Quiz 1
(10 points, 30 minutes)

Student name: Solution

1. (2 credits) **Particle Horizon**

If the dominant form of mass-energy at early times scales as $\rho \propto a^{-\alpha}$, where a is the scale factor, then what is the minimum value of α, for a particle horizon to exist?

The particle horizon is
$$d_{max}(t) = a(t) \int_0^t \frac{c\, dt'}{a(t')}$$

$$= a(t) \int_0^a \frac{c\, da'}{a'^2 H(a')}$$

$$H(a') \propto a'^{1-\alpha/2}$$

$$[a'^2 H(a')]^{-1} \propto a'^{-2+\frac{\alpha}{2}}$$

$$d_{max}(t) \propto a(t) \times \frac{1}{-1+\frac{\alpha}{2}} a'^{-1+\frac{\alpha}{2}} \Big|_0^a$$

Therefore, the integral converges only when

$$-1 + \frac{\alpha}{2} > 0$$

So

$$\boxed{\alpha > 2}.$$

Note: I didn't conside the exponential expansion case, $\alpha = 0$. But this case has divergent d_{max}, too.

2. (3 credits) Find the **deceleration parameter** $q = -\ddot{a}a/\dot{a}^2$

- in a matter-dominated universe.
- in a radiation-dominated universe (i.e. the limiting case when $z \to \infty$).
- in a cosmological constant dominated universe (i.e. the limiting case when $t \to \infty$).

For $a \propto t^{\alpha}$,

$$\dot{a} = \alpha a/t \quad \propto t^{\alpha-1}$$

$$\ddot{a} = \frac{d\dot{a}}{dt} = (\alpha-1)\frac{\dot{a}}{t} = \alpha(\alpha-1)\frac{a}{t^2}$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{-\alpha(\alpha-1)\,a^2/t^2}{\alpha^2\,a^2/t^2} = \frac{\alpha(1-\alpha)}{\alpha^2} = \frac{1-\alpha}{\alpha}$$

- In a matter-dominated universe. $a \propto t^{2/3}$, $\alpha = \frac{2}{3}$. $\boxed{q = \frac{1}{2}}$
- In a radiation-dominated universe, $a \propto t^{1/2}$, $\alpha = 1/2$, $\boxed{q = 1}$
- In a Λ-dominated universe, $a \propto e^{Ht}$

$$\dot{a} = Ha$$

$$\ddot{a} = H^2 a$$

$$\boxed{q = -\frac{\ddot{a}a}{\dot{a}^2} = -1}.$$

3. (5=2+2+1 credits) It is possible to have a **completely empty, curved universe** (i.e. $\Omega_m = \Omega_\Lambda = \Omega_R = 0$).

(1) Solve the Friedman equation and find $a(t)$ as a function of time.

(2) [Suppose there was just enough matter to have an observer and a couple of galaxies. $\rho = 0$ would still be a good approximation, but now we can observe stars and measure their redshift.] What in this universe is the relation between comoving distance χ and redshift z?

(3) What about angular diameter distance $d_A(z)$ and luminosity distance $d_L(z)$ as a function of redshift z?

(1) In a completely empty, curved universe, $\Omega_m = \Omega_R = \Omega_\Lambda = 0$,

then $\Omega_K = 1$, $K = -1$ (open universe)

$$a_0 = \frac{1}{\sqrt{|\Omega_K|}\,H_0} = \frac{1}{H_0} \quad \left(\text{or } a_0 = \frac{c}{H_0} \text{ if adding "c"}\right)$$

$$\frac{da/dt}{a} = H = H_0\sqrt{\Omega_K (a/a_0)^{-2}} = H_0\left(\frac{a}{a_0}\right)^{-1}$$

$$d(a/a_0) = H_0\,dt$$

$$a/a_0 = H_0 t \quad (\text{the integration constant} = 0 \text{ because } a(t=0)=0)$$

$$\boxed{a = t \times a_0 H_0 = t} \quad (\text{If adding "c", then } a = ct)$$

(2)
$$\chi = c\int_t^{t_0} \frac{dt'}{a(t')/a_0} = c\int_0^z \frac{dz'}{H(z')} = c\int_0^z \frac{dz'}{H_0(1+z')}$$

$$= \frac{c}{H_0}\ln(1+z)$$

(3) Because $K=-1$, the comoving transverse distance

$$S_K(\chi) = a_0 \sinh[\chi/a_0] = \frac{c}{H_0}\sinh[\ln(1+z)]$$

angular diameter distance $d_A(z) = \frac{1}{1+z} S_K(\chi) = \frac{c}{H_0(1+z)}\sinh[\ln(1+z)]$

luminosity distance $d_L(z) = (1+z) S_K(\chi) = \frac{(1+z)c}{H_0}\sinh[\ln(1+z)]$.