suppose that the baryond density in the universe today is larger than we think. more [illegible]. [illegible]. more n

b) Suppose that there is a fourth neutrino flavor with mass much less than 1 MeV. (It must be "sterile," e.g. right-handed, not to have been detected in Z^0 decay at LEP.) Pauli: blocking, enhancement

less n. less 4He less 2H

c) Suppose that there are many more neutrinos than antineutrinos or photons in the cosmic background today.

d) Suppose that there are many more antineutrinos than neutrinos or photons in the cosmic background today.

e) Suppose that there is a significant contribution of gravitational radiation to the total energy density of the universe, comparable in magnitude with the energy density of the microwave background radiation.

Solution: (b) (c) can produce less 4He, (a) (d) can produce less D. ✓

The idea is to analysis the reaction

$$\begin{cases} p + e^- \rightleftarrows n + \nu_e & (1) \\ p + n \rightleftarrows D + \gamma & (2) \\ D + D \rightleftarrows {}^3He + n, \quad {}^3He + D \rightleftarrows {}^4He + p & (3) \end{cases}$$

(a) more baryon, so more n in (1), then (2) (3) process is quicker, produce more 4He. less D will remain. ✓

(b) a fourth type of neutrino will make the inverse "←" reaction of (1) is quicker, so less n, less 4He production, and more D remains. ✗

(c) More neutrino than anti neutrino, so the inverse "←" reaction of (1) is quicker, less n. less 4He production, more D remains. ✓

(d) More antineutrino, so the reaction "→" of (1) is quicker, more n, more 4He production, and less D remains. ✓

(e) gravitational radiation (it is tensor perturbation) should not couple to matter component, so it should not change the baryon production. ✗

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Problem 5. Neutrinos. (3 points)

Suppose neutrinos have a mass of 1 eV. Then when their interactions freeze out in the early Universe at a temperature $T \approx$ MeV, they are relativistic. At approximately what redshift would the neutrinos become non-relativistic? If they became non-relativistic, then what, (qualitatively) would the energy distribution of neutrinos look like today?

Solution :

After freeze out, the temperature of neutrinos is

$$T_\nu = \frac{T_\gamma}{1.401} = \frac{T_{CMB}(1+z)}{1.401}$$

If the temperature is T_ν such that

$$kT_\nu \simeq m_\nu c^2$$

then they would become non-relativistic. So at this epoch $T_\nu \simeq \frac{m_\nu c^2}{k} \simeq 10^4 K$.

$$z \simeq \frac{1.401\, T_\nu}{T_{CMB}} \simeq 5140$$

If they become non-relativistic, the velocity/momentum distribution would be Maxwell distribution

$$f(p)\,dp \propto e^{-\frac{p^2}{2mkT_\nu}}\,dp$$

and thus

$$f(E)\,dE \propto e^{-\frac{E}{kT_\nu}} \frac{1}{\sqrt{E}}\,dE$$

However in this case $T_\nu \propto a^{-2}$.

✓

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Problem 6. "Sound horizon." (4 points)

For a relativistic gas, the "effective" mass density $\rho_r = u/c^2$, where u is the energy density. The pressure is $p_r = u/3$, and the properly relativistic sound speed is given by $c_s^2 = (4/3)\,p_r/(\rho_r + p_r/c^2) = c^2/3$. During the radiation-dominated era, assume this is coupled to a non-relativistic baryon fluid with mass density ρ_b, so the baryons contribute to the mass density but are negligible in the pressure. The sound speed is then $c_s^2 = (4/3)\,p_r/(\rho_b + \rho_r + p_r/c^2) = (c^2/3)\,(1 + (3/4)\,(\rho_b/\rho_r))^{-1}$. Estimate the co-moving distance a sound wave has propagated since the big bang during the radiation-dominated era (just assume a flat universe with $\Omega_r = 1$, at all times, until the epoch of radiation-matter equality at $z \approx 3000$, at which time $\rho_b = \rho_r$ – before this, $\rho_b \propto a^{-3}$, $\rho_r \propto a^{-4}$).

Solution:

The sound speed is $c_s = \frac{c}{\sqrt{3}} \frac{1}{\sqrt{\frac{3}{4}\frac{\rho_b}{\rho_r} + 1}}$ $\Rightarrow$

because

$\frac{\rho_b}{\rho_r} \propto \frac{a^{-3}}{a^{-4}} \propto a$, so $\frac{\rho_b}{\rho_r} = \frac{\rho_b(eq)}{\rho_r(eq)} \frac{a}{a_{eq}} = \frac{a}{a_{eq}}$

$$c_s = \frac{c}{\sqrt{3}} \frac{1}{\sqrt{\frac{3}{4}\frac{a}{a_{eq}} + 1}}$$

The comoving sound horizon is then

$$\text{Horizon} = \int_0^{t_{eq}} \frac{c_s\,dt}{a} = \int_0^{a_{eq}} \frac{c_s\,da}{H\,a^2} = \int_0^{a_{eq}} \frac{c_s\,da}{H_0}$$

$$= \int_0^{a_{eq}} \frac{1}{H_0} \frac{c}{\sqrt{3}} \frac{1}{\sqrt{\frac{3}{4}\frac{a}{a_{eq}} + 1}}\,da = \frac{c}{H_0} \frac{8}{3} \frac{1}{\sqrt{3}} \left(\sqrt{\frac{7}{4}} - 1\right) \frac{1}{1+z_{eq}}$$

when we set $z_{eq} = 3000$, then

$$\text{Horizon} \simeq \frac{c}{H_0} \times 1.66 \times 10^{-4}$$

✓

4 Problem 7. Magnetic monopoles. (4 points)

Magnetic monopoles are generically produced in grand unied theories (GUT) of the strong and electroweak interactions due to spontaneous symmetry breaking as the temperature drops below $E_{\mathrm{GUT}} = kT_{\mathrm{GUT}} \sim 10^{15}$ GeV. Since the fields whose symmetry is broken cannot be correlated on scales larger than the horizon at the time of symmetry breaking, this "phase transition" leads to the formation of about one "topological defect" (in this case, a monopole) per horizon volume (more detailed calculations give one per $\sim 3-4$ horizon volumes, but lets go with astronomical accuracy and say $3 \sim 1$). This is the horizon volume *at that time.*

If the monopole mass is $M \sim E_{\mathrm{GUT}}/c^2$, ✓ estimate the ratio of monopole number density to photon number density at that time (assume a flat radiation-dominated universe with $\Omega_r = 1$). At this density, monopole-antimonopole annihilation is not significant, so their co-moving number density should be conserved towards later times. Use this to estimate (order-of-magnitude) $\Omega_{\mathrm{MP}} = \rho_{\mathrm{monopole}}/\rho_{\mathrm{crit}}$ *today.* You should find a very large number ($\sim 10^{13}$), clearly wildly different from what's observed! Explain qualitatively how this might be solved by inflation. This is, in fact, historically one of the major arguments for inflation, and a major requirement in any inflationary model is that the temperature to which the Universe is re-heated after inflation must be $T_{\mathrm{re-heat}} \ll T_{\mathrm{GUT}}$ (or else the monopoles just come back!).

Solution: (1) For radiation dominated Universe

$$H = H_0 \sqrt{\Omega_R a^{-4}} \simeq H_0 a^{-2}$$
$$\Rightarrow \frac{1}{aH} \simeq \frac{a}{H_0}$$

The comoving horizon is

$$\chi(a) = \frac{c}{aH} \simeq \frac{c\,a}{H_0} \simeq \frac{c}{H_0}\frac{T_{CMB}}{T}$$

So the number density (in comoving scale) of monopole is

$$n_{monopole} a^3 = \frac{1}{\frac{4}{3}\pi \chi^3_{(a_{GUT})}} = \frac{1}{\frac{4}{3}\pi\left(\frac{c}{H_0}\frac{T_{CMB}}{T_{GUT}}\right)^3} \simeq 5.13\times10^{-6}\ cm^{-3}$$

where the photon number density is

$$n_\gamma a^3_{GUT} = n_{\gamma 0} = \int_0^\infty n(\omega)d\omega = 20.28\left(\frac{T_{CMB}}{K}\right)^3 cm^{-3} \doteq 410\ cm^{-3}$$

The ratio of these two number density is

$$\frac{n_{monopole}\, a^3_{GUT}}{n_\gamma\, a^3_{GUT}} \doteq 1.25\times10^{-8}$$

✓ ✗

(2) At present,

$$n_{monopole,0} \simeq n_{monopole}\, a^3_{GUT} = 5.13\times10^{-6}$$

So

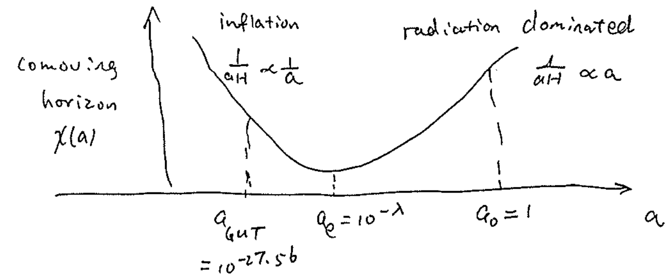
$$\Omega_{MP} = \frac{\rho_{monopole,0}}{\rho_{crit,0}} \simeq \frac{n_{monopole,0}\cdot M_{GUT}}{\frac{3H_0^2}{8\pi G}} = \frac{9.13\times10^{-15}\ g/cm^3}{9.21\times10^{-30}\ g/cm^3} \simeq 10^{15}$$ quite large. ✓

(3) Let's set a_e = scale factor at the end of inflation

a_{GUT} = scale factor where $kT = 10^{15}$ GeV $\simeq 10^{-27.56}$

$a_0 = 1$ is scale factor today.

If $a_{GUT} < a_e < a_0$, then the monopole will be generated before the end of inflation. At this time, the horizon could be very large, so the comoving number density $n_{monopole} a^3$ could be very small. That will result the monopole hard to be detected at today, Let's do some estimation, if we set $a_e \simeq 10^{-\lambda}$, where $0 < \lambda < 27.56$, the history of evolution of $\frac{1}{aH}$ is



we may have $\frac{\chi(a_{GUT})}{\chi(a_e)} = 10^{-\lambda + 27.56}$, $\frac{\chi(\text{present})}{\chi(a_e)} = 10^{\lambda}$

So we know the horizon volume is $V(a_{GUT}) = 10^{3(27.56-2\lambda)} \cdot \frac{4}{3}\pi \left(\frac{c}{H_0}\right)^3$

$$\Omega_{mp} = \frac{\frac{M_{GUT}}{V(GUT)}}{\frac{3H_0^2}{8\pi G}} \doteq \frac{10^{15}}{10^{6\times(27.56-\lambda)}}$$

If we need $\Omega_{mp} \ll 10^{-5}$, we just need $\lambda \ll 27.56 - \frac{20}{6} \simeq 24.23$

That means, if the inflation ended after $a_e = 10^{-24.23}$, the observed Ω_{mp} will be not detected.

6

Problem 8. Derive the Boltzmann equation for photons. (6 points)

Note: you may follow the derivation of *Dodelson*'s book. However, you must give every detail.

The original form of Boltzmann equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\dot{x}^i + \frac{\partial f}{\partial p}\frac{dp}{dt} + \underbrace{\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}}_{\text{at least } 2^{nd} \text{ order}} = C[f]$$

① we first derive $\dot{x}^i$:

from
$$\begin{cases} \delta_{ij}\hat{p}^i\hat{p}^j = 1 \\ p^2 = g_{ij}p^ip^j \\ g_{\mu\nu}p^\mu p^\nu = 0 \\ g_{\mu\nu} = \begin{pmatrix} -1-2\Psi & \\ & a^2(1+2\Phi)\delta_{ij} \end{pmatrix} \end{cases}$$

It can be shown

$$p^\mu = \left(p(1-\Psi),\ \frac{p\hat{p}^i}{a}(1-\Phi)\right)$$

Then
$$\dot{x}^i = \frac{dx^i}{dt} = \frac{1}{p^0}\frac{dx^i}{d\tau} = \frac{p^i}{p^0} = \frac{\hat{p}^i}{a}(1-\Phi+\Psi)$$

② we then derive $\frac{dp}{dt}$:

From geodesic equation, zero component is $\frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta$.

The lhs can be written by

$$lhs = \frac{dt}{d\lambda}\frac{dp^0}{dt} = p^0\frac{dp^0}{dt} = p(1-\Psi)\frac{d}{dt}\left(p(1-\Psi)\right)$$

to 1st order

$$= p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi) - p\left(\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right)\right]$$

The rhs can be written by

$$(-1)\times rhs = \frac{1}{2}g^{0\mu}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\right)p^\alpha p^\beta$$

$$= -\frac{1}{2}(1-2\Psi)\left(2\partial_\alpha g_{0\beta}\, p^\alpha p^\beta - \frac{\partial}{\partial t} g_{\alpha\beta}\cdot p^\alpha p^\beta\right)$$

$$= -\frac{1}{2}(1-2\Psi)\left(-4\partial_\alpha \Psi p^\alpha p^0 - \frac{\partial}{\partial t} g_{\alpha\beta}p^\alpha p^\beta\right)$$

$$= -\frac{1}{2}(1-2\Psi)\left(-4\frac{\partial\Psi}{\partial t}p^{0\,2} - 4\frac{\partial\Psi}{\partial x^i}p^ip^0 - \frac{\partial}{\partial t}(g_{00})p^{0\,2} - \frac{\partial g_{ij}}{\partial t}p^ip^j\right)$$

to 1st order

$$= -p^2\left(-\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t} - \frac{2}{a}\frac{\partial\Psi}{\partial x^i}\hat{p}^i - H + 2H\Psi\right)$$

Compare lhs and rhs we obtain

$$\frac{dp}{dt} = -p\left[H + \frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

Using ①②, the Boltzmann equation can be convert to

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right] = C[f] \text{------} \quad (\bullet 1)$$

For the distribution of photon, let $T(t)$ = mean temperature, then the perturbed temperature is $T(t,\vec{x},\hat{p}^i) = T(t)\left(1+\Theta(t,\vec{x},\hat{p}^i)\right)$. The distribution function

$$f(p) = \frac{1}{e^{p/T(t,\vec{x},\hat{p}^i)}-1} = f^{(0)}(p) + \frac{\partial f^{(0)}}{\partial T}T(t)\Theta$$

using $\frac{\partial f^{(0)}}{\partial T}T = -p\frac{\partial f^{(0)}}{\partial p}$

$$= f^{(0)}(p) - p\frac{\partial f^{(0)}}{\partial p}\Theta(t,\vec{x},\hat{p}^i) \quad \text{-----} \quad (\bullet 2)$$

③ Now, by combining $(\bullet 1)(\bullet 2)$, we can write the 0-th order and 1-st order term of Boltzmann equation

The 0-th order equation is

$$\left.\frac{df}{dt}\right|_{0th} \overset{0th\ order}{=\!=} \frac{\partial f^{(0)}}{\partial t} - p\frac{\partial f^{(0)}}{\partial p}H = 0$$

it is easy to convert it into $\frac{1}{T}\frac{dT}{dt} = -\frac{1}{a}\frac{da}{dt}$, which gives the solution like the un-perturbed case $T \propto a^{-1}$.

The 1-st order equation is

$$\left.\frac{df}{dt}\right|_{1th} \overset{1st\ order}{=\!=} \frac{\partial}{\partial t}\left[-\frac{\partial f^{(0)}}{\partial p}\cdot p\cdot\Theta\right] - p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right]$$

$$-p\frac{\partial}{\partial p}\left[-p\frac{\partial f^{(0)}}{\partial p}\right]\Theta H$$

$$\text{the first term} + \text{the last term} = -p\Theta\left[\frac{\partial}{\partial p}\underbrace{\left(\frac{\partial}{\partial t}f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}H\right)}_{=0}\right] + \frac{\partial\Theta}{\partial t}\left(-p\frac{\partial f^{(0)}}{\partial p}\right)$$

$$= -\frac{\partial\Theta}{\partial t}p\frac{\partial f^{(0)}}{\partial p}$$

So we obtain

$$\left.\frac{df}{dt}\right|_{1th} = -p\frac{\partial f^{(0)}}{\partial p}\left(\frac{\partial\Theta}{\partial t} + \frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a} + \frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a}\right) = C[f]\Big|_{1th} \quad (\bullet 3)$$

The Scattering term for Compton Scattering $\underset{(\vec{p})}{\gamma} + \underset{(\vec{q})}{e} \leftrightarrows \underset{(\vec{p}')}{\gamma'} + \underset{(\vec{q}')}{e'}$ is

$$C[f]\Big|_{1th} = \frac{1}{p}\int\frac{d^3q}{(2\pi)^3 2E_e(q)}\int\frac{d^3q'}{(2\pi)^3 2E_e(q')}\int\frac{d^3p'}{(2\pi)^3 2E(p')}$$

$$|M|^2(2\pi)^4\delta^3(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\delta\left(E(p)+E_e(q)-E(p')-E_e(q')\right)$$

$$\times\left\{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right\}$$

the Bose enhancement and Pauli Blocking are ignored.

for photon $\begin{cases} E(p) = p \\ E(p') = p' \end{cases}$, for electron $E(q) \simeq m_e$, $E(q') \simeq m_e$

$$\frac{q^2}{2m_e} - \frac{(\vec{q}+\vec{p}-\vec{p}')^2}{2m_e} \simeq \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e}$$

By integral over $\vec{q}'$, and using approximation above, also we expand $\delta\left(|\vec{p}| + \frac{q^2}{2m_e} - |\vec{p}'| - \frac{(\vec{q}+\vec{p}-\vec{p}')^2}{2m_e}\right) \simeq \delta(p-p') + \frac{\partial\delta(p-p')}{\partial p'}\frac{\vec{p}-\vec{p}'}{m}\cdot\vec{q}$

And by setting $|M|^2 \simeq 8\pi\sigma_T m_e^2$, we obtain

$$C[f]\Big|_{1th} = \frac{2\pi^2\sigma_T n_e}{p}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3 p'}\left\{\delta(p-p') + \frac{\partial\delta(p-p')}{\partial p'}\frac{\vec{p}-\vec{p}'}{m_e}\cdot\vec{q}\right\}f(\vec{q})$$

$$\times\left\{f(\vec{p}') - f(\vec{p})\right\}$$

By integral over $\vec{q}$, and use $n_e = \int\frac{d^3q}{(2\pi)^3}f(\vec{q})$ we obtain

$$\vec{v}_b = \frac{1}{n_e}\int\frac{d^3q}{(2\pi)^3}f(\vec{q})\,\vec{q}$$

$$C[f]\Big|_{1st} = \frac{2\pi^2 \sigma_T n_e}{p}\int \frac{d^3p'}{(2\pi)^3 p'}\left\{\delta(p-p') + \frac{\partial\delta(p-p')}{\partial p'}(\vec{p}-\vec{p}')\cdot\vec{v}_b\right\}\left(f(\vec{p}') - f(\vec{p})\right)$$

The first term $\delta(p-p')$ is of order 1. So we expand $f(\vec{p}')-f(\vec{p})$ into $-p'\frac{\partial f^{(0)}(p')}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p}) + f^{(0)}(p') - f^{(0)}(p)$, the second term has $\vec{v}_b$, which is small enough, so we only keep zero-order of $f(\vec{p}')-f(\vec{p}) = f^{(0)}(\vec{p}') - f^{(0)}(\vec{p})$, by integral over all Ω. and use the monopole $\Theta_0(t,\vec{x}) = \int\frac{d\Omega}{4\pi}\Theta(t,\vec{x},\hat{p})$, we obtain

$$\begin{aligned}
C[f]\Big|_{1st} &= \frac{\sigma_T n_e}{p}\int p'dp'\Big\{\delta(p-p')\Big[-p'\frac{\partial f^{(0)}}{\partial p'}\Theta_0 + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\Big] \\
&\qquad + \frac{\partial\delta(p-p')}{\partial p'}\vec{p}\cdot\vec{v}_b\left\{f^{(0)}(p') - f^{(0)}(p)\right\}\Big\} \\
&= \frac{\sigma_T n_e}{p}\left[-p^2\frac{\partial f^{(0)}}{\partial p}\left(\Theta_0 - \Theta(\hat{p})\right) + \vec{p}\cdot\vec{v}_b\left(-p\frac{\partial f^{(0)}(p)}{\partial p}\right)\right] \\
&= -p\frac{\partial f^{(0)}}{\partial p}\sigma_T n_e\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]
\end{aligned}$$

Now the equation (*3) can be written as

$$\frac{\partial\Theta}{\partial t} + \frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a} + \frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} = \sigma_T n_e\left[\Theta_0 - \Theta(\hat{p}) - \hat{p}\cdot\vec{v}_b\right]$$

Using conformal time $a\,d\eta = dt$, and let $\vec{v}_b \parallel \vec{k}$, $\cos\langle\vec{k},\hat{p}\rangle = \mu$, and make fourier transformation by $\frac{\partial}{\partial x^i} \to ik^i$, and use

$$\tau = \int_\eta^{\eta_0}\sigma_T n_e\,dt = \int_\eta^{\eta_0}\sigma_T n_e\,a\,d\eta \quad\Rightarrow\quad -\dot{\tau} = \sigma_T n_e a$$

We finally obtain

$$\dot{\tilde{\Theta}} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} + ik\mu\tilde{\Theta} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta}(\hat{p}) + \mu\tilde{v}_b\right]$$

$4x+3 = 120 + 2x = 1kx$

$G_9\ G_{15}\ G_{12}$